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F. D. M. BEZERRA
A. N. CARVALHO
J. W. CHOLEWA
M. J. D. NASCIMENTO

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PARABOLIC APPROXIMATION OF DAMPED WAVE EQUATIONS VIA FRACTIONAL POWERS: FAST GROWING NONLINEARITIES AND CONTINUITY OF THE DYNAMICS

F. D. M. BEZERRA†, A. N. CARVALHO‡, J. W. CHOLEWA♯, AND M. J. D. NASCIMENTO⋆

Abstract. In this paper we consider a semilinear damped wave equation with supercritically fast growing nonlinearity using parabolic approximations governed by the fractional powers of the wave operator.

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Contents

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1. Introduction

We consider the problem of the form

$$\begin{cases} u_{tt} + au_t - \Delta_D u = f(u), & t > 0,\ x \in \Omega, \\ u(0,x) = u_0(x),\ u_t(0,x) = v_0(x), & x \in \Omega, \\ u(t,x) = 0, & t \geqslant 0,\ x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $a > 0$, Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geqslant 3$, and $f \in C^1(\mathbb{R})$ satisfies

$$|f'(s)| \leqslant C(1 + |s|^{\rho-1}), \quad s \in \mathbb{R}, \tag{1.2}$$

for some

$$\frac{N}{N-2} < \rho < \frac{N+2}{N-2}, \tag{1.3}$$

and

$$\limsup_{|s|\to\infty} \frac{f(s)}{s} < \mu_1 \tag{1.4}$$

with μ_1 being the first eigenvalue of the negative Dirichlet Laplacian $-\Delta_D$ in $L^2(\Omega)$.

Semilinear wave equation have been considered by many authors; see e.g. Arrieta, Carvalho and Hale [2], Babin and Vishik [3, 4], Chueshov, Lasiecka and Toundykov [10], Ghidaglia and Temam [11], Khanmamedov [13], Pata and Zelik [15] and references therein.

In this paper we study (1.1) using approximation by parabolic type problems of "lower" order. This complements in particular some earlier results in this direction by Carvalho, Cholewa and Dlotko [7], where (1.1) was considered with a supercritical exponent (1.3) as a limit as $\eta \searrow 0$ of a strongly damped wave equation involving term $2\eta(-\Delta_D)^{\frac{1}{2}}$ as in Chen and Triggiani [8].

Recall that if $X = L^2(\Omega)$ and $A : D(A) \subset X \to X$ is defined by

$$Au = -\Delta_D u \ \text{ for } \ u \in D(A) = H^2(\Omega) \cap H^1_0(\Omega), \tag{1.5}$$

then A is a positive self-adjoint operator and $-A$ generates a compact analytic C^0-semigroup in X.

Denote by X^α the fractional power spaces associated to operator A; that is, $X^\alpha = D(A^\alpha)$ with the norm $\|A^\alpha \cdot\|_X : X^\alpha \to \mathbb{R}^+$. For $\alpha > 0$ define also $X^{-\alpha}$ as the completion of X with the norm $\|A^{-\alpha} \cdot\|_X$. Observe that with this notation $X^{\frac{1}{2}} = H^1_0(\Omega)$ and $X^1 = H^2(\Omega) \cap H^1_0(\Omega)$. Observe also from [1, Chapter V] that $X^{-\alpha} = (X^\alpha)'$.

With the above set-up the problem (1.1) can be rewritten as an abstract Cauchy problem

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} + \varLambda \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ f^e(u) - av \end{bmatrix},\ t > 0, \qquad \begin{bmatrix} u \\ v \end{bmatrix}_{t=0} = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \tag{1.6}$$

where

$$\begin{aligned} &\Lambda : D(\Lambda) \subset X^{\frac{1}{2}} \times X \to X^{\frac{1}{2}} \times X, \\ &\Lambda \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \begin{bmatrix} 0 & -I \\ A & 0 \end{bmatrix} \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \quad \text{for} \quad \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in D(\Lambda) = X^1 \times X^{\frac{1}{2}} \end{aligned} \tag{1.7}$$

and f^e is given by

$$f^e(\varphi)(\cdot) := f(\varphi(\cdot)) \tag{1.8}$$

for φ in any suitable function space which will be specified below (see Lemma 3.1).

Given Λ as in (1.7) we consider next a family of fractional powers Λ^α, $\alpha \in (0,1)$. In particular, we prove the following.

Proposition 1.1. *i) For each $\alpha \in (0,1)$ the operator Λ^α is a negative generator of an analytic C^0 semigroup $\{e^{-\Lambda^\alpha t} : t \geqslant 0\}$.*
ii) The eigenvalues of $-\Lambda^\alpha$ converge as $\alpha \nearrow 1$ to the corresponding eigenvalues of $-\Lambda$.
iii) The linear semigroups generated by $-\Lambda^\alpha$ behave continuously as $\alpha \nearrow 1$, that is, given any $\theta \in (0,1]$,

$$e^{-\Lambda^\alpha t} \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \overset{X^{\frac{\theta}{2}} \times X^{\frac{\theta-1}{2}}}{\longrightarrow} e^{-\Lambda t} \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \quad \text{as} \quad \alpha \nearrow 1$$

uniformly for t in bounded time intervals and for $\begin{bmatrix} \varphi \\ \psi \end{bmatrix}$ in compact subsets of $X^{\frac{\theta}{2}} \times X^{\frac{\theta-1}{2}}$.

Since (1.1) can be viewed in the form (1.6), due to Proposition 1.1 it is natural to consider with (1.6) a family of problems

$$\frac{d}{dt} \begin{bmatrix} u^\alpha \\ v^\alpha \end{bmatrix} + \Lambda^\alpha \begin{bmatrix} u^\alpha \\ v^\alpha \end{bmatrix} = \begin{bmatrix} 0 \\ f^e(u^\alpha) - a v^\alpha \end{bmatrix}, \ t > 0, \qquad \begin{bmatrix} u^\alpha \\ v^\alpha \end{bmatrix}_{t=0} = \begin{bmatrix} u_{0\alpha} \\ v_{0\alpha} \end{bmatrix} \tag{1.9}$$

with $\alpha \nearrow 1$.

Exploiting parabolic structure of (1.9) we prove local well posedness of (1.9) for all $\alpha < 1$ close enough to 1 in a suitably large phase space of initial data containing the energy space $H^1_0(\Omega) \times L^2(\Omega)$ for (1.1).

Theorem 1.2. *Assume (1.2)-(1.3) and fix any number s satisfying $\frac{N}{2}(1 - \frac{1}{\rho}) - \frac{1}{\rho} < s < 1$.*

Then, for each $\alpha \in [\frac{N}{2}(\rho - 1) - \rho s, 1)$ the following hold.
i) For any $\begin{bmatrix} u_{0\alpha} \\ v_{0\alpha} \end{bmatrix} \in X^{\frac{s}{2}} \times X^{\frac{s-1}{2}}$ there exists a unique mild solution $\begin{bmatrix} u^\alpha \\ v^\alpha \end{bmatrix} \in C([0, \tau_{u_{0\alpha},v_{0\alpha}}), X^{\frac{s}{2}} \times X^{\frac{s-1}{2}})$ of (1.9) defined on a maximal interval of existence $[0, \tau_{u_{0\alpha},v_{0\alpha}})$. This solution depends continuously on the initial data and satisfies a blow up alternative in $X^{\frac{s}{2}} \times X^{\frac{s-1}{2}}$. In particular, if $\| \begin{bmatrix} u^\alpha \\ v^\alpha \end{bmatrix} \|_{X^{\frac{s}{2}} \times X^{\frac{s-1}{2}}}$−norm remains bounded as long as the solution exists then $\tau_{u_{0\alpha},v_{0\alpha}} = \infty$.
ii) The solution in part i) above is a regular solution. Namely,

$$\begin{bmatrix} u^\alpha \\ v^\alpha \end{bmatrix} \in C((0, \tau_{u_{0\alpha},v_{0\alpha}}), X^{\frac{1+\alpha}{2}} \times X^{\frac{\alpha}{2}}) \cap C^1((0, \tau_{u_{0\alpha},v_{0\alpha}}), X^{\frac{1+\sigma}{2}} \times X^{\frac{\sigma}{2}}) \ \text{ for each } \sigma < \alpha \tag{1.10}$$

and $\begin{bmatrix} u^\alpha \\ v^\alpha \end{bmatrix}$ satisfies (1.9).
iii) Actually, for any set B bounded in $X^{\frac{s}{2}} \times X^{\frac{s-1}{2}}$ there is a certain time $\tau_B > 0$ such that for each $\begin{bmatrix} u_{0\alpha} \\ v_{0\alpha} \end{bmatrix} \in B$ the solution $\begin{bmatrix} u^\alpha \\ v^\alpha \end{bmatrix}$ through $\begin{bmatrix} u_{0\alpha} \\ v_{0\alpha} \end{bmatrix}$ in part i) exists (at least) until τ_B and

given any $\tau \in (0, \tau_B]$ *there is a positive constant* $M = M(\tau, B)$ *such that*

$$\sup_{\left[\begin{smallmatrix} u_{0\alpha} \\ v_{0\alpha} \end{smallmatrix}\right] \in B} \left\| \left[\begin{smallmatrix} u^\alpha(\tau) \\ v^\alpha(\tau) \end{smallmatrix}\right] \right\|_{X^{\frac{1+\alpha}{2}} \times X^{\frac{\alpha}{2}}} \leqslant M. \tag{1.11}$$

Using (1.4) and exploiting gradient structure of (1.9) we establish global well posedness of (1.9) and obtain the existence of global attractors.

Theorem 1.3. *Assume (1.2)-(1.3) and fix any number* s *satisfying* $\frac{N}{2}(1-\frac{1}{\rho}) - \frac{1}{\rho} < s < 1$. *Assume also (1.4).*

For all $\alpha < 1$ *close enough to* 1 *we then have the following.*

i) For any $\left[\begin{smallmatrix} u_{0\alpha} \\ v_{0\alpha} \end{smallmatrix}\right] \in X^{\frac{s}{2}} \times X^{\frac{s-1}{2}}$ *the solution* $\left[\begin{smallmatrix} u^\alpha \\ v^\alpha \end{smallmatrix}\right]$ *of (3.1) obtained in Theorem 1.2 exists globally in time and satisfies for* $\tau > 0$ *the estimate*

$$\left\| \left[\begin{smallmatrix} u^\alpha(t) \\ v^\alpha(t) \end{smallmatrix}\right] \right\|_{X^{\frac{1+\alpha}{4}} \times X^{\frac{-1+\alpha}{4}}} \leqslant C\Big(\tau, \left[\begin{smallmatrix} u_{0\alpha} \\ v_{0\alpha} \end{smallmatrix}\right]\Big), \ t > \tau, \tag{1.12}$$

where C *is a positive constant which can be chosen uniformly for* $\left[\begin{smallmatrix} u_{0\alpha} \\ v_{0\alpha} \end{smallmatrix}\right]$ *in bounded subsets of* $X^{\frac{s}{2}} \times X^{\frac{s-1}{2}}$.

ii) The family of maps

$$S_\alpha(t) \left[\begin{smallmatrix} u_{0\alpha} \\ v_{0\alpha} \end{smallmatrix}\right] = \left[\begin{smallmatrix} u^\alpha(t) \\ v^\alpha(t) \end{smallmatrix}\right], \ \left[\begin{smallmatrix} u_{0\alpha} \\ v_{0\alpha} \end{smallmatrix}\right] \in X^{\frac{s}{2}} \times X^{\frac{s-1}{2}}, \ t \geqslant 0, \quad \textit{where } \left[\begin{smallmatrix} u^\alpha \\ v^\alpha \end{smallmatrix}\right] \textit{ is a solution of } (3.1),$$

is a compact semigroup in $X^{\frac{s}{2}} \times X^{\frac{s-1}{2}}$ *and* $S_\alpha(\tau)\gamma^+(B) = \cup_{t \geqslant \tau} S_\alpha(t)B$ *is bounded in* $X^{\frac{1+\alpha}{4}} \times X^{\frac{-1+\alpha}{4}}$ *for any* B *bounded in* $X^{\frac{s}{2}} \times X^{\frac{s-1}{2}}$ *and for any* $\tau > 0$.

iii) There exists a global attractor $\mathcal{A}_\alpha$ *for* $\{S_\alpha(t) : t \geqslant 0\}$ *in* $X^{\frac{s}{2}} \times X^{\frac{s-1}{2}}$.

iv) $\mathcal{A}_\alpha$ *is bounded in* $X^{\frac{1+\alpha}{2}} \times X^{\frac{\alpha}{2}}$ *and, given any* $\sigma \in [s-1, \alpha)$, $\mathcal{A}_\alpha$ *attracts under* $\{S_\alpha(t) : t \geqslant 0\}$ *bounded sets of* $X^{\frac{s}{2}} \times X^{\frac{s-1}{2}}$ *in* $X^{\frac{1+\sigma}{2}} \times X^{\frac{\sigma}{2}}$ *norm, that is,* $\mathcal{A}_\alpha$ *is for* $\sigma \in [s-1, \alpha)$ *a global* $(X^{\frac{s}{2}} \times X^{\frac{s-1}{2}} - X^{\frac{1+\sigma}{2}} \times X^{\frac{\sigma}{2}})$ *attractor.*

We then derive some bounds for the solutions when $\alpha < 1$ is close enough to 1. If $\alpha \nearrow 1$, $\left[\begin{smallmatrix} u_{0\alpha} \\ v_{0\alpha} \end{smallmatrix}\right] \in X^{\frac{1+\alpha}{4}} \times X^{\frac{-1+\alpha}{4}}$ and

$$\limsup_{\alpha \nearrow 1} \| \left[\begin{smallmatrix} u_{0\alpha} \\ v_{0\alpha} \end{smallmatrix}\right] \|_{X^{\frac{1+\alpha}{4}} \times X^{\frac{-1+\alpha}{4}}} < \infty,$$

we show that there exists $\alpha^* < 1$ such that for all $\alpha \in (\alpha^*, 1)$ the global solutions $\left[\begin{smallmatrix} u^\alpha \\ v^\alpha \end{smallmatrix}\right]$ of (3.1) through $\left[\begin{smallmatrix} u_{0\alpha} \\ v_{0\alpha} \end{smallmatrix}\right]$ are defined as in Theorem 1.3 and satisfy

$$\sup_{t \geqslant 0} \sup_{\alpha \in (\alpha^*, 1)} \left\| \left[\begin{smallmatrix} u^\alpha(t) \\ v^\alpha(t) \end{smallmatrix}\right] \right\|_{X^{\frac{1+\alpha}{4}} \times X^{\frac{-1+\alpha}{4}}} < \infty$$

(see Lemma 4.3). This enables us to obtain solutions of (1.6) with initial data in the "limit" space $X^{\frac{1}{2}}_{\lim} \times X_{\lim}$, where

$$X^{\frac{1}{2}}_{\lim} = \cap_{0 \leqslant \alpha < 1} X^{\frac{1+\alpha}{4}}, \qquad X_{\lim} = \cap_{0 \leqslant \alpha < 1} X^{\frac{-1+\alpha}{4}} \tag{1.13}$$

are normed, respectively, by

$$\| \cdot \|_{X^{\frac{1}{2}}_{\lim}} = \lim_{\alpha \nearrow 1} \| \cdot \|_{X^{\frac{1+\alpha}{4}}} \ \text{ and } \ \| \cdot \|_{X_{\lim}} = \lim_{\alpha \nearrow 1} \| \cdot \|_{X^{\frac{-1+\alpha}{4}}}. \tag{1.14}$$

Note that the energy space $X^{\frac{1}{2}} \times X$ for (1.1) is contained in $X^{\frac{1}{2}}_{\text{lim}} \times X_{\text{lim}}$ and that the latter space is contained in $X^{\frac{s}{2}} \times X^{\frac{s-1}{2}}$ for every $s < 1$ (see Lemma 4.1).

Definition 1.4. *Given* $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in X \times X^{-\frac{1}{2}}$ *we say that* $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$ *is a global mild solution of* (1.6) *provided that* $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] \in C([0,\infty), X \times X^{-\frac{1}{2}})$, $f^e(u) \in C([0,\infty), X^{-\frac{1}{2}})$ *and* $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$ *satisfies for* $t > 0$ *the integral equation*

$$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = e^{-\Lambda t} \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] + \int_0^t e^{-\Lambda(t-s)} \left[\begin{smallmatrix} 0 \\ f^e(u(s)) - av(s) \end{smallmatrix}\right] ds.$$

Theorem 1.5. *Assume (1.2)-(1.3) and (1.4).*

Given any $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in X^{\frac{1}{2}}_{\text{lim}} \times X_{\text{lim}}$ *there then exists a global mild solution* $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$ *of* (1.6) *which on each time interval* $[0,T]$ *and for any* $\zeta \in [-1,1)$ *is the uniform limit in* $X^{\frac{1+\zeta}{4}} \times X^{\frac{-1+\zeta}{4}}$ *of a certain sequence* $\{\left[\begin{smallmatrix} u^{\alpha_n} \\ v^{\alpha_n} \end{smallmatrix}\right]\}$*, where* $\alpha_n \nearrow 1$ *and* $\left[\begin{smallmatrix} u^{\alpha_n} \\ v^{\alpha_n} \end{smallmatrix}\right]$ *is a solution of (3.1) through* $\left[\begin{smallmatrix} u_{0\alpha_n} \\ v_{0\alpha_n} \end{smallmatrix}\right] = \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]$ *as in Theorem 1.3.*

For the solutions constructed via limiting procedure as in Theorem 1.5 we prove the existence of an absorbing set.

Theorem 1.6. *Assume (1.2)-(1.3) and (1.4).*

There then exists a ball B_0 *in* $X^{\frac{1}{2}}_{\text{lim}} \times X_{\text{lim}}$ *such that, given any bounded subset* B *of* $X^{\frac{1}{2}}_{\text{lim}} \times X_{\text{lim}}$*, a global mild solution* $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$ *of* (1.6) *through* $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in B$ *obtained via limiting procedure in Theorem 1.5 satisfies*

$$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} \in B_0 \ \text{ for all } \ t \geqslant t_B,$$

where t_B *is independent of* $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in B$.

We finally exhibit the existence of an attractor $\mathcal{A}_1$ for (1.1) in the sense of Theorem 5.5 which, in particular, is a compact set in $X^{\frac{1+\zeta}{4}} \times X^{\frac{-1+\zeta}{4}}$ for any $\zeta \in [-1,1)$. Furthermore, we show that the attractors $\mathcal{A}_\alpha$ for (3.1) as in Theorem 1.3 has the property that

$$\sup_{\alpha \in [\alpha_0,1)} \sup_{\left[\begin{smallmatrix} u_{0\alpha} \\ v_{0\alpha} \end{smallmatrix}\right] \in \mathcal{A}_\alpha} \left\| \left[\begin{smallmatrix} u_{0\alpha} \\ v_{0\alpha} \end{smallmatrix}\right] \right\|_{X^{\frac{1+\alpha}{4}} \times X^{\frac{-1+\alpha}{4}}} \leqslant R, \tag{1.15}$$

for some constants $\alpha_0 \in (0,1)$ and $R > 0$. We then obtain upper semicontinuity of the dynamics proving that

$$\lim_{\alpha \nearrow 1} d_{X^{\frac{1+\zeta}{4}} \times X^{\frac{-1+\zeta}{4}}} (\mathcal{A}_\alpha, \mathcal{A}_1) = 0 \quad \text{ for each } \ \zeta \in [-1,1) \tag{1.16}$$

(see Theorem 5.8), where $d_{X^{\frac{1+\zeta}{4}} \times X^{\frac{-1+\zeta}{4}}}$ is the Hausdorff semidistance of sets as in (5.23).

The following Section 2 is devoted to abstract linear wave operator, its fractional powers, associated extrapolated fractional power scale and to the proof Proposition 1.1.

In Section 3 we construct a family of approximate solutions for (1.1) proving Theorems 1.2 and 1.3.

In Section 4, following the limiting procedure, we show the existence of global mild solutions to (1.6) and prove Theorem 1.5.

Section 5 is devoted to a long time behavior of mild solutions to (1.6). In particular, we prove therein Theorem 1.6 and exhibit properties of an attractor $\mathcal{A}_1$ for (1.6). We also show that the family of attractors behave upper semicontinuously as $\alpha \nearrow 1$ proving (1.15) and (1.16).

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2. Analysis of linear wave operator

In this section we proceed with the analysis of the operator Λ which was specified in (1.5).

2.1. Fractional powers and associated linear semigroups.

Lemma 2.1. *If A and Λ are as in (1.5) and in (1.7) respectively then we have all the following.*
i) $0 \in \rho(\Lambda)$ *and*

$$\Lambda^{-1} = \begin{bmatrix} 0 & A^{-1} \\ -I & 0 \end{bmatrix}.$$

ii) The adjoint Λ^* of Λ is given by

$$\Lambda^* = \begin{bmatrix} 0 & I \\ -A & 0 \end{bmatrix} = -\Lambda.$$

iii) Operator $i\Lambda$ is self-adjoint and Λ is the infinitesimal generator of a C_0-group $\{e^{\Lambda t} : t \geqslant 0\}$ of unitary operators in $X^{\frac{1}{2}} \times X$.
iv) Fractional powers Λ^α can be defined for $\alpha \in (0,1)$ through

$$\Lambda^{-\alpha} = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \lambda^{-\alpha}(\lambda I + \Lambda)^{-1} d\lambda. \tag{2.1}$$

v) For each $\alpha \in (0,1)$ the operator Λ^α is a negative generator of an analytic C^0-semigroup $\{e^{-\Lambda^\alpha t} : t \geqslant 0\}$.
vi) Given any $0 < \alpha < 1$ we have that

$$\Lambda^{-\alpha} = \begin{bmatrix} \cos \frac{\pi\alpha}{2} A^{-\frac{\alpha}{2}} & \sin \frac{\pi\alpha}{2} A^{\frac{-1-\alpha}{2}} \\ & \\ -\sin \frac{\pi\alpha}{2} A^{\frac{1-\alpha}{2}} & \cos \frac{\pi\alpha}{2} A^{-\frac{\alpha}{2}} \end{bmatrix}$$

and

$$\Lambda^{\alpha} = \begin{bmatrix} \cos \frac{\pi\alpha}{2} A^{\frac{\alpha}{2}} & -\sin \frac{\pi\alpha}{2} A^{\frac{-1+\alpha}{2}} \\ & \\ \sin \frac{\pi\alpha}{2} A^{\frac{1+\alpha}{2}} & \cos \frac{\pi\alpha}{2} A^{\frac{\alpha}{2}} \end{bmatrix}. \tag{2.2}$$

vii) For each $\alpha \in (0,1]$ the spectrum of $-\Lambda^\alpha$ is a point spectrum consisting of eigenvalues

$$\lambda_{\alpha,n}^{\pm} = e^{\pm i \frac{\pi(2-\alpha)}{2}} (\mu_n)^{\frac{\alpha}{2}},\ n \in \mathbb{N}, \tag{2.3}$$

where $\{\mu_n\}_{n\in\mathbb{N}}$ *denotes the ordered sequence of eigenvalues of* A *including their multiplicity.*
viii) $\Lambda^{-\alpha}$ *converges to* Λ^{-1} *in* $\mathcal{L}_s(X^{\frac{1}{2}}\times X)$ *as* $\alpha\nearrow 1$.
ix) For each $\left[\begin{smallmatrix}\varphi\\ \psi\end{smallmatrix}\right]\in X^1\times X^{\frac{1}{2}}$

$$\Lambda^{\alpha}\left[\begin{smallmatrix}\varphi\\ \psi\end{smallmatrix}\right]\to\Lambda\left[\begin{smallmatrix}\varphi\\ \psi\end{smallmatrix}\right]\quad in\ \ X^{\frac{1}{2}}\times X\quad as\ \ \alpha\nearrow 1.$$

Proof: Part i) is a consequence of (1.5) and (1.7).

Part iii) comes froms ii) and Stone's theorem (see Pazy [17, Theorem 10.8]).

Parts iv) and v) follows from [14, Theorems 1, 2].

Concerning part vi) note that given $\lambda\in\mathbb{C}$ we have

$$\lambda I+\Lambda=\left[\begin{smallmatrix}\lambda I & -I\\ A & \lambda I\end{smallmatrix}\right]$$

and

$$(\lambda I+\Lambda)^{-1}=\begin{bmatrix}\lambda(\lambda^2I+A)^{-1} & (\lambda^2I+A)^{-1}\\ \\ -A(\lambda^2I+A)^{-1} & \lambda(\lambda^2I+A)^{-1}\end{bmatrix}\quad\text{for all }\lambda\in\rho(-\Lambda).$$

Due to (2.1) for any $0<\alpha<1$ we get

$$\Lambda^{-\alpha}=\begin{bmatrix}\cos\frac{\pi\alpha}{2}A^{-\frac{\alpha}{2}} & \sin\frac{\pi\alpha}{2}A^{\frac{-1-\alpha}{2}}\\ \\ -\sin\frac{\pi\alpha}{2}A^{\frac{1-\alpha}{2}} & \cos\frac{\pi\alpha}{2}A^{-\frac{\alpha}{2}}\end{bmatrix}$$

which leads to (2.2).

Concerning vii) observe that $\lambda\in\mathbb{C}$ is an eigenvalue of $-\Lambda^{\alpha}$ if and only if there exists a nontrivial solution of

$$\begin{cases}-\cos\dfrac{\pi\alpha}{2}A^{\frac{\alpha}{2}}\varphi+\sin\dfrac{\pi\alpha}{2}A^{\frac{-1+\alpha}{2}}\psi=\lambda\varphi\\ \\ -\sin\dfrac{\pi\alpha}{2}A^{\frac{1+\alpha}{2}}\varphi-\cos\dfrac{\pi\alpha}{2}A^{\frac{\alpha}{2}}\psi=\lambda\psi\end{cases}$$

which in turn holds if and only if

$$\lambda^2+2\lambda\cos\frac{\pi\alpha}{2}A^{\frac{\alpha}{2}}+A^{\alpha}=(\lambda-e^{i\frac{\pi(2-\alpha)}{2}}A^{\frac{\alpha}{2}})(\lambda-e^{-i\frac{\pi(2-\alpha)}{2}}A^{\frac{\alpha}{2}})$$

is not injective. The eigenvalues λ of $-\Lambda^{\alpha}$ are thus solutions of

$$(\lambda-e^{i\frac{\pi(2-\alpha)}{2}}\mu_n^{\frac{\alpha}{2}})(\lambda-e^{-i\frac{\pi(2-\alpha)}{2}}\mu_n^{\frac{\alpha}{2}})=0,$$

that is, $\lambda^{\pm}_{\alpha,n}$ are as in (2.3).

Part viii) follows from [1, Theorem III.4.6.2].

To prove part ix) we fix $\left[\begin{smallmatrix}\varphi\\ \psi\end{smallmatrix}\right]\in X^1\times X^{\frac{1}{2}}$ and observe that

$$\cos\frac{\pi\alpha}{2}A^{\frac{\alpha}{2}}\varphi\xrightarrow{X^{\frac{1}{2}}}0\quad\text{ and }\quad\cos\frac{\pi\alpha}{2}A^{\frac{\alpha}{2}}\psi\xrightarrow{X}0\ \text{ as }\alpha\nearrow 1$$

because on the one hand $\cos\dfrac{\pi\alpha}{2}\to 0$ and, on the other, due to viii),

$$A^{\frac{1}{2}}A^{\frac{\alpha}{2}}\varphi=A^{-\frac{1-\alpha}{2}}A^1\varphi\xrightarrow{X}A^1\varphi\quad\text{ and }\quad A^{\frac{\alpha}{2}}\psi=A^{-\frac{1-\alpha}{2}}A^{\frac{1}{2}}\psi\xrightarrow{X}A^{\frac{1}{2}}\varphi\ \text{ as }\ \alpha\nearrow 1.$$

In a similar manner, using that $\sin\dfrac{\pi\alpha}{2}\to 1$ as $\alpha\nearrow 1$ and that, due to viii),

$$A^{\frac{1+\alpha}{2}}\varphi = A^{\frac{-1+\alpha}{2}}A^{1}\varphi \xrightarrow{X} A\phi \quad \text{and} \quad A^{\frac{-1+\alpha}{2}}A^{\frac12}\psi \xrightarrow{X} A^{\frac12}\psi$$

we get

$$\sin\frac{\pi\alpha}{2}A^{\frac{1+\alpha}{2}}\varphi \xrightarrow{X} A\phi \quad \text{and} \quad \sin\frac{\pi\alpha}{2}A^{\frac{-1+\alpha}{2}}\psi \xrightarrow{X^{\frac12}} \psi \ \text{ as } \alpha\nearrow 1.$$

Hence we get

$$\begin{bmatrix} \cos\frac{\pi\alpha}{2}A^{\frac{\alpha}{2}} & -\sin\frac{\pi\alpha}{2}A^{\frac{-1+\alpha}{2}} \\ \sin\frac{\pi\alpha}{2}A^{\frac{1+\alpha}{2}} & \cos\frac{\pi\alpha}{2}A^{\frac{\alpha}{2}}\end{bmatrix}\begin{bmatrix}\varphi\\ \psi\end{bmatrix} \xrightarrow{X^{\frac12}\times X} \begin{bmatrix} 0 & -I\\ A & 0\end{bmatrix}\begin{bmatrix}\phi\\ \varphi\end{bmatrix} \text{ for each } \begin{bmatrix}\varphi\\ \psi\end{bmatrix}\in X^1\times X^{\frac12},$$

which gives the result. $\square$

2.2. **Convergence of linear semigroups: proof of Proposition 1.1.** Proposition 1.1 follows from the analysis of the wave operator carried out in Lemma 2.1. Namely, part i) is contained in Lemma 2.1 v). Part iii) follows from (2.3). Finally part ii) is a consequence of Lemma 2.1 ix) and [17, Chapter 3, Theorem 4.5]. $\square$

We include below Figure 1 which reflects, in particular, the loss of a "good" sectoriality property as $\alpha\nearrow 1$.

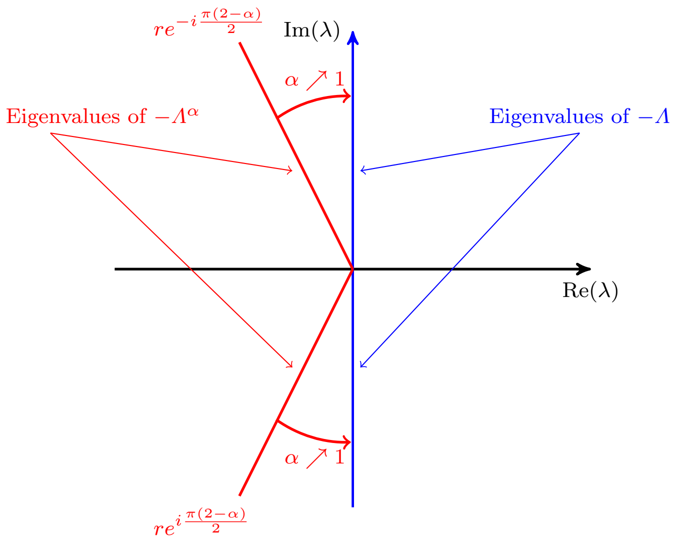

FIGURE 1. Note that eigenvalues of $-\Lambda^{\alpha}$ lie on the edges $\{re^{\pm i\frac{\pi(2-\alpha)}{2}} : r\geqslant 0\}$ of a sector of angle $\frac{\pi(2-\alpha)}{2}$ which approaches the half-plane $\{\lambda\in\mathbb{C} : \mathrm{Re}\lambda\geqslant 0\}$ as $\alpha\to 1$

2.3. **Associated extrapolated fractional power scale.**

Lemma 2.2. *The extrapolation space of $X^{\frac12}\times X$ generated by Λ coincides with $X\times X^{-\frac12}$.*

Proof: Using Lemma 2.1 i) we have that

$$\|\Lambda^{-1} \left[\begin{smallmatrix}\varphi\\ \psi\end{smallmatrix}\right]\|_{X^{\frac12}\times X} = \left\|\left[\begin{smallmatrix}\Lambda^{-1}\psi\\ -\varphi\end{smallmatrix}\right]\right\|_{X^{\frac12}\times X} = \|\left[\begin{smallmatrix}\varphi\\ \psi\end{smallmatrix}\right]\|_{X\times X^{-\frac12}} \quad \text{for any} \quad \left[\begin{smallmatrix}\varphi\\ \psi\end{smallmatrix}\right] \in X^{\frac12}\times X$$

and taking the completion we get the result (see [5, Lemma 2] for a similar argument). □

Lemma 2.3. *Let $\alpha \in (0,1]$ be fixed and let $\{E^\theta(\alpha), \theta \in [-1,\infty)\}$ be the extrapolated fractional power scale of order 1 generated by $(X^{\frac12}\times X, \Lambda^\alpha)$.*

Then

$$E^\theta(\alpha) = X^{\frac{1+\alpha\theta}{2}} \times X^{\frac{\alpha\theta}{2}} \quad \textit{for each} \quad \theta \in [-1,1]. \tag{2.4}$$

Proof: Consider first $\alpha = 1$. Due to Lemma 2.1 and Lumer-Phillips theorem (see [17]) Λ is a maximal accretive operator with zero in the resolvent set. Hence Λ is of the class $\mathcal{B}IP(1,\frac{\pi}{2})$ (see [1, §III.4.7.3]) and its fractional power spaces can be characterized with the aid of complex interpolation. Using this and repeating the proof of [5, Theorem 2] (see also [5, Remark 2]) we get

$$E^\sigma(1) = X^{\frac{1+\sigma}{2}} \times X^{\frac{\sigma}{2}} \quad \text{for each} \quad \sigma \in [-1,1], \tag{2.5}$$

which proves (2.4) for $\alpha = 1$.

In particular, given $\sigma \in [-1,0]$ we have from [1, Theorem V.1.4.12] that $E^\sigma(1) = (E^{-\sigma}(1))'$, or equivalently,

$$X^{\frac{1+\sigma}{2}} \times X^{\frac{\sigma}{2}} = (D(\Lambda^{-\sigma}))' = (X^{\frac{1-\sigma}{2}} \times X^{-\frac{\sigma}{2}})' \qquad \text{for} \quad \sigma \in [-1,0]. \tag{2.6}$$

Now for $\alpha \in (0,1)$ we have, raising power to power, that

$$D((\Lambda^\alpha)^\theta) = D(\Lambda^{\alpha\theta}) \quad \text{for any } \theta \in (0,1). \tag{2.7}$$

Using (2.7) and (2.5) with $\sigma = \alpha\theta$ we get

$$E^\theta(\alpha) = X^{\frac{1+\alpha\theta}{2}} \times X^{\frac{\alpha\theta}{2}} \quad \text{for each} \quad \theta \in (0,1].$$

Finally, if $\alpha \in (0,1)$ and $\theta \in [-1,0)$ then due to [1, Theorem V.1.4.12] and (2.6) we obtain using (2.7) and (2.6) with $\sigma = \alpha\theta$ that

$$E^\theta(\alpha) = (E^{-\theta}(\alpha))' = (D(\Lambda^{-\alpha\theta}))' = (X^{\frac{1-\alpha\theta}{2}} \times X^{-\frac{\alpha\theta}{2}})' = X^{\frac{1+\alpha\theta}{2}} \times X^{\frac{\alpha\theta}{2}},$$

which completes the proof. □

3. Solutions of perturbed problems and their properties

In this section we consider a Cauchy problem

$$\frac{d}{dt}\left[\begin{smallmatrix}u^\alpha\\ v^\alpha\end{smallmatrix}\right] + \Lambda^\alpha \left[\begin{smallmatrix}u^\alpha\\ v^\alpha\end{smallmatrix}\right] = F(\left[\begin{smallmatrix}u^\alpha\\ v^\alpha\end{smallmatrix}\right]), \quad t>0, \quad \left[\begin{smallmatrix}u^\alpha\\ v^\alpha\end{smallmatrix}\right]_{t=0} = \left[\begin{smallmatrix}u_{0\alpha}\\ v_{0\alpha}\end{smallmatrix}\right] \tag{3.1}$$

with Λ^α as in Lemma 2.1 and with F defined by

$$F\left(\left[\begin{smallmatrix}\varphi\\ \psi\end{smallmatrix}\right]\right) := \left[\begin{smallmatrix}0\\ f^e(\varphi)-a\psi\end{smallmatrix}\right] \tag{3.2}$$

for $\left[\begin{smallmatrix}\varphi\\ \psi\end{smallmatrix}\right]$ in any suitable function space which will be specified in Lemma 3.2 below.

3.1. **Action of nonlinear right hand side in extrapolated fractional power scale.** To obtain local well posedness of (3.1) we will use the analytic semigroup approach and a concept of the extrapolated fractional power scale developed in [2, Chapter V]. The following two lemmas will be useful in this.

Lemma 3.1. *Assume (1.2)-(1.3).*

If $\frac{N}{2}(1-\frac{1}{\rho})-\frac{1}{\rho} < s \leqslant \frac{N}{2}(1-\frac{1}{\rho})$ *then* $\sigma = \frac{N}{2}(\rho-1)-\rho s \in [0,1)$ *and* f^e *in (1.8) satisfies*

$$\|f^e(\varphi)\|_{X^{-\frac{\sigma}{2}}} \leqslant c(1+\|\varphi\|^{\rho}_{X^{\frac{s}{2}}}) \text{ for each } \varphi \in X^{\frac{s}{2}} \tag{3.3}$$

and

$$\|f^e(\varphi_1)-f^e(\varphi_2)\|_{X^{-\frac{\sigma}{2}}} \leqslant c\|\varphi_1-\varphi_2\|_{X^{\frac{s}{2}}}(1+\|\varphi_1\|^{\rho-1}_{X^{\frac{s}{2}}}+\|\varphi_2\|^{\rho-1}_{X^{\frac{s}{2}}}) \text{ for each } \varphi_1,\varphi_2 \in X^{\frac{s}{2}}, \tag{3.4}$$

where c is a certain positive constant.

Proof: Due to (1.3) we have that $0 < \frac{N}{2}(1-\frac{1}{\rho})-\frac{1}{\rho} < 1 < \frac{N}{2}(1-\frac{1}{\rho})$.

If $s \in (\frac{N}{2}(1-\frac{1}{\rho})-\frac{1}{\rho}, \frac{N}{2}(1-\frac{1}{\rho})]$ then $s = \frac{N}{2}(1-\frac{1}{\rho})-\frac{\sigma}{\rho}$ for a certain $\sigma \in [0,1)$. Observe that $\sigma = \frac{N}{2}(\rho-1)-\rho s$ and define $p = \frac{2N}{N+2\sigma}$. For such parameters $X^{\frac{s}{2}} \hookrightarrow L^{p\rho}(\Omega)$, $L^p(\Omega) \hookrightarrow X^{-\frac{\sigma}{2}}$ and

$$\|f^e(\varphi)\|_{X^{-\frac{\sigma}{2}}} \leqslant \|C(1+|\varphi|^{\rho})\|_{L^p(\Omega)} \leqslant C(|\Omega|^{\frac{1}{p}}+\|\varphi\|^{\rho}_{L^{p\rho}(\Omega)}) \leqslant c(1+\|\varphi\|^{\rho}_{X^{\frac{s}{2}}}), \quad \varphi \in X^{\frac{s}{2}},$$

which proves (3.3).

In a similar manner we get (3.4). □

Lemma 3.2. *Assume (1.2)-(1.3) and let $E^{\theta}(\alpha)$ (with $\theta \in [-1,1]$ and $\alpha \in (0,1]$) be as in (2.4).*

If $\frac{N}{2}(1-\frac{1}{\rho})-\frac{1}{\rho} < s \leqslant \frac{N}{2}(1-\frac{1}{\rho})$ *then for any* $\alpha \in [s-1+\frac{N}{2}(\rho-1)-\rho s, 1)$ *satisfying* $\frac{N}{2}(\rho-1)-\rho s < \alpha$ *there exist* $\theta_1 = \frac{-\frac{N}{2}(\rho-1)+\rho s}{\alpha} \in [-1,0)$ *and* $\theta_2 = \frac{s-1}{\alpha} \in (\theta_1,1)$ *such that* $\theta_2-\theta_1<1$ *and for F in (3.2) we have*

$$\left\|F\left(\left[\begin{smallmatrix}\varphi\\ \psi\end{smallmatrix}\right]\right)\right\|_{E^{\theta_1}(\alpha)} \leqslant c(1+\left\|\left[\begin{smallmatrix}\varphi\\ \psi\end{smallmatrix}\right]\right\|^{\rho}_{E^{\theta_2}(\alpha)}),\ \left[\begin{smallmatrix}\varphi\\ \psi\end{smallmatrix}\right] \in E^{\theta_2}(\alpha)$$

and

$$\begin{aligned}&\left\|F\left(\left[\begin{smallmatrix}\varphi_1\\ \psi_1\end{smallmatrix}\right]\right)-F\left(\left[\begin{smallmatrix}\varphi_2\\ \psi_2\end{smallmatrix}\right]\right)\right\|_{E^{\theta_1}(\alpha)}\\ &\quad\leqslant c\left\|\left[\begin{smallmatrix}\varphi_1\\ \psi_1\end{smallmatrix}\right]-\left[\begin{smallmatrix}\varphi_2\\ \psi_2\end{smallmatrix}\right]\right\|_{E^{\theta_2}(\alpha)}(1+\left\|\left[\begin{smallmatrix}\varphi_1\\ \psi_1\end{smallmatrix}\right]\right\|^{\rho-1}_{E^{\theta_2}(\alpha)}+\left\|\left[\begin{smallmatrix}\varphi_2\\ \psi_2\end{smallmatrix}\right]\right\|^{\rho-1}_{E^{\theta_2}(\alpha)}),\ \left[\begin{smallmatrix}\varphi_1\\ \psi_1\end{smallmatrix}\right],\left[\begin{smallmatrix}\varphi_2\\ \psi_2\end{smallmatrix}\right] \in E^{\theta_2}(\alpha).\end{aligned}$$

Proof: Recall that due to (1.3) we have $0 < \frac{N}{2}(1-\frac{1}{\rho})-\frac{1}{\rho} < 1 < \frac{N}{2}(1-\frac{1}{\rho})$. Recall also that $\frac{N}{2}(\rho-1)-\rho s \in [0,1)$ whenever $s \in (\frac{N}{2}(1-\frac{1}{\rho})-\frac{1}{\rho}, \frac{N}{2}(1-\frac{1}{\rho})]$. Furthermore, if $s \in (\frac{N}{2}(1-\frac{1}{\rho})-\frac{1}{\rho}, \frac{N}{2}(1-\frac{1}{\rho})]$ and $\frac{N}{2}(\rho-1)-\rho s < \alpha \in [s-1+\frac{N}{2}(\rho-1)-\rho s, 1)$ then $-1 < \frac{-\frac{N}{2}(\rho-1)+\rho s}{\alpha} \leqslant 0$ and $0 < \frac{s-1}{\alpha} - \frac{-\frac{N}{2}(\rho-1)+\rho s}{\alpha} < 1$.

We now apply (3.3) with $\sigma = \frac{N}{2}(\rho-1)-\rho s = -\alpha\theta_1$ and $s = 1+\alpha\theta_2$ where $\theta_1 = \frac{-\frac{N}{2}(\rho-1)+\rho s}{\alpha}$ and $\theta_2 = \frac{s-1}{\alpha}$ to get

$$\|f^e(\varphi)\|_{X^{\frac{\alpha\theta_1}{2}}} \leqslant c(1+\|\varphi\|^{\rho}_{X^{\frac{1+\alpha\theta_2}{2}}}) \text{ for each } \varphi \in X^{\frac{1+\alpha\theta_2}{2}}$$

and then, due to (2.4),

$$\left\|F\left(\left[\begin{smallmatrix}\varphi\\ \psi\end{smallmatrix}\right]\right)\right\|_{E^{\theta_1}(\alpha)} \leqslant c\left(1+\left\|\left[\begin{smallmatrix}\varphi\\ \psi\end{smallmatrix}\right]\right\|^{\rho}_{E^{\theta_2}(\alpha)}\right),\ \left[\begin{smallmatrix}\varphi\\ \psi\end{smallmatrix}\right] \in E^{\theta_2}(\alpha).$$

Applying then (3.4) with $\sigma = \frac{N}{2}(\rho-1)-\rho s = -\alpha\theta_1$ and $s = 1+\alpha\theta_2$ where $\theta_1 = \frac{-\frac{N}{2}(\rho-1)+\rho s}{\alpha}$ and $\theta_2 = \frac{s-1}{\alpha}$, and proceeding in a similar way as above, we get the result. □

3.2. **Local well posedness of (3.1): proof of Theorem 1.2.** Fix any number s satisfying $\frac{N}{2}(1-\frac{1}{\rho})-\frac{1}{\rho} < s < 1$. Observe that, due to Lemma 3.2, F is Lipschitz continuous on bounded sets from $E^{\theta_2}(\alpha)$ into $E^{\theta_1}(\alpha)$ where $\theta_1 = \frac{-\frac{N}{2}(\rho-1)+\rho s}{\alpha}$ and $\theta_2 = \frac{s-1}{\alpha}$. Observe also that the analytic semigroup approach of [12] applies for the extension to extrapolated spaces of the semigroup $\{e^{-\Lambda^{\alpha}t} : t \geqslant 0\}$ obtained in part v) of Lemma 2.1. Hence we get the result in part i) and, in addition, the solution satisfies

$$\left[\begin{smallmatrix}u^{\alpha}\\ v^{\alpha}\end{smallmatrix}\right] \in C((0,\tau_{u_{0\alpha},v_{0\alpha}}), X^{\frac{1+\alpha(\theta_1(s)+1)}{2}} \times X^{\frac{\alpha(\theta_1(s)+1)}{2}}) \cap C^1((0,\tau_{u_{0\alpha},v_{0\alpha}}), X^{\frac{1+\sigma}{2}} \times X^{\frac{\sigma}{2}})$$

for each $\sigma < \alpha(\theta_1+1)$.

To prove part ii) we now restart the solution at arbitrarily small positive time $t_0 \in (0,\tau_{u_{0\alpha},v_{0\alpha}})$, that is, with the initial data $\left[\begin{smallmatrix}u_{0\alpha}\\ v_{0\alpha}\end{smallmatrix}\right] = \begin{bmatrix}u_\alpha(t_0)\\ v_\alpha(t_0)\end{bmatrix} \in X^{\frac{s_1}{2}} \times X^{\frac{s_1-1}{2}}$ where $s_1 = 1+\alpha(\theta_1(s_1)+1)$. Hence we get

$$\left[\begin{smallmatrix}u^{\alpha}\\ v^{\alpha}\end{smallmatrix}\right] \in C((0,\tau_{u_{0\alpha},v_{0\alpha}}), X^{\frac{1+\alpha(\theta_1(s_1)+1)}{2}} \times X^{\frac{\alpha(\theta_1(s_1)+1)}{2}}) \cap C^1((0,\tau_{u_{0\alpha},v_{0\alpha}}), X^{\frac{1+\sigma}{2}} \times X^{\frac{\sigma}{2}})$$

for each $\sigma < \alpha(\theta_1(s_1)+1)$.

Note that repeating this step by step we obtain for $j = 1,\dots,k$ that

$$\left[\begin{smallmatrix}u^{\alpha}\\ v^{\alpha}\end{smallmatrix}\right] \in C((0,\tau_{u_{0\alpha},v_{0\alpha}}), X^{\frac{1+\alpha(\theta_1(s_j)+1)}{2}} \times X^{\frac{\alpha(\theta_1(s_j)+1)}{2}}) \cap C^1((0,\tau_{u_{0\alpha},v_{0\alpha}}), X^{\frac{1+\sigma}{2}} \times X^{\frac{\sigma}{2}})$$

for each $\sigma < \alpha(\theta_1(s_j)+1)$.

Now observe that $s_{j+1} = 1+\alpha(\theta_1(s_1)+1)$ is an increasing sequence which needs to exceed $s^* := \frac{N}{2}(1-\frac{1}{\rho})$ as otherwise it would be bounded and its limit would satisfy a false relation.

The above ensures that in a final number of steps the solution enters $X^{\frac{s^*}{2}} \times X^{\frac{s^*-1}{2}}$ in which case, due to Lemma 3.2, F is Lipschitz continuous on bounded sets from $E^{\theta_2(s^*)}(\alpha)$ into $E^{\theta_1(s^*)}(\alpha)$ where $\theta_1(s^*) = 0$, that is, $1+\alpha(\theta_1(s^*)+1) = 1+\alpha$. Using this we get

$$\left[\begin{smallmatrix}u^{\alpha}\\ v^{\alpha}\end{smallmatrix}\right] \in C((t_0,\tau_{u_{0\alpha},v_{0\alpha}}), X^{\frac{1+\alpha}{2}} \times X^{\frac{\alpha}{2}}) \cap C^1((t_0,\tau_{u_{0\alpha},v_{0\alpha}}), X^{\frac{1+\sigma}{2}} \times X^{\frac{\sigma}{2}})$$

for each $\sigma < \alpha$ and since t_0 can be chosen arbitrarily small we obtain (1.10).

Concerning part iii) we recall from [6, Theorem 5] that the regularizing properties of the local solution exhibited in the proof of part ii) above are actually uniform on bounded sets. Hence we get (1.11). □

3.3. **Lyapunov functionals associated with perturbed problems.** Local solution to (3.1) in Theorem 1.2 satisfies (see (1.10)

$$\begin{cases} u^\alpha_t + \cos\tfrac{\pi\alpha}{2} A^{\frac{\alpha}{2}} u^\alpha - \sin\tfrac{\pi\alpha}{2} A^{\frac{-1+\alpha}{2}} v^\alpha = 0,\\ v^\alpha_t + \sin\tfrac{\pi\alpha}{2} A^{\frac{1+\alpha}{2}} u^\alpha + \cos\tfrac{\pi\alpha}{2} A^{\frac{\alpha}{2}} v^\alpha + a v^\alpha = f(u^\alpha). \end{cases} \tag{3.5}$$

From the first equation we obtain that $\sin\frac{\pi\alpha}{2} v^\alpha = A^{\frac{1-\alpha}{2}}\Big(u^\alpha_t + \cos\frac{\pi\alpha}{2} A^{\frac{\alpha}{2}} u^\alpha\Big)$ which leads to

$$\sin\frac{\pi\alpha}{2} v^\alpha_t = A^{\frac{1-\alpha}{2}}\Big(u^\alpha_{tt} + \cos\frac{\pi\alpha}{2} A^{\frac{\alpha}{2}} u^\alpha_t\Big). \tag{3.6}$$

Substituting (3.6) to the second equation in (3.5), after some calculations we get

$$A^{\frac{1-\alpha}{2}} u^\alpha_{tt} + 2\cos\frac{\pi\alpha}{2} A^{\frac12} u^\alpha_t + A^{\frac{1+\alpha}{2}} u^\alpha + aA^{\frac{1-\alpha}{2}} u^\alpha_t + a\cos\frac{\pi\alpha}{2} A^{\frac12} u^\alpha = \sin\frac{\pi\alpha}{2} f(u^\alpha). \tag{3.7}$$

Multiplying the above equation by u_t and integrating, we have that function $\mathcal{P}_\alpha(u^\alpha, u^\alpha_t))$, where

$$\mathcal{P}_\alpha(u^\alpha, u^\alpha_t) = \frac12 \|u^\alpha_t\|^2_{X^{\frac{1-\alpha}{4}}} + \frac12 \|u^\alpha\|^2_{X^{\frac{1+\alpha}{4}}} + \frac{a}{2}\cos\frac{\pi\alpha}{2}\|u^\alpha\|^2_{X^{\frac14}} - \sin\frac{\pi\alpha}{2}\int_\Omega\int_0^{u^\alpha} f(s)dsdx \tag{3.8}$$

satisfies the differential equation

$$\frac{d}{dt}\left(\mathcal{P}_\alpha(u^\alpha, u^\alpha_t)\right) = -2\cos\frac{\pi\alpha}{2}\|u^\alpha_t\|^2_{X^{\frac14}} - a\|u^\alpha_t\|^2_{X^{\frac{1-\alpha}{4}}}. \tag{3.9}$$

Since $u^\alpha_t = \sin\frac{\pi\alpha}{2} A^{\frac{-1+\alpha}{2}} v^\alpha - \cos\frac{\pi\alpha}{2} A^{\frac{\alpha}{2}} u^\alpha$ (see (3.5)), what was said above leads to the consideration of a functional $\mathcal{L}_\alpha$,

$$\begin{aligned} \mathcal{L}_\alpha\left(\left[\begin{smallmatrix} w\\ z\end{smallmatrix}\right]\right) &= \frac12\|w\|^2_{X^{\frac{1+\alpha}{4}}} + \frac12\Big\|\sin\frac{\pi\alpha}{2} A^{\frac{-1+\alpha}{2}} z - \cos\frac{\pi\alpha}{2} A^{\frac{\alpha}{2}} w\Big\|^2_{X^{\frac{1-\alpha}{4}}}\\ &\quad + \frac{a}{2}\cos\frac{\pi\alpha}{2}\|w\|^2_{X^{\frac14}} - \sin\frac{\pi\alpha}{2}\int_\Omega\int_0^w f(s)dsdx\\ &= \frac12\|w\|^2_{X^{\frac{1+\alpha}{4}}} + \frac12\Big\|\sin\frac{\pi\alpha}{2} A^{\frac{-1+\alpha}{4}} z - \cos\frac{\pi\alpha}{2} A^{\frac{1+\alpha}{4}} w\Big\|^2_{X}\\ &\quad + \frac{a}{2}\cos\frac{\pi\alpha}{2}\|w\|^2_{X^{\frac14}} - \sin\frac{\pi\alpha}{2}\int_\Omega\int_0^w f(s)dsdx, \end{aligned} \tag{3.10}$$

defined on the domain

$$D(\mathcal{L}_\alpha) = \Big\{\left[\begin{smallmatrix} w\\ z\end{smallmatrix}\right] \in X^{\frac{1+\alpha}{4}} \times X^{\frac{-1+\alpha}{4}} : \int_0^w f(s)ds \in L^1(\Omega)\Big\}.$$

Remark 3.3. *i) Observe that $D(\mathcal{L}_\alpha) = X^{\frac{1+\alpha}{4}} \times X^{\frac{-1+\alpha}{4}}$ provided that α is close enough to 1. Actually, there is a positive constant c_0 such that for all α close enough to 1 we have*

$$\mathcal{L}_\alpha\left(\left[\begin{smallmatrix} w\\ z\end{smallmatrix}\right]\right) \leqslant c_0\Big(1 + \Big\|\left[\begin{smallmatrix} w\\ z\end{smallmatrix}\right]\Big\|^{\rho+1}_{X^{\frac{1+\alpha}{4}}\times X^{\frac{-1+\alpha}{4}}}\Big),\ \left[\begin{smallmatrix} w\\ z\end{smallmatrix}\right] \in X^{\frac{1+\alpha}{4}} \times X^{\frac{-1+\alpha}{4}}. \tag{3.11}$$

ii) In particular, if $s \in \big(\frac{N}{2}(1-\frac{1}{\rho}) - \frac{1}{\rho}, 1\big)$ then, due to (1.10), $\mathcal{L}_\alpha\Big(\Big[\begin{smallmatrix} u^\alpha(t)\\ v^\alpha(t)\end{smallmatrix}\Big]\Big)$ is well defined for all $\alpha \in [\frac{N}{2}(\rho-1) - \rho s, 1)$ and $t \in (0, \tau_{u_{0\alpha}, v_{0\alpha}})$ along each solution $\left[\begin{smallmatrix} u^\alpha\\ v^\alpha\end{smallmatrix}\right]$ through $\left[\begin{smallmatrix} u_{0\alpha}\\ v_{0\alpha}\end{smallmatrix}\right] \in X^{\frac{s}{2}} \times X^{\frac{s-1}{2}}$ from Theorem 1.2.

iii) Actually, for positive times and as long as the solutions exist we have

$$\mathcal{P}_\alpha(u^\alpha, u_t^\alpha) = \mathcal{L}_\alpha\left(\left[\begin{smallmatrix} u^\alpha(t) \\ v^\alpha(t)\end{smallmatrix}\right]\right)$$

and hence

$$\frac{d}{dt}\left(\mathcal{P}_\alpha(u^\alpha, u_t^\alpha)\right) = \frac{d}{dt}\left(\mathcal{L}_\alpha\left(\left[\begin{smallmatrix} u^\alpha(t) \\ v^\alpha(t)\end{smallmatrix}\right]\right)\right) = -2\cos\frac{\pi\alpha}{2}\|u_t^\alpha\|^2_{X^{\frac14}} - a\|u_t^\alpha\|^2_{X^{\frac{1-\alpha}{4}}} \leqslant 0. \tag{3.12}$$

We now prove that the functional is bounded from below as stated in the following lemma.

Lemma 3.4. *Assume (1.2)-(1.3) and (1.4).*

There are positive constants c_1, c_2 such that for all $\alpha < 1$ close enough to 1 we have that $\mathcal{L}_\alpha$ in (3.10) satisfies the estimate

$$\mathcal{L}_\alpha\left(\left[\begin{smallmatrix} w \\ z\end{smallmatrix}\right]\right) \geqslant c_1 \left\|\left[\begin{smallmatrix} w \\ z\end{smallmatrix}\right]\right\|_{X^{\frac{1+\alpha}{4}}\times X^{\frac{-1+\alpha}{4}}} - c_2 \quad \textit{for any} \quad \left[\begin{smallmatrix} w \\ z\end{smallmatrix}\right] \in X^{\frac{1+\alpha}{4}} \times X^{\frac{-1+\alpha}{4}}. \tag{3.13}$$

Proof: For $\mu < \mu_1$ close enough to μ_1 we get from (1.4)

$$\mathcal{L}_\alpha\left(\left[\begin{smallmatrix} w \\ z\end{smallmatrix}\right]\right) \geqslant \frac12\|w\|^2_{X^{\frac{1+\alpha}{4}}} + \frac12\|\sin\frac{\pi\alpha}{2}A^{\frac{-1+\alpha}{4}}z - \cos\frac{\pi\alpha}{2}A^{\frac{1+\alpha}{4}}w\|^2_X - c_\mu - \frac{\mu}{2}\|w\|^2_X,$$

where due to Poincaré's inequality we have that

$$-\frac{\mu}{2}\|w\|^2_X \geqslant -\frac12\mu_1^{-\frac{1+\alpha}{2}}\mu\|w\|^2_{X^{\frac{1+\alpha}{4}}}.$$

For fixed $\epsilon \in (0,1)$ we also have

$$\begin{aligned}\|\sin\frac{\pi\alpha}{2}A^{\frac{-1+\alpha}{4}}z - \cos\frac{\pi\alpha}{2}A^{\frac{1+\alpha}{4}}w\|^2_X &\geqslant |\sin\frac{\pi\alpha}{2}\|z\|_{X^{\frac{-1+\alpha}{4}}} - \cos\frac{\pi\alpha}{2}\|w\|_{X^{\frac{1+\alpha}{4}}}|^2 \\ &\geqslant (1-\epsilon)\sin^2\frac{\pi\alpha}{2}\|z\|^2_{X^{\frac{-1+\alpha}{4}}} + (1-\frac{1}{\epsilon})\cos^2\frac{\pi\alpha}{2}\|w\|^2_{X^{\frac{1+\alpha}{4}}},\end{aligned}$$

because

$$-2\sin\frac{\pi\alpha}{2}\|z\|_{X^{\frac{-1+\alpha}{4}}}\cos\frac{\pi\alpha}{2}\|w\|_{X^{\frac{1+\alpha}{4}}} \geqslant -\epsilon\sin^2\frac{\pi\alpha}{2}\|z\|^2_{X^{\frac{-1+\alpha}{4}}} - \frac{1}{\epsilon}\cos^2\frac{\pi\alpha}{2}\|w\|^2_{X^{\frac{1+\alpha}{4}}}.$$

Consequently we get

$$\mathcal{L}_\alpha\left(\left[\begin{smallmatrix} w \\ z\end{smallmatrix}\right]\right) \geqslant \left(\frac12 - \frac12\mu_1^{-\frac{1+\alpha}{2}}\mu - \frac{1-\epsilon}{\epsilon}\cos^2\frac{\pi\alpha}{2}\right)\|w\|^2_{X^{\frac{1+\alpha}{4}}} + (1-\epsilon)\sin^2\frac{\pi\alpha}{2}\|z\|^2_{X^{\frac{-1+\alpha}{4}}} - c_\mu.$$

Since $\mu_1^{-1}\mu < 1$, given $\nu \in (\mu_1^{-1}\mu, 1)$ there exists $\alpha_1 < 1$ such that for all $\alpha \in (\alpha_1, 1)$

$$\mu_1^{-\frac{1+\alpha}{2}}\mu < \nu, \qquad \frac{1-\nu}{4} > \frac{1-\epsilon}{\epsilon}\cos^2\frac{\pi\alpha}{2} \quad \text{and} \quad \sin\frac{\pi\alpha}{2} > \frac14.$$

For ϵ, ν fixed as above and for any $\alpha \in (\alpha_1, 1)$ we thus have

$$\mathcal{L}_\alpha\left(\left[\begin{smallmatrix} w \\ z\end{smallmatrix}\right]\right) \geqslant \frac14(1-\nu)\|w\|^2_{X^{\frac{1+\alpha}{4}}} + \frac{1-\epsilon}{4}\|z\|^2_{X^{\frac{-1+\alpha}{4}}} - c_\mu,$$

which gives the result. $\square$

Remark 3.5. *Since* (3.7) *rewrites for* $\mathbb{A} = A^\alpha$ *as*

$$u^\alpha_{tt} + 2\cos\frac{\pi\alpha}{2}\mathbb{A}^{\frac12}u^\alpha_t + \mathbb{A}u^\alpha + au^\alpha_t + a\cos\frac{\pi\alpha}{2}\mathbb{A}^{\frac12}u^\alpha = \sin\frac{\pi\alpha}{2}\mathbb{A}^{-\frac{1-\alpha}{2\alpha}}f(u^\alpha),$$

this latter equation can be viewed as an approximation of (1.1) *(see* [8] *and* [5, 6] *for the extensive studies of the strongly damped wave equations).*

3.4. **Global well posedness and global attractors for (3.1): proof of Theorem 1.3.** Due to part iii) of Theorem 1.2 if B is bounded in $X^{\frac{s}{2}} \times X^{\frac{s-1}{2}}$ there is a certain time $\tau_B > 0$ such that for each $\left[\begin{smallmatrix} u_{0\alpha} \\ v_{0\alpha}\end{smallmatrix}\right] \in B$ the solution $\left[\begin{smallmatrix} u^\alpha \\ v^\alpha\end{smallmatrix}\right]$ through $\left[\begin{smallmatrix} u_{0\alpha} \\ v_{0\alpha}\end{smallmatrix}\right]$ exists until τ_B and (1.11) holds. Recalling Remark 3.3 i)-ii) and using (1.11) and (3.11) we get that for each $\alpha < 1$ close enough to 1 a constant $c_\alpha > 0$ exists such that

$$\mathcal{L}_\alpha\Big(\left[\begin{smallmatrix} u^\alpha(\tau) \\ v^\alpha(\tau)\end{smallmatrix}\right]\Big) \leqslant c_\alpha \Big\| \left[\begin{smallmatrix} u^\alpha(\tau) \\ v^\alpha(\tau)\end{smallmatrix}\right] \Big\|_{X^{\frac{1+\alpha}{2}} \times X^{\frac{\alpha}{2}}} \leqslant c_\alpha M(\tau, B).$$

On the other hand, we have from (3.12) that

$$\mathcal{L}_\alpha\Big(\left[\begin{smallmatrix} u^\alpha(t) \\ v^\alpha(t)\end{smallmatrix}\right]\Big) \leqslant \mathcal{L}_\alpha\Big(\left[\begin{smallmatrix} u^\alpha(\tau) \\ v^\alpha(\tau)\end{smallmatrix}\right]\Big),\ \tau \leqslant t < \tau_{u_{0\alpha},v_{0\alpha}},$$

whereas from (3.13) we obtain that

$$c_1 \Big\| \left[\begin{smallmatrix} u^\alpha(t) \\ v^\alpha(t)\end{smallmatrix}\right] \Big\|_{X^{\frac{1+\alpha}{4}} \times X^{\frac{-1+\alpha}{4}}} \leqslant \mathcal{L}_\alpha\Big(\left[\begin{smallmatrix} u^\alpha(\tau) \\ v^\alpha(\tau)\end{smallmatrix}\right]\Big) + c_2,\ \tau \leqslant t < \tau_{u_{0\alpha},v_{0\alpha}}. \tag{3.14}$$

Since

$$X^{\frac{1+\alpha}{4}} \times X^{\frac{-1+\alpha}{4}} \hookrightarrow X^{\frac{s}{2}} \times X^{\frac{s-1}{2}} \quad \text{for all} \quad \alpha \in (2s-1, 1), \tag{3.15}$$

we now conclude all results of part i).

Part ii) follows from part i) and from compactness of the embedding (3.15).

Concerning part iii) we first note that the set of equilibria of (3.1) is bounded. Indeed, the first coordinate of equilibrium satisfies

$$A^{\frac{1+\alpha}{2}}u + a\cos\frac{\pi\alpha}{2}A^{\frac12}u = \sin\frac{\pi\alpha}{2}f(u)$$

(see (3.7)) which, after multiplying by u and using (1.4) gives the bound of u in $X^{\frac{1+\alpha}{4}}$. Using this bound in the equation for equilibria which comes from (3.5), that is in

$$\cos\frac{\pi\alpha}{2}A^{\frac{\alpha}{2}}u - \sin\frac{\pi\alpha}{2}A^{\frac{-1+\alpha}{2}}v = 0,$$

we obtain the bound on the second coordinate of equilibrium, v, in $X^{\frac{-1+\alpha}{4}}$.

Using boundedness of equilibria, the properties of the semigroup in part ii) and using that $\mathcal{L}_\alpha$ is a Lyapunov functional we get the existence of a global attractor as stated in part iii).

Concerning part iv) note that since $\mathcal{A}_\alpha$ is invariant it is bounded in $X^{\frac{1+\alpha}{2}} \times X^{\frac{\alpha}{2}}$ because of part iii) of Theorem 1.2. Actually, due to Theorem 1.2 iii), $S_\alpha(t)$ is compact at any positive time t from $X^{\frac{s}{2}} \times X^{\frac{s-1}{2}}$ into any space in which $X^{\frac{1+\alpha}{2}} \times X^{\frac{\alpha}{2}}$ is compactly embedded. Hence, given any $\sigma \in (s-1, \alpha)$ and using [9, Corollary 4.3] we get that $\mathcal{A}_\alpha$ is actually a global $(X^{\frac{s}{2}} \times X^{\frac{s-1}{2}} - X^{\frac{1+\sigma}{2}} \times X^{\frac{\sigma}{2}})$ attractor. $\square$

4. Existence of global mild solutions to (1.6)

In this section, we will show that as $\alpha \nearrow 1$ the solutions of (3.1) suitably converge along subsequences to global mild solutions of (1.6).

4.1. **Limit spaces.** We now analyze briefly the spaces defined in (1.13)-(1.14) .

Lemma 4.1. *If*

$$X_{\lim}^{\frac{\pm 1+1}{4}} := \cap_{0\leqslant\alpha<1} X^{\frac{\pm 1+\alpha}{4}}$$

then we have the following.
i) For each $w \in X_{\lim}^{\frac{\pm 1+1}{4}}$

$$\liminf_{\alpha\nearrow 1} \|w\|_{X^{\frac{\pm 1+\alpha}{4}}} = \limsup_{\alpha\nearrow 1} \|w\|_{X^{\frac{\pm 1+\alpha}{4}}}. \tag{4.1}$$

In particular, for each $u \in X_{\lim}^{\frac{\pm 1+1}{4}}$ *there exists* $\lim_{\alpha\nearrow 1} \|w\|_{X^{\frac{\pm 1+\alpha}{4}}}$.
ii) Expressions

$$\lim_{\alpha\nearrow 1} \|w\|_{X^{\frac{\pm 1+\alpha}{4}}} \quad \textit{and} \quad \sup_{\alpha\in(0,1)} \|w\|_{X^{\frac{\pm 1+\alpha}{4}}},$$

define equivalent norms in $X_{\lim}^{\frac{\pm 1+1}{4}}$.
iii) $X_{\lim}^{\frac{\pm 1+1}{4}}$ *with the norm* $\lim_{\alpha\nearrow 1} \|w\|_{X^{\frac{\pm 1+\alpha}{4}}}$ *is a Hilbert space.*
iv) $X^{\frac{\pm 1+1}{4}}$ *is embedded in* $X_{\lim}^{\frac{\pm 1+1}{4}}$ *and*

$$\|w\|_{X^{\frac{\pm 1+1}{4}}} = \lim_{\alpha\nearrow 1} \|w\|_{X^{\frac{\pm 1+\alpha}{4}}} \quad \textit{for each} \quad u \in \|w\|_{X^{\frac{\pm 1+1}{4}}}.$$

v) $X_{\lim}^{\frac{\pm 1+1}{4}}$ *is compactly embedded into* $X^{\frac{\pm 1+\alpha}{4}}$ *for every* $\alpha < 1$.

Proof: We prove the result for *plus* sign. For the *minus* sign the proof is similar.

We start from a straightforward inequality

$$\liminf_{\alpha\nearrow 1} \|w\|_{X^{\frac{1+\alpha}{4}}} \leqslant \limsup_{\alpha\nearrow 1} \|w\|_{X^{\frac{1+\alpha}{4}}} \leqslant \sup_{\alpha\in(0,1)} \|w\|_{X^{\frac{1+\alpha}{4}}}, \ u \in X_{\lim}^{\frac{1}{2}}.$$

Now, if $0 < \beta < \alpha < 1$ and $w \in X_{\lim}^{\frac{1}{2}}$ is such that $\|w\|_X = 1$ then using Fourier series (see [18, Lemma 3.27]) we have

$$\|w\|_{X^{\frac{1+\beta}{4}}} \leqslant \|w\|_{X^{\frac{1+\alpha}{4}}}^{\frac{1+\beta}{1+\alpha}} \|w\|_X^{\frac{\alpha-\beta}{1+\alpha}} = \|w\|_{X^{\frac{1+\alpha}{4}}}^{\frac{1+\beta}{1+\alpha}}. \tag{4.2}$$

From this we get

$$\|w\|_{X^{\frac{1+\beta}{4}}} \leqslant (\liminf_{\alpha\nearrow 1} \|w\|_{X^{\frac{1+\alpha}{4}}})^{\frac{1+\beta}{2}}$$

and, consequently,

$$\limsup_{\beta\nearrow 1} \|w\|_{X^{\frac{1+\beta}{4}}} \leqslant \liminf_{\alpha\nearrow 1} \|w\|_{X^{\frac{1+\alpha}{4}}}.$$

This proves that (4.1) holds for all $w \in X_{\lim}^{\frac{1}{2}}$ with $\|w\|_X = 1$, which in turn implies that (4.1) holds for all $w \in X_{\lim}^{\frac{1}{2}}$. Hence we get part i).

Using next first inequality in (4.2) and Poincaré's inequality for any $w \in X^{\frac{1}{2}}_{\lim}$ we have

$$\|w\|_{X^{\frac{1+\beta}{4}}} \leqslant \mu_1^{-\frac{\alpha-\beta}{4}} \|w\|_{X^{\frac{1+\alpha}{4}}}$$

and thus

$$\|w\|_{X^{\frac{1+\beta}{4}}} \leqslant \mu_1^{-\frac{1-\beta}{4}} \lim_{\alpha\nearrow 1} \|w\|_{X^{\frac{1+\alpha}{4}}}.$$

Consequently, we obtain that $\sup_{\beta\in(0,1)} \|w\|_{X^{\frac{1+\beta}{4}}}$ is bounded by a multiple of $\lim_{\alpha\nearrow 1} \|w\|_{X^{\frac{1+\alpha}{4}}}$, which completes the proof of part ii).

Concerning part iii) we first observe that if $w, z \in X^{\frac{1}{2}}_{\lim}$ then, given any $\alpha < 1$ and $w, z \in X^{\frac{1+\alpha}{4}}$, since $X^{\frac{1+\alpha}{4}}$ is a Hilbert space, we have

$$\|w+z\|^2_{X^{\frac{1+\alpha}{4}}} + \|w-z\|^2_{X^{\frac{1+\alpha}{4}}} = 2\|w\|^2_{X^{\frac{1+\alpha}{4}}} + 2\|z\|^2_{X^{\frac{1+\alpha}{4}}}. \tag{4.3}$$

Passing to the limit as $\alpha \nearrow 1$ we get from (4.3) that

$$\|w+z\|^2_{X^{\frac{1}{2}}_{\lim}} + \|w-z\|^2_{X^{\frac{1}{2}}_{\lim}} = 2\|w\|^2_{X^{\frac{1}{2}}_{\lim}} + 2\|z\|^2_{X^{\frac{1}{2}}_{\lim}}.$$

Therefore, $X^{\frac{1}{2}}_{\lim}$ with the norm $\lim_{\alpha\nearrow 1} \|w\|_{X^{\frac{\pm 1+\alpha}{4}}}$ is a pre-Hilbert space (see [20]). Using the equivalent norm $\sup_{\alpha\in(0,1)} \|w\|_{X^{\frac{1+\alpha}{4}}}$ we observe that the space is complete, which proves part iii).

Note that, due to [1, Theorem III.4.6.2], for any $w \in X^{\frac{1}{2}}$ we get

$$\lim_{\alpha\nearrow 1} \|A^{\frac{1+\alpha}{4}} w\|_X = \lim_{\alpha\nearrow 1} \|A^{\frac{\alpha-1}{4}} A^{\frac{1}{2}} w\|_X = \|A^{\frac{1}{2}} w\|_X$$

which gives the result of part iv).

Finally, part v) follows using supremum norm in $X^{\frac{1}{2}}_{\lim}$ and compactness of the scale. □

Remark 4.2. *Observe from* [1, (2.11.4), p. 36] *that if* $0 < \alpha < 1$ *then*

$$(X, X^1)_{\alpha,1} \subset X^\alpha \subset (X, X^1)_{\alpha,\infty}$$

whereas, due to [19, Theorem 1.3.3],

$$X^{\frac{1}{2}} \subset (X, X^1)_{\frac{1}{2},\infty} = (X^1, X)_{\frac{1}{2},\infty} \subset (X^1, X)_{\beta,1} = (X, X^1)_{1-\beta,1} \subset X^{1-\beta} \quad \textit{for all } \beta \in (\frac{1}{2}, 1)\ .$$

This leads to the inclusions

$$X^{\frac{1}{2}} \subset (X, X^1)_{\frac{1}{2},\infty} \subset X^{\frac{1}{2}}_{\lim}$$

which in turn indicate that $X^{\frac{1}{2}} \subsetneq X^{\frac{1}{2}}_{\lim}$.

4.2. **Estimate for solutions of (3.1) as $\alpha \nearrow 1$.** We now obtain estimate for the solutions of (3.1) uniformly for initial data in bounded subsets of $X^{\frac{1}{2}}_{\mathrm{lim}} \times X_{\mathrm{lim}}$.

Lemma 4.3. *Assume (1.2)-(1.3) and (1.4).*

There then exists $\alpha^* < 1$ and given B bounded in $X^{\frac{1}{2}}_{\mathrm{lim}} \times X_{\mathrm{lim}}$ there also exists $M = M(B) > 0$ such that for any $\alpha \in (\alpha^*, 1)$ and any $\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right] \in B$ there is a global solution $\left[\begin{smallmatrix} u^\alpha \\ v^\alpha \end{smallmatrix}\right]$ of (3.1) through $\left[\begin{smallmatrix} u_{0\alpha} \\ v_{0\alpha} \end{smallmatrix}\right] = \left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right]$ as in Theorem 1.3 and

$$\sup_{t \geqslant 0} \left\| \left[\begin{smallmatrix} u^\alpha(t) \\ v^\alpha(t) \end{smallmatrix}\right] \right\|_{X^{\frac{1+\alpha}{4}} \times X^{\frac{-1+\alpha}{4}}} \leqslant M.$$

Proof: Observe that $B \subset D(\mathcal{L}_\alpha)$. Hence, for any $\alpha < 1$ close enough to 1, Theorem 1.3 applies with $s = \frac{(1+\alpha)}{2}$, so in addition the solution $\left[\begin{smallmatrix} u^\alpha \\ v^\alpha \end{smallmatrix}\right]$ of (3.1) through $\left[\begin{smallmatrix} u_{0\alpha} \\ v_{0\alpha} \end{smallmatrix}\right] = \left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right]$ satisfies (3.14) with $\tau = 0$. Hence we get

$$\left\| \left[\begin{smallmatrix} u^\alpha \\ v^\alpha \end{smallmatrix}\right] \right\|_{X^{\frac{1+\alpha}{4}} \times X^{\frac{-1+\alpha}{4}}} \leqslant c\left(1 + \mathcal{L}_\alpha(\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right])\right) \tag{4.4}$$

for some positive constant c which does not depend on α, t and $\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right] \in B$. Since B is bounded in $X^{\frac{1}{2}}_{\mathrm{lim}} \times X_{\mathrm{lim}}$, we have that there is a positive constant c_B independent of $\alpha \in [-1, 1)$ and such that

$$\left\| \left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right] \right\|_{X^{\frac{1+\alpha}{4}} \times X^{\frac{-1+\alpha}{4}}} \leqslant c_B \quad \text{for all } \left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right] \in B.$$

Hence $\mathcal{L}_\alpha(\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right])$ is bounded from above by a constant independent of $\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right] \in B$ and of α close enough to 1 (see (3.11) in Remark 3.3)), which together with (4.4) gives the result. $\square$

Corollary 4.4. *Assume (1.2)-(1.3) and (1.4).*

There is a certain $\alpha^* \in (0, 1)$ such that if $\alpha_n \nearrow 1$ and $\{\left[\begin{smallmatrix} w_n \\ z_n \end{smallmatrix}\right]\}$ is a sequence of elements $\left[\begin{smallmatrix} w_n \\ z_n \end{smallmatrix}\right] \in X^{\frac{1+\alpha_n}{4}} \times X^{\frac{-1+\alpha_n}{4}}$ satisfying

$$\sup_{n \in \mathbb{N}} \left\| \left[\begin{smallmatrix} w_n \\ z_n \end{smallmatrix}\right] \right\|_{X^{\frac{1+\alpha_n}{4}} \times X^{\frac{-1+\alpha_n}{4}}} \leqslant r$$

then a global solution $\left[\begin{smallmatrix} u^{\alpha_n} \\ v^{\alpha_n} \end{smallmatrix}\right]$ of (3.1) through $\left[\begin{smallmatrix} u_{0\alpha_n} \\ v_{0\alpha_n} \end{smallmatrix}\right] = \left[\begin{smallmatrix} w_n \\ z_n \end{smallmatrix}\right]$ as in Theorem 1.3 exists for all n large enough and

$$\sup_{\{n \in \mathbb{N}:\ \alpha_n \geqslant \alpha^*\}} \sup_{t \geqslant 0} \left\| \left[\begin{smallmatrix} u^{\alpha_n}(t) \\ v^{\alpha_n}(t) \end{smallmatrix}\right] \right\|_{X^{\frac{1+\alpha_n}{4}} \times X^{\frac{-1+\alpha_n}{4}}} \leqslant M(r).$$

Proof: Following the proof of Lemma 4.3 we get (4.4) with $\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right]$ replaced now by $\left[\begin{smallmatrix} w_n \\ z_n \end{smallmatrix}\right]$ where, by assumption and (3.11), $\mathcal{L}_\alpha(\left[\begin{smallmatrix} w_n \\ z_n \end{smallmatrix}\right])$ is bounded from above uniformly for n. $\square$

4.3. **Limiting procedure: proof of Theorem 1.5.** Theorem 1.5 is a consequence of the following result.

Theorem 4.5. *Assume (1.2)-(1.3) and (1.4).*

If $r > 0$, $\alpha_n \nearrow 1$, a sequence $\{\left[\begin{smallmatrix} w_n \\ z_n \end{smallmatrix}\right]\}$ of elements satisfying $\left\| \left[\begin{smallmatrix} w_n \\ z_n \end{smallmatrix}\right] \right\|_{X^{\frac{1+\alpha_n}{4}} \times X^{\frac{-1+\alpha_n}{4}}} < r$ is convergent for some $\zeta \in (-1, 1)$ in $X^{\frac{1+\zeta}{4}} \times X^{\frac{-1+\zeta}{4}}$ to $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in X^{\frac{1}{2}}_{\mathrm{lim}} \times X_{\mathrm{lim}}$ and if $\left[\begin{smallmatrix} u^{\alpha_n} \\ v^{\alpha_n} \end{smallmatrix}\right]$ is a solution of (3.1) through $\left[\begin{smallmatrix} u_{0\alpha_n} \\ v_{0\alpha_n} \end{smallmatrix}\right] = \left[\begin{smallmatrix} w_n \\ z_n \end{smallmatrix}\right]$ as in Theorem 1.3, then there is a subsequence

$\left\{\left[\begin{smallmatrix} u^{\alpha_{n_k}} \\ v^{\alpha_{n_k}} \end{smallmatrix}\right]\right\}$ *of* $\{\left[\begin{smallmatrix} u^{\alpha_n} \\ v^{\alpha_n} \end{smallmatrix}\right]\}$ *and there exists a bounded function* $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] : [0,\infty) \to X^{\frac{1}{2}}_{\rm lim} \times X_{\rm lim}$ *such that, given any* $\zeta \in [-1,1)$,

$$\sup_{t\in[0,T]} \left\| \begin{bmatrix} u^{\alpha_{n_k}}(t) \\ v^{\alpha_{n_k}}(t) \end{bmatrix} - \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} \right\|_{X^{\frac{1+\zeta}{4}} \times X^{\frac{-1+\zeta}{4}}} \to 0 \ \ \textit{for every } T > 0 \tag{4.5}$$

and $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$ *is a global weak solution of* (1.6) *as in Definition 1.4*

Proof: By assumption on $\{\left[\begin{smallmatrix} w_n \\ z_n \end{smallmatrix}\right]\}$ we infer from Corollary 4.4 that

$$\sup_{t\geqslant 0}\sup_{n\geqslant n_0} \left\| \begin{bmatrix} u^{\alpha_n}(t) \\ v^{\alpha_n}(t) \end{bmatrix} \right\|_{X^{\frac{1+\alpha_n}{4}} \times X^{\frac{-1+\alpha_n}{4}}} \leqslant M \tag{4.6}$$

for some positive constant M. Using (4.6) and (1.2)-(1.3) we then infer from (3.5) that

$$\sup_{t\geqslant 0}\sup_{n\geqslant n_0} \left\| \begin{bmatrix} u_t^{\alpha_n}(t) \\ v_t^{\alpha_n}(t) \end{bmatrix} \right\|_{X \times X^{-\frac{1}{2}}} < \tilde{M},$$

where $\tilde{M}$ is a multiple of $1 + M^{\rho}$. Hence Arzelá-Ascoli Theorem applies (see [16, Section 7.10]) and there is a subsequence $\{\alpha_{n_k}\}$ and a function $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] \in C([0,\infty), X \times X^{-\frac{1}{2}})$ such that

$$\sup_{t\in[0,T]} \left\| \begin{bmatrix} u^{\alpha_{n_k}}(t) \\ v^{\alpha_{n_k}}(t) \end{bmatrix} - \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} \right\|_{X \times X^{-\frac{1}{2}}} \to 0 \ \text{ for every } T > 0. \tag{4.7}$$

As a consequence of (4.6) and embedding properties of the scale a certain $\hat{M} > 0$ exists such that for any $\zeta \in [-1,1)$ there is a number $n_\zeta \in \mathbb{N}$ such that

$$\sup_{t\geqslant 0}\sup_{n_k\geqslant n_\zeta} \left\| \begin{bmatrix} u^{\alpha_{n_k}}(t) \\ v^{\alpha_{n_k}}(t) \end{bmatrix} \right\|_{X^{\frac{1+\zeta}{4}} \times X^{\frac{-1+\zeta}{4}}} \leqslant \hat{M}. \tag{4.8}$$

Combining (4.7)-(4.8) and using interpolation inequality we infer that, given any $\zeta \in [-1,1)$,

$$\sup_{t\in[0,T]} \left\| \begin{bmatrix} u^{\alpha_{n_k}}(t) \\ v^{\alpha_{n_k}}(t) \end{bmatrix} - \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} \right\|_{X^{\frac{1+\zeta}{4}} \times X^{\frac{-1+\zeta}{4}}} \to 0 \ \text{ for every } T > 0. \tag{4.9}$$

Using now (4.7) Lemma 3.1 with $s = \frac{1+\zeta}{2} \in (\frac{N}{2}(1-\frac{1}{\rho}) - \frac{1}{\rho}, \frac{N}{2}(1-\frac{1}{\rho}))$ we have for $\sigma = \frac{N}{2}(\rho-1) - \rho s \in [0,1)$ that

$$\sup_{t\in[0,T]} \left\| F\left(\begin{bmatrix} u^{\alpha_{n_k}}(t) \\ v^{\alpha_{n_k}}(t) \end{bmatrix} \right) - F\left(\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} \right) \right\|_{X^{\frac{1-\sigma}{2}} \times X^{-\frac{\sigma}{2}}} \to 0 \text{ whenever } T > 0.$$

Observe that due to (4.8)-(4.9) we get

$$\left\| \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} \right\|_{X^{\frac{1+\zeta}{4}} \times X^{\frac{-1+\zeta}{4}}} \leqslant \hat{M} \ \text{ for every } t \geqslant 0 \ \text{ and } \ \zeta \in [-1,1)$$

with constant $\hat{M}$ independent on t and ζ. Hence, on the one hand, we have

$$\left\| \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} \right\|_{X^{\frac{1}{2}}_{\rm lim} \times X_{\rm lim}} \leqslant \hat{M} \ \text{ for every } t \geqslant 0,$$

and, on the other, since $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] \in C([0,\infty), X \times X^{-\frac{1}{2}})$ and the scale is compactly embedded,

$$\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] \in C([0,\infty), X^{\frac{1+\zeta}{4}} \times X^{\frac{-1+\zeta}{4}}) \text{ for every } \ \zeta \in [-1,1).$$

Consequently, using once more (4.7) Lemma 3.1 with $s = \frac{1+\zeta}{2} \in (\frac{N}{2}(1-\frac{1}{\rho}) - \frac{1}{\rho}, \frac{N}{2}(1-\frac{1}{\rho}))$ we have for $\sigma = \frac{N}{2}(\rho - 1) - \rho s \in [0,1)$ that

$$F\left(\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]\right) \in C([0,\infty), X^{\frac{1-\sigma}{2}} \times X^{-\frac{\sigma}{2}}). \tag{4.10}$$

It remains to prove that $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$ satisfies

$$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = e^{-\Lambda t} \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] + \int_0^t e^{-\Lambda(t-s)} F\left(\begin{bmatrix} u(s) \\ v(s) \end{bmatrix}\right) ds,\ t \geqslant 0.$$

This follows using that functions $\left[\begin{smallmatrix} u^{\alpha_{n_k}} \\ v^{\alpha_{n_k}} \end{smallmatrix}\right]$ satisfy

$$\begin{bmatrix} u^{\alpha_{n_k}}(t) \\ v^{\alpha_{n_k}}(t) \end{bmatrix} = e^{-\Lambda^{\alpha_{n_k}} t} \left[\begin{smallmatrix} w_{n_k} \\ z_{n_k} \end{smallmatrix}\right] + \int_0^t e^{-\Lambda^{\alpha_{n_k}}(t-s)} F\left(\begin{bmatrix} u^{\alpha_{n_k}}(s) \\ v^{\alpha_{n_k}}(s) \end{bmatrix}\right) ds,\ t \geqslant 0$$

and using (4.9)-(4.10) together with convergence of the linear semigroups in Proposition 1.1 iii) (see [7], where a similar argument was used). $\square$

5. Long time behavior of global mild solutions of (1.6)

This section is devoted to asymptotic behavior of global mild solutions of (1.6) obtained via limiting procedure exhibited in the proof of Theorem 4.5.

5.1. **Additional estimates for the solutions of (3.1).** Using $\mathcal{L}_\alpha$ as in (3.10) we will consider here the functional

$$\mathcal{L}_{\delta,\alpha}\left(\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right]\right) = \mathcal{L}_\alpha\left(\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right]\right) + \mathcal{V}_{\delta,\alpha}\left(\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right]\right), \tag{5.1}$$

where

$$\mathcal{V}_{\delta,\alpha}\left(\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right]\right) = \delta \int_\Omega A^{\frac{1-\alpha}{2}} w (\sin\frac{\pi\alpha}{2} A^{\frac{-1+\alpha}{2}} z - \cos\frac{\pi\alpha}{2} A^{\frac{\alpha}{2}} w) dx. \tag{5.2}$$

Remark 5.1. *i) Note that for all $\alpha < 1$ close enough to 1 the real map $\mathcal{L}_{\delta,\alpha}$ is well defined in $X^{\frac{1+\alpha}{4}} \times X^{\frac{-1+\alpha}{4}}$ (see Remark 3.3) and*

$$\mathcal{V}_{\delta,\alpha}\left(\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right]\right) = \delta \sin\frac{\pi\alpha}{2} \int_\Omega A^{\frac{1-\alpha}{4}} w A^{\frac{-1+\alpha}{4}} z dx - \delta\cos\frac{\pi\alpha}{2} \|A^{\frac14} w\|_X^2.$$

ii) Thus note that due to (3.11) and Lemma 3.4 there are constants $c_0, c_1, c_2 > 0$ and $\delta_0 < 1$ such that for all $\delta \in (0,\delta_0)$ and any $\alpha < 1$ close enough to 1 we have

$$\mathcal{L}_{\delta,\alpha}\left(\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right]\right) \leqslant c_0(1 + \|\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right]\|^{\rho+1}_{X^{\frac{1+\alpha}{4}} \times X^{\frac{-1+\alpha}{4}}}),\ \left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right] \in X^{\frac{1+\alpha}{4}} \times X^{\frac{-1+\alpha}{4}}, \tag{5.3}$$

and

$$\mathcal{L}_{\delta,\alpha}\left(\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right]\right) \geqslant \frac12 c_1 \|\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right]\|_{X^{\frac{1+\alpha}{4}} \times X^{\frac{-1+\alpha}{4}}} - c_2,\ \left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right] \in X^{\frac{1+\alpha}{4}} \times X^{\frac{-1+\alpha}{4}}. \tag{5.4}$$

Lemma 5.2. *Assume (1.2)-(1.3) and (1.4).*

There then exists $R_0 > 0$ such that for each $\alpha < 1$ close enough to 1, and given any $r > 0$ and any $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]$ in a ball $B_{\lim}(r)$ of radius r around zero in $X^{\frac12}_{\lim} \times X_{\lim}$, we have that the solution $\left[\begin{smallmatrix} u^\alpha \\ v^\alpha \end{smallmatrix}\right]$ of (3.1) through $\left[\begin{smallmatrix} u_{0\alpha} \\ v_{0\alpha} \end{smallmatrix}\right] = \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]$ as in Theorem 1.3 enters the ball $B_\alpha(R)$ of radius

R_0 around zero in $X^{\frac{1+\alpha}{4}}\times X^{\frac{-1+\alpha}{4}}$ at a certain positive time t_r (independent of $\alpha<1$ close enough to 1 and $\left[\begin{smallmatrix}u_0\\ v_0\end{smallmatrix}\right]\in B_{\lim}(r)$) and remains in $B_\alpha(R_0)$ for all $t\geqslant t_r$.

Proof: Going to (3.1) and using (3.7) we get

$$\begin{aligned}\frac{d}{dt}\int_\Omega A^{\frac{1-\alpha}{2}}u^\alpha u_t^\alpha dx &= \langle A^{\frac{1-\alpha}{2}}u_t^\alpha, u_t^\alpha\rangle_{L^2(\Omega)}+\langle u^\alpha, A^{\frac{1-\alpha}{2}}u_{tt}^\alpha\rangle_{L^2(\Omega)}\\
&=\|u_t^\alpha\|^2_{X^{\frac{1-\alpha}{4}}}-\langle u^\alpha, 2\cos\frac{\pi\alpha}{2}A^{\frac12}u_t^\alpha\rangle_{L^2(\Omega)}-\langle u^\alpha, A^{\frac{1+\alpha}{2}}u^\alpha\rangle_{L^2(\Omega)}\\
&-\langle u^\alpha, aA^{\frac{1-\alpha}{2}}u_t^\alpha\rangle_{L^2(\Omega)}-\langle u^\alpha, a\cos\frac{\pi\alpha}{2}A^{\frac12}u^\alpha\rangle_{L^2(\Omega)}+\langle u^\alpha,\sin\frac{\pi\alpha}{2}f(u^\alpha)\rangle_{L^2(\Omega)}\\
&=\|u_t^\alpha\|^2_{X^{\frac{1-\alpha}{4}}}-2\cos\frac{\pi\alpha}{2}\int_\Omega u^\alpha A^{\frac12}u_t^\alpha dx-\|u^\alpha\|^2_{X^{\frac{1+\alpha}{4}}}\\
&-a\int_\Omega u^\alpha A^{\frac{1-\alpha}{2}}u_t^\alpha dx-a\cos\frac{\pi\alpha}{2}\|u^\alpha\|^2_{X^{\frac14}}+\sin\frac{\pi\alpha}{2}\int_\Omega f(u^\alpha)u^\alpha dx.\end{aligned}$$

From (1.4) we infer that for each $\mu<\mu_1$ close enough to μ_1

$$\sin\frac{\pi\alpha}{2}\int_\Omega u^\alpha f(u^\alpha)dx\leqslant\mu\|u^\alpha\|_X^2+c_\mu\leqslant\mu\mu_1^{-\frac{1+\alpha}{2}}\|u^\alpha\|^2_{X^{\frac{1+\alpha}{4}}}+C_\mu,$$

whereas for any $\varepsilon>0$ we also have

$$2\left|\cos\frac{\pi\alpha}{2}\int_\Omega u^\alpha A^{\frac12}u_t^\alpha dx\right|=2\left|\cos\frac{\pi\alpha}{2}\int_\Omega A^{\frac{1+\alpha}{4}}u^\alpha A^{\frac{1-\alpha}{4}}u_t^\alpha dx\right|\leqslant\varepsilon\|u^\alpha\|^2_{X^{\frac{1+\alpha}{4}}}+\frac{1}{\varepsilon}\|u_t^\alpha\|^2_{X^{\frac{1-\alpha}{4}}},$$

and

$$a\left|\int_\Omega u^\alpha A^{\frac{1-\alpha}{2}}u_t^\alpha dx\right|=a\left|\int_\Omega A^{\frac{1+\alpha}{4}}u^\alpha A^{\frac{1-3\alpha}{4}}u_t^\alpha dx\right|\leqslant\varepsilon\|u^\alpha\|^2_{X^{\frac{1+\alpha}{4}}}+C_\varepsilon\|u_t^\alpha\|^2_{X^{\frac{1-\alpha}{4}}}.$$

Consequently, we obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega A^{\frac{1-\alpha}{2}}u^\alpha u_t^\alpha dx&\leqslant-\Big(1-2\varepsilon-\mu\mu_1^{-\frac{1+\alpha}{2}}\Big)\|u^\alpha\|^2_{X^{\frac{1+\alpha}{4}}}-a\cos\frac{\pi\alpha}{2}\|u^\alpha\|^2_{X^{\frac14}}\\
&+\left(1+\frac{1}{\varepsilon}+C_\varepsilon\right)\|u_t^\alpha\|^2_{X^{\frac{1-\alpha}{4}}}+C_\mu.\end{aligned}\tag{5.5}$$

Now, recalling the expression for $\mathcal{P}_\alpha(u^\alpha,u_t^\alpha)$ in (3.8) and letting

$$\mathcal{P}_{\delta,\alpha}(u^\alpha,u_t^\alpha)=\mathcal{P}_\alpha(u^\alpha,u_t^\alpha)+\delta\int_\Omega A^{\frac{1-\alpha}{2}}u^\alpha u_t^\alpha dx,\tag{5.6}$$

we observe that (3.9) and (5.5)-(5.6) give

$$\begin{aligned}\frac{d}{dt}\mathcal{P}_{\delta,\alpha}(u^\alpha,u_t^\alpha)&\leqslant-\delta\Big(1-2\varepsilon-\mu\mu_1^{-\frac{1+\alpha}{2}}\Big)\|u^\alpha\|^2_{X^{\frac{1+\alpha}{4}}}-\delta a\cos\frac{\pi\alpha}{2}\|u^\alpha\|^2_{X^{\frac14}}\\
&-\left(a-\delta-\frac{\delta}{\varepsilon}-\delta C_\epsilon\right)\|u_t^\alpha\|^2_{X^{\frac{1-\alpha}{4}}}+\delta C_\mu.\end{aligned}$$

Since $\mu_1^{-1}\mu<1$, given $\nu\in(\mu_1^{-1}\mu,1)$ there exists $\alpha_1<1$ such that for all $\alpha\in(\alpha_1,1)$

$$\mu_1^{-\frac{1+\alpha}{2}}\mu<\nu.$$

For suitably small $\varepsilon > 0$ we then have

$$1 - 2\varepsilon - \mu\mu_1^{-\frac{1+\alpha}{2}} > \frac{1-\nu}{2}.$$

Having fixed ν, ε as above for any $\delta > 0$ small enough we have $a - \delta - \frac{\delta}{\varepsilon} - \delta C_\epsilon > \frac{a}{2} + \frac{\delta}{2}$, $C_\mu > \delta C_\mu$ and hence

$$\frac{d}{dt}\mathcal{P}_{\delta,\alpha}(u^\alpha, u_t^\alpha) \leqslant -\delta\frac{1-\nu}{2}\|u^\alpha\|^2_{X^{\frac{1+\alpha}{4}}} - \delta a \cos\frac{\pi\alpha}{2}\|u^\alpha\|^2_{X^{\frac{1}{4}}} - \frac{\delta}{2}\|u_t\|^2_{X^{\frac{1-\alpha}{4}}} - \frac{a}{2}\|u_t\|^2_{X^{\frac{1-\alpha}{4}}} + C_\mu.$$

For $\eta = \min\{\delta(1-\nu), 2\delta, 1\}$ this in turn implies that

$$\begin{aligned}\frac{d}{dt}\mathcal{P}_{\delta,\alpha}(u^\alpha, u_t^\alpha) \leqslant -\eta(&\frac{1}{2}\|u^\alpha\|^2_{X^{\frac{1+\alpha}{4}}} + \frac{a}{2}\cos\frac{\pi\alpha}{2}\|u^\alpha\|^2_{X^{\frac{1}{4}}} \\ &+ \frac{1}{2}\|u_t\|^2_{X^{\frac{1-\alpha}{4}}}) - \frac{a}{2}\|u_t\|^2_{X^{\frac{1-\alpha}{4}}} + C_\mu.\end{aligned} \tag{5.7}$$

On the other hand, due to Lemma 4.3, there exists $M = M(r) > 0$ such that the solutions to (3.1) satisfy

$$\sup_{t\geqslant 0}\left\| \begin{bmatrix} u^\alpha(t) \\ v^\alpha(t)\end{bmatrix} \right\|_{X^{\frac{1+\alpha}{4}}\times X^{\frac{-1+\alpha}{4}}} \leqslant M(r) \quad \text{for all} \quad \alpha\in(\alpha^*,1),\ \begin{bmatrix} u_{0\alpha} \\ v_{0\alpha}\end{bmatrix} = \begin{bmatrix} u_0 \\ v_0\end{bmatrix} \in B_{X^{\frac{1}{2}}_{\lim}\times X_{\lim}}(r). \tag{5.8}$$

Letting $f_0 = f - f(0)$ and using (1.2) we can find a constant $\bar{c} > 1$ such that

$$-\int_\Omega\int_0^{u^\alpha} f_0(s)dsdx \leqslant \bar{c}\|u^\alpha\|^2_{X^{\frac{1+\alpha}{4}}}(1 + \|u^\alpha\|^{\rho-1}_{X^{\frac{1+\alpha}{4}}}). \tag{5.9}$$

Hence we obtain

$$-\bar{d}\int_\Omega\int_0^{u^\alpha} f_0(s)dsdx \leqslant \|u^\alpha\|^2_{X^{\frac{1+\alpha}{4}}} \quad \text{for all} \quad \alpha\in(\alpha^*,1),\ \begin{bmatrix} u_{0\alpha} \\ v_{0\alpha}\end{bmatrix} = \begin{bmatrix} u_0 \\ v_0\end{bmatrix} \in B_{X^{\frac{1}{2}}_{\lim}\times X_{\lim}}(r) \tag{5.10}$$

with a constant

$$\bar{d} = \frac{1}{\bar{c}(1 + M(r)^{\rho-1})} \leqslant 1 \tag{5.11}$$

and consequently

$$\begin{aligned}-\frac{\eta}{4}\|u^\alpha\|^2_{X^{\frac{1+\alpha}{4}}} &\leqslant \frac{\eta\bar{d}}{4}\int_\Omega\int_0^{u^\alpha} f_0(s)dsdx = \frac{\eta\bar{d}}{4}\int_\Omega\int_0^{u^\alpha}(f(s)-f(0))dsdx \\ &\leqslant \frac{\eta\bar{d}}{4}\int_\Omega\int_0^{u^\alpha} f(s)dsdx + \frac{\eta\bar{d}}{4}|f(0)|(1 + \|u^\alpha\|^2_{L^2(\Omega)}) \\ &\leqslant \frac{\eta\bar{d}}{4}\int_\Omega\int_0^{u^\alpha} f(s)dsdx + D,\end{aligned} \tag{5.12}$$

for $D = \frac{\eta\bar{d}}{4}|f(0)|(1 + \sup_{\alpha\in(0,1)}\mu_1^{-\frac{1+\alpha}{2}} M^2(r))$.

Since $\sin\frac{\pi\alpha}{2} < 1$ we actually have from (5.12)

$$-\frac{\eta}{4}\|u^\alpha\|^2_{X^{\frac{1+\alpha}{4}}} \leqslant -\frac{\eta}{4}\sin\frac{\pi\alpha}{4}\|u^\alpha\|^2_{X^{\frac{1+\alpha}{4}}} \leqslant \frac{\eta\bar{d}}{2}\sin\frac{\pi\alpha}{4}\int_\Omega\int_0^{u^\alpha} f(s)dsdx + D. \tag{5.13}$$

Connecting (5.7) and (5.13) we get

$$\frac{d}{dt}\mathcal{P}_{\delta,\alpha}(u^\alpha,u_t^\alpha)\leqslant-\frac{\eta}{2}\Big(\frac{1}{2}\|u^\alpha\|^2_{X^{\frac{1+\alpha}{4}}}+\frac{1}{2}\|u_t^\alpha\|^2_{X^{\frac{1-\alpha}{4}}}+\frac{a}{2}\cos\frac{\pi\alpha}{2}\|u\|^2_{X^{\frac{1}{4}}}\Big)-\frac{a}{2}\|u_t\|^2_{X^{\frac{1-\alpha}{4}}}+C_\mu$$
$$+\frac{\eta\bar{d}}{4}\sin\frac{\pi\alpha}{2}\int_\Omega\int_0^{u^\alpha}f(s)dsdx+D$$

and, since $\eta\bar{d}\leqslant\eta$ (see (5.11)),

$$\frac{d}{dt}\mathcal{P}_{\delta,\alpha}(u^\alpha,u_t^\alpha)\leqslant-\frac{\eta\bar{d}}{2}\Big(\frac{1}{2}\|u^\alpha\|^2_{X^{\frac{1+\alpha}{4}}}+\frac{1}{2}\|u_t^\alpha\|^2_{X^{\frac{1-\alpha}{4}}}+\frac{a}{2}\cos\frac{\pi\alpha}{2}\|u\|^2_{X^{\frac{1}{4}}}\Big)-\frac{a}{2}\|u_t\|^2_{X^{\frac{1-\alpha}{4}}}+C_\mu$$
$$+\frac{\eta\bar{d}}{4}\sin\frac{\pi\alpha}{2}\int_\Omega\int_0^{u^\alpha}f(s)dsdx+D$$

We also have

$$\frac{\eta\delta\bar{d}}{4}\int_\Omega A^{\frac{1-\alpha}{2}}u^\alpha u_t^\alpha dx\leqslant\frac{\eta\delta\bar{d}}{4}\left|\int_\Omega A^{\frac{1+\alpha}{4}}u^\alpha A^{\frac{1-3\alpha}{4}}u_t^\alpha dx\right|\leqslant\frac{\eta\delta\bar{d}}{4}\|u^\alpha\|^2_{X^{\frac{1+\alpha}{4}}}+\frac{\eta\delta\bar{d}}{4}b\|u_t^\alpha\|^2_{X^{\frac{1-\alpha}{4}}},$$

where $b=\sup_{\alpha\in(0,1)}\mu_1^{-\alpha}$. Hence, taking into account that $\delta>0$ can be small enough, we get

$$\frac{\eta\delta\bar{d}}{4}\int_\Omega A^{\frac{1-\alpha}{2}}u^\alpha u_t^\alpha dx\leqslant\frac{\eta\delta\bar{d}}{4}\|u^\alpha\|^2_{X^{\frac{1+\alpha}{4}}}+\frac{a}{2}\|u_t^\alpha\|^2_{X^{\frac{1-\alpha}{4}}},$$

so that

$$\frac{d}{dt}\mathcal{P}_{\delta,\alpha}(u^\alpha,u_t^\alpha)\leqslant-\frac{\eta\bar{d}}{4}\Big(\frac{1}{2}\|u^\alpha\|^2_{X^{\frac{1+\alpha}{4}}}+\frac{1}{2}\|u_t^\alpha\|^2_{X^{\frac{1-\alpha}{4}}}+\frac{a}{2}\cos\frac{\pi\alpha}{2}\|u\|^2_{X^{\frac{1}{4}}}\Big)$$
$$-\frac{\eta\delta\bar{d}}{4}\int_\Omega A^{\frac{1-\alpha}{2}}u^\alpha u_t^\alpha dx+C_\mu+\frac{\eta\bar{d}}{4}\sin\frac{\pi\alpha}{2}\int_\Omega\int_0^{u^\alpha}f(s)dsdx+D$$

This ensures the inequality

$$\frac{d}{dt}\left(\mathcal{P}_{\delta,\alpha}(u^\alpha,u_t^\alpha)\right)\leqslant-\frac{\eta\bar{d}}{4}\mathcal{P}_{\delta,\alpha}(u^\alpha,u_t^\alpha)+C_\mu+D. \tag{5.14}$$

Since we have

$$\mathcal{L}_{\delta,\alpha}\left(\left[\begin{smallmatrix}u^\alpha\\ v^\alpha\end{smallmatrix}\right]\right)=\mathcal{P}_{\delta,\alpha}(u^\alpha,u_t^\alpha)\ \text{ and }\ \frac{d}{dt}\left(\mathcal{L}_{\delta,\alpha}\left(\left[\begin{smallmatrix}u^\alpha\\ v^\alpha\end{smallmatrix}\right]\right)\right)=\frac{d}{dt}\left(\mathcal{P}_{\delta,\alpha}(u^\alpha,u_t^\alpha)\right), \tag{5.15}$$

(see (3.5), (5.1)-(5.2) and (5.6)), using (5.14)-(5.15) we actually get

$$\frac{d}{dt}\left(\mathcal{L}_{\delta,\alpha}\left(\left[\begin{smallmatrix}u^\alpha\\ v^\alpha\end{smallmatrix}\right]\right)\right)\leqslant-\frac{\eta\bar{d}}{4}\mathcal{L}_{\delta,\alpha}\left(\left[\begin{smallmatrix}u^\alpha\\ v^\alpha\end{smallmatrix}\right]\right)+C_\mu+D.$$

which together with (5.4) (see Remark 5.1) implies that

$$\frac{1}{2}c_1\left\|\left[\begin{smallmatrix}u^\alpha\\ v^\alpha\end{smallmatrix}\right]\right\|_{X^{\frac{1+\alpha}{4}}\times X^{\frac{-1+\alpha}{4}}}-c_2\leqslant\mathcal{L}_{\delta,\alpha}\left(\left[\begin{smallmatrix}u_0\\ v_0\end{smallmatrix}\right]\right)e^{-\frac{\eta\bar{d}}{4}t}+\frac{4(C_\mu+D)}{\eta\bar{d}}.$$

Taking into account that $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]$ is in a ball $B_{X^{\frac{1}{2}}_{\rm lim}\times X_{\rm lim}}(r)$ and using (5.3) we have from all above that, for a certain $t_r>0$,

$$\frac{1}{2}c_1 \left\|\left[\begin{smallmatrix} u^\alpha \\ v^\alpha \end{smallmatrix}\right]\right\|_{X^{\frac{1+\alpha}{4}}\times X^{\frac{-1+\alpha}{4}}} - c_2 \leqslant 1 + \frac{4(C_\mu + D)}{\eta\bar{d}},\ t\geqslant t_r,$$

which gives the result.

Corollary 5.3. *Assume (1.2)-(1.3) and (1.4).*

There then exist $R_0>0$, $\alpha^*\in(0,1)$ and given any $r>0$ there is a certain $t_r>0$ such that for each sequence $\alpha_n \nearrow 1$, and any sequence $\{\left[\begin{smallmatrix} w_n \\ z_n \end{smallmatrix}\right]\}$ of elements $\left[\begin{smallmatrix} w_n \\ z_n \end{smallmatrix}\right]\in X^{\frac{1+\alpha_n}{4}}\times X^{\frac{-1+\alpha_n}{4}}$ satisfying

$$\sup_{n\in\mathbb{N}}\left\|\left[\begin{smallmatrix} w_n \\ z_n \end{smallmatrix}\right]\right\|_{X^{\frac{1+\alpha_n}{4}}\times X^{\frac{-1+\alpha_n}{4}}}\leqslant r,$$

we have that the solution $\left[\begin{smallmatrix} u^{\alpha_n} \\ v^{\alpha_n} \end{smallmatrix}\right]$ of (3.1) through $\left[\begin{smallmatrix} u_{0\alpha_n} \\ v_{0\alpha_n} \end{smallmatrix}\right]=\left[\begin{smallmatrix} w_n \\ z_n \end{smallmatrix}\right]$ as in Theorem 1.3 exists for all n large enough and

$$\sup_{\{n\in\mathbb{N}:\ \alpha_n\geqslant\alpha^*\}}\ \sup_{t\geqslant t_r}\left\|\left[\begin{smallmatrix} u^{\alpha_n} \\ v^{\alpha_n} \end{smallmatrix}\right]\right\|_{X^{\frac{1+\alpha_n}{4}}\times X^{\frac{-1+\alpha_n}{4}}}\leqslant R_0.$$

Proof: The proof follows the lines of the proof of Lemma 5.2 with the only differences that $\left[\begin{smallmatrix} u^\alpha \\ v^\alpha \end{smallmatrix}\right]$ is replaced by the solution $\left[\begin{smallmatrix} u^{\alpha_n} \\ v^{\alpha_n} \end{smallmatrix}\right]$ through $\left[\begin{smallmatrix} u_{0\alpha_n} \\ v_{0\alpha_n} \end{smallmatrix}\right]=\left[\begin{smallmatrix} w_n \\ z_n \end{smallmatrix}\right]$ and that (instead of using Lemma 4.3 to get (5.8)) we use Corollary 4.4 to get that there is $M=M(r)>0$ such that

$$\sup_{t\geqslant 0}\left\|\begin{bmatrix} u^{\alpha_n}(t) \\ v^{\alpha_n}(t) \end{bmatrix}\right\|_{X^{\frac{1+\alpha_n}{4}}\times X^{\frac{-1+\alpha_n}{4}}}\leqslant M(r) \text{ for all } \alpha_n\geqslant\alpha^*,\ \left\|\left[\begin{smallmatrix} w_n \\ z_n \end{smallmatrix}\right]\right\|_{X^{\frac{1+\alpha_n}{4}}\times X^{\frac{-1+\alpha_n}{4}}}\leqslant r,$$

which plays a role of the counterpart of (5.8). The rest is unchanged. □

5.2. **Absorbing set for (1.6): proof of Theorem 1.6.** Let B be bounded in $X^{\frac{1}{2}}_{\rm lim}\times X_{\rm lim}$ and $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$ is a global mild solution of (1.6) through $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]\in B$ obtained via limiting procedure as in Theorem 1.5. Then, for each $\zeta\in[-1,1)$, $\begin{bmatrix} u(t) \\ v(t) \end{bmatrix}$ is a limit in $X^{\frac{1+\zeta}{4}}\times X^{\frac{-1+\zeta}{4}}$ of a sequence $\begin{bmatrix} u^{\alpha_{n_k}}(t) \\ v^{\alpha_{n_k}}(t) \end{bmatrix}$ of solutions to (3.1) through $\left[\begin{smallmatrix} u_{0\alpha} \\ v_{0\alpha} \end{smallmatrix}\right]=\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]$ (see (4.5)). Observe that for R_0 as in Lemma 5.2

$$\sup_{k\in\mathbb{N}}\left\|\begin{bmatrix} u^{\alpha_{n_k}}(t) \\ v^{\alpha_{n_k}}(t) \end{bmatrix}\right\|_{X^{\frac{1+\zeta}{4}}\times X^{\frac{-1+\zeta}{4}}}\leqslant R_0 \quad\text{for every}\quad t\geqslant t_r.$$

Due to embedding properties of the scale a certain $c_0>0$ exists such that for any $\zeta\in[-1,1)$ there is a number $n_\zeta\in\mathbb{N}$ such that

$$\sup_{n_k\geqslant n_\zeta}\left\|\begin{bmatrix} u^{\alpha_{n_k}}(t) \\ v^{\alpha_{n_k}}(t) \end{bmatrix}\right\|_{X^{\frac{1+\zeta}{4}}\times X^{\frac{-1+\zeta}{4}}}\leqslant c_0R_0 \quad\text{for every}\quad t\geqslant t_r. \tag{5.16}$$

Combining (5.16) and (4.5) we get

$$\left\|\begin{bmatrix} u(t) \\ v(t) \end{bmatrix}\right\|_{X^{\frac{1+\zeta}{4}}\times X^{\frac{-1+\zeta}{4}}}\leqslant c_0R_0 \quad\text{for every}\quad t\geqslant t_r \quad\text{and}\quad \zeta\in[-1,1),$$

which leads to the result of Theorem 1.6. □

5.3. **Attractor for (1.6).** Given $R_0 > 0$ as in Corollary 5.3 let $B_\alpha(R_0)$ be the ball in $X^{\frac{1+\alpha}{4}} \times X^{\frac{-1+\alpha}{4}}$ of radius R_0 around zero.

Denoting by $\left[\begin{smallmatrix} u^{\alpha_n} \\ v^{\alpha_n} \end{smallmatrix}\right](t_n, \left[\begin{smallmatrix} w_n \\ z_n \end{smallmatrix}\right])$ the solution of (3.1) through $\left[\begin{smallmatrix} u_{0\alpha_n} \\ v_{0\alpha_n} \end{smallmatrix}\right] = \left[\begin{smallmatrix} w_n \\ z_n \end{smallmatrix}\right]$ we let

$$
\begin{aligned}
\mathcal{A}_1 := \Big\{ \left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right] \in X^{\frac12}_{\lim} \times X_{\lim} : \ & \text{there are sequences } t_n \to \infty, \\
& \alpha_n \nearrow 1, \ \text{and} \ \left[\begin{smallmatrix} w_n \\ z_n \end{smallmatrix}\right] \in B_{\alpha_n}(R_0), \\
& \text{such that for each } \zeta \in [-1,1) \\
& \left[\begin{smallmatrix} u^{\alpha_n} \\ v^{\alpha_n} \end{smallmatrix}\right](t_n, \left[\begin{smallmatrix} w_n \\ z_n \end{smallmatrix}\right]) \xrightarrow{X^{\frac{1+\zeta}{4}} \times X^{\frac{-1+\zeta}{4}}} \left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right] \Big\}.
\end{aligned} \tag{5.17}
$$

We also define the following class of global mild solutions to (3.1).

Definition 5.4. *We say that* $\left[\begin{smallmatrix} \phi \\ \varphi \end{smallmatrix}\right] \in \mathcal{LS}$ *if and only if one of the following conditions holds.*

(i) $\left[\begin{smallmatrix} \phi \\ \varphi \end{smallmatrix}\right](0) \in \mathcal{A}_1$ *and* $\left[\begin{smallmatrix} \phi \\ \varphi \end{smallmatrix}\right]$ *is a global weak solution of (1.6) being for each* $\zeta \in [-1,1)$ *(uniform for t in compact subsets of* $[0,\infty)$*) limit in* $X^{\frac{1+\zeta}{4}} \times X^{\frac{-1+\zeta}{4}}$ *of a sequence of solutions of* (3.1) *of the form* $\left[\begin{smallmatrix} u^{\alpha_n} \\ v^{\alpha_n} \end{smallmatrix}\right](\cdot, \left[\begin{smallmatrix} u^{\alpha_n} \\ v^{\alpha_n} \end{smallmatrix}\right](t_n, \left[\begin{smallmatrix} w_n \\ z_n \end{smallmatrix}\right]))$*, where* $t_n \to \infty$*,* $\alpha_n \nearrow 1$ *and* $\left[\begin{smallmatrix} w_n \\ z_n \end{smallmatrix}\right] \in B_{\alpha_n}(R_0)$*;*

(ii) $\left[\begin{smallmatrix} \phi \\ \varphi \end{smallmatrix}\right](0) \in X^{\frac12}_{\lim} \times X_{\lim} \setminus \mathcal{A}_1$ *and* $\left[\begin{smallmatrix} \phi \\ \varphi \end{smallmatrix}\right]$ *is a global weak solution of* (1.6) *being for each* $\zeta \in [-1,1)$ *(uniform for t in compact subsets of* $[0,\infty)$*) limit in* $X^{\frac{1+\zeta}{4}} \times X^{\frac{-1+\zeta}{4}}$ *of a sequence of solutions of* (3.1) *of the form* $\left[\begin{smallmatrix} u^{\alpha_n} \\ v^{\alpha_n} \end{smallmatrix}\right](\cdot, \left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right](0))$*, where* $t_n \to \infty$ *and* $\alpha_n \nearrow 1$.

With this set-up we have the following result.

Theorem 5.5. *Assume (1.2)-(1.3) and (1.4) and let* $\mathcal{LS}$ *be as in Definition 5.4.*

Then all below hold.

i) (*Existence*) *Given* $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in X^{\frac12}_{\lim} \times X_{\lim}$ *there exists* $\left[\begin{smallmatrix} \phi \\ \varphi \end{smallmatrix}\right] \in \mathcal{LS}$ *with* $\left[\begin{smallmatrix} \phi \\ \varphi \end{smallmatrix}\right](0) = \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]$.

ii) (*Bounded dissipative*) *There is a bounded subset* B_0 *of* $X^{\frac12}_{\lim} \times X_{\lim}$ *such that for any* B *bounded in* $X^{\frac12}_{\lim} \times X_{\lim}$*, each* $\left[\begin{smallmatrix} \phi \\ \varphi \end{smallmatrix}\right]$ *from the class* $\mathcal{LS}$ *with* $\left[\begin{smallmatrix} \phi \\ \varphi \end{smallmatrix}\right](0) \in B$ *enters* B_0 *in a certain time* $\tau_B \geqslant 0$ *and stays in* B_0 *for all* $t \geqslant \tau_B$.

iii) (*Attractor*) $\mathcal{A}_1$ *is a bounded and closed subset in* $X^{\frac12}_{\lim} \times X_{\lim}$ *which satisfies*

a) (*Compactness*) $\mathcal{A}_1$ *is compact in* $X^{\frac{1+\zeta}{4}} \times X^{\frac{-1+\zeta}{4}}$ *for any* $\zeta \in [-1,1)$*,*

b) (*Positive invariance*) $\left\{\left[\begin{smallmatrix} \phi \\ \varphi \end{smallmatrix}\right](t);\ \left[\begin{smallmatrix} \phi \\ \varphi \end{smallmatrix}\right] \in \mathcal{LS}, \left[\begin{smallmatrix} \phi \\ \varphi \end{smallmatrix}\right](0) \in \mathcal{A}_1,\ t \geqslant 0\right\} \subset \mathcal{A}_1$*,*
(*Negative invariance*) *for each* $\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right] \in \mathcal{A}_1$ *and for any* $t \geqslant 0$ *there is a certain* $\left[\begin{smallmatrix} \bar\phi \\ \bar\varphi \end{smallmatrix}\right] \in \mathcal{LS}$ *with* $\left[\begin{smallmatrix} \bar\phi \\ \bar\varphi \end{smallmatrix}\right](0) \in \mathcal{A}_1$ *such that* $\left[\begin{smallmatrix} \bar\phi(t) \\ \bar\varphi(t) \end{smallmatrix}\right] = \left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right]$*,*

c) (*Attracting property*) *for any* B *bounded in* $X^{\frac12}_{\lim} \times X_{\lim}$ *and any* $\zeta \in [-1,1)$

$$
\sup_{\left[\begin{smallmatrix} \phi \\ \varphi \end{smallmatrix}\right] \in \mathcal{LS} \text{ with } \left[\begin{smallmatrix} \phi \\ \varphi \end{smallmatrix}\right](0) \in B} \ \inf_{\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right] \in \mathcal{A}_1} \left\| \left[\begin{smallmatrix} \phi \\ \varphi \end{smallmatrix}\right](t) - \left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right] \right\|_{X^{\frac{1+\zeta}{4}} \times X^{\frac{-1+\zeta}{4}}} \overset{t\to\infty}{\longrightarrow} 0. \tag{5.18}
$$

Proof: Part i) is a consequence of Theorem 4.5 and part ii) comes from Theorem 1.6.

If $\left[\begin{smallmatrix} w\\ z\end{smallmatrix}\right]\in\mathcal{A}_1$ then $\left[\begin{smallmatrix} w\\ z\end{smallmatrix}\right]$ is approximated by a suitable sequence $\{\left[\begin{smallmatrix} u^{\alpha_n}\\ v^{\alpha_n}\end{smallmatrix}\right](t_n,\left[\begin{smallmatrix} w_n\\ z_n\end{smallmatrix}\right])\}$ as in (5.17) and hence there is $c_0>0$ and, given any $\zeta\in[-1,1)$ there is also $n_\zeta\in\mathbb{N}$ such that

$$\begin{aligned}\|\left[\begin{smallmatrix} w\\ z\end{smallmatrix}\right]\|_{X^{\frac{1+\zeta}{4}}\times X^{\frac{-1+\zeta}{4}}} &= \lim\|\left[\begin{smallmatrix} u^{\alpha_n}\\ v^{\alpha_n}\end{smallmatrix}\right](t_n,\left[\begin{smallmatrix} w_n\\ z_n\end{smallmatrix}\right])\|_{X^{\frac{1+\zeta}{4}}\times X^{\frac{-1+\zeta}{4}}}\\ &\leqslant c_0\sup_{n\geqslant n_\zeta}\|\left[\begin{smallmatrix} u^{\alpha_n}\\ v^{\alpha_n}\end{smallmatrix}\right](t_n,\left[\begin{smallmatrix} w_n\\ z_n\end{smallmatrix}\right])\|_{X^{\frac{1+\alpha_n}{4}}\times X^{\frac{-1+\alpha_n}{4}}}.\end{aligned}\tag{5.19}$$

Using that $t_n\to\infty$, $\left[\begin{smallmatrix} w_n\\ z_n\end{smallmatrix}\right]\in B_{\alpha_n}(R_0)$ and applying Corollary 5.3 with $r=R_0$ we can actualy replace the supremum on the right hand side of (5.19) by c_0R_0 and hence we get

$$\|\left[\begin{smallmatrix} w\\ z\end{smallmatrix}\right]\|_{X^{\frac{1+\zeta}{4}}\times X^{\frac{-1+\zeta}{4}}}\leqslant c_0R_0.\tag{5.20}$$

Since this can be done for every $\zeta\in[-1,1)$ and c_0,R_0 are independent of such ζ we conclude that $\mathcal{A}_1$ is bounded in $X^{\frac12}_{\lim}\times X_{\lim}$.

If $\{\left[\begin{smallmatrix} a_n\\ b_n\end{smallmatrix}\right]\}\subset\mathcal{A}_1$ and if we choose some $\zeta\in[-1,1)$ then in $X^{\frac{1+\zeta}{4}}\times X^{\frac{-1+\zeta}{4}}$ each element $\left[\begin{smallmatrix} a_n\\ b_n\end{smallmatrix}\right]$ is, in particular, a limit as $m\to\infty$ of a sequence $\begin{bmatrix} u^{\alpha_m}_{(n)}\\ v^{\alpha_m}_{(n)}\end{bmatrix}\left(t^{(n)}_m,\begin{bmatrix} w^{(n)}_m\\ z^{(n)}_m\end{bmatrix}\right)$ where $\begin{bmatrix} w^{(n)}_m\\ z^{(n)}_m\end{bmatrix}\in B_{\alpha_m}(R_0)$ and $\alpha_m\nearrow 1$, $t^{(n)}_m\to\infty$. Thus the distance of $\left[\begin{smallmatrix} a_n\\ b_n\end{smallmatrix}\right]$ from some element $\begin{bmatrix} u^{\alpha_{m_n}}_{(n)}\\ v^{\alpha_{m_n}}_{(n)}\end{bmatrix}\left(t^{(n)}_{m_n},\begin{bmatrix} w^{(n)}_{m_n}\\ z^{(n)}_{m_n}\end{bmatrix}\right)$ is in this space less than $\frac1n$, where $\begin{bmatrix} w^{(n)}_{m_n}\\ z^{(n)}_{m_n}\end{bmatrix}\in B_{\alpha_{m_n}}(R_0)$ and we can choose $\{t^{(n)}_{m_n}\}$ increasing to ∞. Due to Corollary 5.3, $\begin{bmatrix} u^{\alpha_{m_n}}_{(n)}\\ v^{\alpha_{m_n}}_{(n)}\end{bmatrix}\left(t^{(n)}_{m_n},\begin{bmatrix} w^{(n)}_{m_n}\\ z^{(n)}_{m_n}\end{bmatrix}\right)\in B_{\alpha_{m_n}}(R_0)$ for almost all m_n. Given any $\tilde\zeta\in[-1,1)$ there then exists $N_{\tilde\zeta}\in\mathbb{N}$ such that $\left\{\begin{bmatrix} u^{\alpha_{m_n}}_{(n)}\\ v^{\alpha_{m_n}}_{(n)}\end{bmatrix}\left(t^{(n)}_{m_n},\begin{bmatrix} w^{(n)}_{m_n}\\ z^{(n)}_{m_n}\end{bmatrix}\right): m_n\geqslant N_{\tilde\zeta}\right\}$ is bounded in $X^{\frac{1+\tilde\zeta}{4}}\times X^{\frac{-1+\tilde\zeta}{4}}$. There is thus a subsequence of $\left\{\begin{bmatrix} u^{\alpha_{m_n}}_{(n)}\\ v^{\alpha_{m_n}}_{(n)}\end{bmatrix}\left(t^{(n)}_{m_n},\begin{bmatrix} w^{(n)}_{m_n}\\ z^{(n)}_{m_n}\end{bmatrix}\right)\right\}$ convergent in $X\times X^{-\frac12}$ to a certain $\left[\begin{smallmatrix} w\\ z\end{smallmatrix}\right]$ and, moreover, from each subsequence of this subsequence we can still choose a subsequence convergent now in $X^{\frac{1+\tilde\zeta}{4}}\times X^{\frac{-1+\tilde\zeta}{4}}$ (thus again to $\left[\begin{smallmatrix} w\\ z\end{smallmatrix}\right]$) where $\tilde\zeta$ can be any number from $[-1,1)$. This proves that a subsequence of $\left\{\begin{bmatrix} u^{\alpha_{m_n}}_{(n)}\\ v^{\alpha_{m_n}}_{(n)}\end{bmatrix}\left(t^{(n)}_{m_n},\begin{bmatrix} w^{(n)}_{m_n}\\ z^{(n)}_{m_n}\end{bmatrix}\right)\right\}$ which converges to $\left[\begin{smallmatrix} w\\ z\end{smallmatrix}\right]$ in $X\times X^{-\frac12}$ actually converges to $\left[\begin{smallmatrix} w\\ z\end{smallmatrix}\right]$ in $X^{\frac{1+\tilde\zeta}{4}}\times X^{\frac{-1+\tilde\zeta}{4}}$ for each $\tilde\zeta\in[-1,1)$. Recalling that $\begin{bmatrix} u^{\alpha_{m_n}}_{(n)}\\ v^{\alpha_{m_n}}_{(n)}\end{bmatrix}\left(t^{(n)}_{m_n},\begin{bmatrix} w^{(n)}_{m_n}\\ z^{(n)}_{m_n}\end{bmatrix}\right)\in B_{\alpha_{m_n}}(R_0)$ and repeating the argument as in (5.19)-(5.20) we obtain that $\left[\begin{smallmatrix} w\\ z\end{smallmatrix}\right]\in X^{\frac12}_{\lim}\times X_{\lim}$. We thus conclude that $\left[\begin{smallmatrix} w\\ z\end{smallmatrix}\right]\in\mathcal{A}_1$. Due to the above argument there is thus a subsequence of $\{\left[\begin{smallmatrix} a_n\\ b_n\end{smallmatrix}\right]\}$ which converges to $\left[\begin{smallmatrix} w\\ z\end{smallmatrix}\right]\in\mathcal{A}_1$ in $X^{\frac{1+\zeta}{4}}\times X^{\frac{-1+\zeta}{4}}$. This proves that $\mathcal{A}_1$ is compact in $X^{\frac{1+\zeta}{4}}\times X^{\frac{-1+\zeta}{4}}$ and since this can be done for arbitrarily chosen $\zeta\in[-1,1)$ we get the result of part a).

Observe that having proved part a) we also have that $\mathcal{A}_1$ is closed in $X^{\frac12}_{\lim}\times X_{\lim}$ as having $\{\left[\begin{smallmatrix} a_n\\ b_n\end{smallmatrix}\right]\}\subset\mathcal{A}_1$ such that $\left[\begin{smallmatrix} a_n\\ b_n\end{smallmatrix}\right]\to\left[\begin{smallmatrix} a\\ b\end{smallmatrix}\right]$ in $X^{\frac12}_{\lim}\times X_{\lim}$ and repeating the proof of part a) above we obtain that a subsequence of $\{\left[\begin{smallmatrix} a_n\\ b_n\end{smallmatrix}\right]\}$ converges to some $\left[\begin{smallmatrix} w\\ z\end{smallmatrix}\right]\in\mathcal{A}_1$ in $X^{\frac{1+\zeta}{4}}\times X^{\frac{-1+\zeta}{4}}$ for some $\zeta\in[-1,1)$. Since $\left[\begin{smallmatrix} a\\ b\end{smallmatrix}\right]$ needs to coincide with $\left[\begin{smallmatrix} w\\ z\end{smallmatrix}\right]$, we obtain that $\left[\begin{smallmatrix} a\\ b\end{smallmatrix}\right]\in\mathcal{A}_1$.

Positive invariance in part b) comes from the fact that, given $\left[\begin{smallmatrix}\phi\\ \varphi\end{smallmatrix}\right](0)\in\mathcal{A}_1$ and given approximating this element sequence $\left[\begin{smallmatrix}u^{\alpha_n}\\ v^{\alpha_n}\end{smallmatrix}\right](t_n,\left[\begin{smallmatrix}w_n\\ z_n\end{smallmatrix}\right])$ as in (5.17), due to Corollary 5.3 for all n large enough and any $t\geqslant 0$ we have that

$$\left[\begin{smallmatrix}u^{\alpha_n}\\ v^{\alpha_n}\end{smallmatrix}\right](t,\left[\begin{smallmatrix}u^{\alpha_n}\\ v^{\alpha_n}\end{smallmatrix}\right](t_n,\left[\begin{smallmatrix}w_n\\ z_n\end{smallmatrix}\right]))=\left[\begin{smallmatrix}u^{\alpha_n}\\ v^{\alpha_n}\end{smallmatrix}\right](t+t_n,\left[\begin{smallmatrix}w_n\\ z_n\end{smallmatrix}\right])\in B_{\alpha_n}(R_0).$$

Hence each value $\left[\begin{smallmatrix}\phi\\ \varphi\end{smallmatrix}\right](t)$ is obtained as in Definition 5.4 (i) so that each value $\left[\begin{smallmatrix}\phi\\ \varphi\end{smallmatrix}\right](t)$ is an element of $\mathcal{A}_1$.

For negative invariance take $\left[\begin{smallmatrix}w\\ z\end{smallmatrix}\right]\in\mathcal{A}_1$ and consider a sequence $\{\left[\begin{smallmatrix}u^{\alpha_n}\\ v^{\alpha_n}\end{smallmatrix}\right](t_n-t,\left[\begin{smallmatrix}w_n\\ z_n\end{smallmatrix}\right])\}$, where $\left[\begin{smallmatrix}u^{\alpha_n}\\ v^{\alpha_n}\end{smallmatrix}\right](t_n,\left[\begin{smallmatrix}w_n\\ z_n\end{smallmatrix}\right])$ approximates $\left[\begin{smallmatrix}w\\ z\end{smallmatrix}\right]$ as in the definition of $\mathcal{A}_1$ (see (5.17)). Given any $t>0$, observe that $\left[\begin{smallmatrix}u^{\alpha_n}\\ v^{\alpha_n}\end{smallmatrix}\right](t_n-t,\left[\begin{smallmatrix}w_n\\ z_n\end{smallmatrix}\right])\in B_{\alpha_n}(R_0)$ for all n large enough (see Corollary 5.3) and hence there is $c_0>0$ and, given any $\zeta\in[-1,1)$ there is also $n_\zeta\in\mathbb{N}$ such that

$$\begin{aligned}\|\left[\begin{smallmatrix}u^{\alpha_n}\\ v^{\alpha_n}\end{smallmatrix}\right](t_n-t,\left[\begin{smallmatrix}w_n\\ z_n\end{smallmatrix}\right])\|_{X^{\frac{1+\zeta}{4}}\times X^{\frac{-1+\zeta}{4}}}&\leqslant c_0\,\|\left[\begin{smallmatrix}u^{\alpha_n}\\ v^{\alpha_n}\end{smallmatrix}\right](t_n-t,\left[\begin{smallmatrix}w_n\\ z_n\end{smallmatrix}\right])\|_{X^{\frac{1+\alpha_n}{4}}\times X^{\frac{-1+\alpha_n}{4}}}\\ &\leqslant c_0R_0\ \text{ whenever }\ \zeta\in[-1,1)\ \text{ and }\ n\geqslant n_\zeta.\end{aligned}\tag{5.21}$$

Since the embeddings are compact we can choose a subsequence $\{\left[\begin{smallmatrix}u^{\alpha_{n_k}}\\ v^{\alpha_{n_k}}\end{smallmatrix}\right]\left(t_{n_k}-t,\left[\begin{smallmatrix}w_{n_k}\\ z_{n_k}\end{smallmatrix}\right]\right)\}$ which converges in $X\times X^{-\frac12}$ to a certain $\left[\begin{smallmatrix}\bar w\\ \bar z\end{smallmatrix}\right]\in X\times X^{-\frac12}$. Then we observe that, due to (5.21), for each $\zeta\in(-1,1)$ and from any subsequence of $\{\left[\begin{smallmatrix}u^{\alpha_{n_k}}\\ v^{\alpha_{n_k}}\end{smallmatrix}\right]\left(t_{n_k}-t,\left[\begin{smallmatrix}w_{n_k}\\ z_{n_k}\end{smallmatrix}\right]\right)\}$ we can choose a subsequence which converges in $X^{\frac{1+\zeta}{4}}\times X^{\frac{-1+\zeta}{4}}$ (thus to $\left[\begin{smallmatrix}\bar w\\ \bar z\end{smallmatrix}\right]$). Hence we have that, on the one hand, the sequence $\{\left[\begin{smallmatrix}u^{\alpha_{n_k}}\\ v^{\alpha_{n_k}}\end{smallmatrix}\right]\left(t_{n_k}-t,\left[\begin{smallmatrix}w_{n_k}\\ z_{n_k}\end{smallmatrix}\right]\right)\}$ actually converges in $X^{\frac{1+\zeta}{4}}\times X^{\frac{-1+\zeta}{4}}$ for each $\zeta\in[-1,1)$ (thus $\left[\begin{smallmatrix}\bar w\\ \bar z\end{smallmatrix}\right]\in X^{\frac{1+\zeta}{4}}\times X^{\frac{-1+\zeta}{4}}$ for each $\zeta\in[-1,1)$) and, on the other, that

$$\|\left[\begin{smallmatrix}\bar w\\ \bar z\end{smallmatrix}\right]\|_{X^{\frac{1+\zeta}{4}}\times X^{\frac{-1+\zeta}{4}}}\leqslant c_0R_0\ \text{ for each }\ \zeta\in[-1,1)$$

(because of (5.21)). Consequently, $\left[\begin{smallmatrix}\bar w\\ \bar z\end{smallmatrix}\right]\in\mathcal{A}_1$ and we apply Theorem 4.5 to get a global mild solution $\begin{bmatrix}\bar\phi\\ \bar\varphi\end{bmatrix}$ of (1.6) being for each $\zeta\in[-1,1)$ (uniform for t in compact subsets of $[0,\infty)$) limit in $X^{\frac{1+\zeta}{4}}\times X^{\frac{-1+\zeta}{4}}$ of a sequence of solutions of (3.1) of the form $\left[\begin{smallmatrix}u^{\alpha_{n_k}}\\ v^{\alpha_{n_k}}\end{smallmatrix}\right]\left(\cdot,\left[\begin{smallmatrix}u^{\alpha_{n_k}}\\ v^{\alpha_{n_k}}\end{smallmatrix}\right]\left(t_{n_k}-t,\left[\begin{smallmatrix}w_{n_k}\\ z_{n_k}\end{smallmatrix}\right]\right)\right)$, where $t_{n_k}\to\infty$, $\alpha_{n_k}\nearrow 1$ and $\left[\begin{smallmatrix}u^{\alpha_{n_k}}\\ v^{\alpha_{n_k}}\end{smallmatrix}\right]\left(t_{n_k}-t,\left[\begin{smallmatrix}w_{n_k}\\ z_{n_k}\end{smallmatrix}\right]\right)\subset B_{\alpha_{n_k}}(R_0)$. This implies, in particular, that $\begin{bmatrix}\bar\phi\\ \bar\varphi\end{bmatrix}(t)=\left[\begin{smallmatrix}w\\ z\end{smallmatrix}\right]$, which concludes part b).

Now observe that if (5.18) fails then there is $\zeta\in[-1,1)$ and there exists a sequence $\{\left[\begin{smallmatrix}\phi_n\\ \varphi_n\end{smallmatrix}\right]\}\subset\mathcal{LS}$ with $\{\left[\begin{smallmatrix}\phi_n\\ \varphi_n\end{smallmatrix}\right](0)\}\subset B$, where B is bounded in $X^{\frac12}_{\lim}\times X_{\lim}$, and there is also a sequence of times $t_n\to\infty$ such that a sequence $\{\left[\begin{smallmatrix}\phi_n\\ \varphi_n\end{smallmatrix}\right](t_n)\}$ is separated from $\mathcal{A}_1$ in $X^{\frac{1+\zeta}{4}}\times X^{\frac{-1+\zeta}{4}}$. Then observe that, due to Theorem 1.6, almost all elements of $\{\left[\begin{smallmatrix}\phi_n\\ \varphi_n\end{smallmatrix}\right](t_n)\}$ are in a bounded subset B_0 of $X^{\frac12}_{\lim}\times X_{\lim}$. In particular, we have from Lemma 4.1 v) that there is a subsequence $\left\{\begin{bmatrix}\phi_{n_k}\\ \varphi_{n_k}\end{bmatrix}(t_{n_k})\right\}$, which converges in $X^{\frac{1+\zeta}{4}}\times X^{\frac{-1+\zeta}{4}}$. Since $\left\{\begin{bmatrix}\phi_{n_k}\\ \varphi_{n_k}\end{bmatrix}\right\}\subset\mathcal{LS}$, it follows from Definition 5.4 that, given $k\in\mathbb{N}$, either the distance of $\begin{bmatrix}\phi_{n_k}\\ \varphi_{n_k}\end{bmatrix}(t_{n_k})$ in $X^{\frac{1+\zeta}{4}}\times X^{\frac{-1+\zeta}{4}}$ is less than $\frac1k$ from some $\begin{bmatrix}u^{\alpha_m}_{(k)}\\ v^{\alpha_m}_{(k)}\end{bmatrix}\left(t_{n_k},\begin{bmatrix}u^{\alpha_m}_{(k)}\\ v^{\alpha_m}_{(k)}\end{bmatrix}\left(t^{(k)}_m,\begin{bmatrix}w^{(k)}_m\\ z^{(k)}_m\end{bmatrix}\right)\right)$, where

$$\begin{bmatrix}u^{\alpha_m}_{(k)}\\ v^{\alpha_m}_{(k)}\end{bmatrix}\left(t_{n_k},\begin{bmatrix}u^{\alpha_m}_{(k)}\\ v^{\alpha_m}_{(k)}\end{bmatrix}\left(t^{(k)}_m,\begin{bmatrix}w^{(k)}_m\\ z^{(k)}_m\end{bmatrix}\right)\right)=\begin{bmatrix}u^{\alpha_m}_{(k)}\\ v^{\alpha_m}_{(k)}\end{bmatrix}\left(t_{n_k}+t^{(k)}_m,\begin{bmatrix}w^{(k)}_m\\ z^{(k)}_m\end{bmatrix}\right)\tag{5.22}$$

(let us call it case (I)) or the distance of $\left[\begin{smallmatrix}\phi_{n_k}\\ \varphi_{n_k}\end{smallmatrix}\right](t_{n_k})$ in $X^{\frac{1+\zeta}{4}}\times X^{\frac{-1+\zeta}{4}}$ is less than $\frac{1}{k}$ from some $\left[\begin{smallmatrix}u^{\alpha_m}_{(k)}\\ v^{\alpha_m}_{(k)}\end{smallmatrix}\right](t^{(k)}_n,\left[\begin{smallmatrix}\phi^{(k)}_m\\ \varphi^{(k)}_m\end{smallmatrix}\right](0))$ (let us call it case (II)). One of these two cases has to happen for infinitely many k. If this is case (I), then using that $\left[\begin{smallmatrix}w_m\\ z_m\end{smallmatrix}\right]\in B_{\alpha_m}(R_0)$ and $t_{n_k}\to\infty$ we have, due to Corollary 5.3, that the right hand side of (5.22) belongs to $B_{\alpha_m}(R_0)$ (which gives, in particular, boundedness of infinitely elements of the right hand side of (5.22) in $X^{\frac{1+\tilde\zeta}{4}}\times X^{\frac{-1+\tilde\zeta}{4}}$ for $\tilde\zeta\in[-1,1)$ similarly as in (5.21)). Therefore, in this case there will be a subsequence of $\left[\begin{smallmatrix}u^{\alpha_m}_{(k)}\\ v^{\alpha_m}_{(k)}\end{smallmatrix}\right]\left(\cdot,\left[\begin{smallmatrix}u^{\alpha_m}_{(k)}\\ v^{\alpha_m}_{(k)}\end{smallmatrix}\right]\left(t^{(k)}_m,\left[\begin{smallmatrix}w^{(k)}_m\\ z^{(k)}_m\end{smallmatrix}\right]\right)\right)$ convergent in $X\times X^{-\frac{1}{2}}$ and thus in $X^{\frac{1+\tilde\zeta}{4}}\times X^{\frac{-1+\tilde\zeta}{4}}$ for any $\tilde\zeta\in[-1,1)$ (thus with a limit point in $\mathcal{A}_1$). If this is case (II), then using that $\left\{\left[\begin{smallmatrix}\phi_{n_k}\\ \varphi_{n_k}\end{smallmatrix}\right](0)\right\}\subset B$ we have, due to Corollary 5.3, that $\left[\begin{smallmatrix}u^{\alpha_m}_{(k)}\\ v^{\alpha_m}_{(k)}\end{smallmatrix}\right](\frac{t_{n_k}}{2},\left[\begin{smallmatrix}\phi^{(k)}_m\\ \varphi^{(k)}_m\end{smallmatrix}\right](0))\in B_{\alpha_m}(R_0)$ and $\left[\begin{smallmatrix}u^{\alpha_m}_{(k)}\\ v^{\alpha_m}_{(k)}\end{smallmatrix}\right](t_{n_k},\left[\begin{smallmatrix}\phi^{(k)}_m\\ \varphi^{(k)}_m\end{smallmatrix}\right](0))\in B_{\alpha_m}(R_0)$ for almost all k (which gives, in particular, boundedness of almost all elements of sequences $\left\{\left[\begin{smallmatrix}u^{\alpha_m}_{(k)}\\ v^{\alpha_m}_{(k)}\end{smallmatrix}\right](\frac{t_{n_k}}{2},\left[\begin{smallmatrix}\phi^{(k)}_m\\ \varphi^{(k)}_m\end{smallmatrix}\right](0))\right\}$ and $\left\{\left[\begin{smallmatrix}u^{\alpha_m}_{(k)}\\ v^{\alpha_m}_{(k)}\end{smallmatrix}\right](t_{n_k},\left[\begin{smallmatrix}\phi^{(k)}_m\\ \varphi^{(k)}_m\end{smallmatrix}\right](0))\right\}$ in $X^{\frac{1+\tilde\zeta}{4}}\times X^{\frac{-1+\tilde\zeta}{4}}$ for every $\tilde\zeta\in[-1,1)$ similarly as in (5.21)). Hence, there will be a subsequence of $\left[\begin{smallmatrix}u^{\alpha_m}_{(k)}\\ v^{\alpha_m}_{(k)}\end{smallmatrix}\right](t_{n_k},\left[\begin{smallmatrix}\phi^{(k)}_m\\ \varphi^{(k)}_m\end{smallmatrix}\right](0))$ convergent in $X\times X^{-\frac{1}{2}}$ and thus in $X^{\frac{1+\tilde\zeta}{4}}\times X^{\frac{-1+\tilde\zeta}{4}}$ for any $\tilde\zeta\in[-1,1)$ to a certain limit point. Observe that this limit point will be in $\mathcal{A}_1$ because we can write $\left[\begin{smallmatrix}u^{\alpha_m}_{(k)}\\ v^{\alpha_m}_{(k)}\end{smallmatrix}\right](t_{n_k},\left[\begin{smallmatrix}\phi^{(k)}_m\\ \varphi^{(k)}_m\end{smallmatrix}\right](0))$ as $\left[\begin{smallmatrix}u^{\alpha_m}_{(k)}\\ v^{\alpha_m}_{(k)}\end{smallmatrix}\right](\frac{t_{n_k}}{2},\left[\begin{smallmatrix}u^{\alpha_m}_{(k)}\\ v^{\alpha_m}_{(k)}\end{smallmatrix}\right](\frac{t_{n_k}}{2},\left[\begin{smallmatrix}\phi^{(k)}_m\\ \varphi^{(k)}_m\end{smallmatrix}\right](0)))$ which is an approximating sequence as required in (5.17). In either case we get that $\{\left[\begin{smallmatrix}\phi_n\\ \varphi_n\end{smallmatrix}\right](t_n)\}$ fails to be separated from $\mathcal{A}_1$ in $X^{\frac{1+\zeta}{4}}\times X^{\frac{-1+\zeta}{4}}$, which is absurd. Hence we have (5.18). $\square$

5.4. **Upper semicontinuity of the dynamics: proofs of (1.15) and (1.16).** We first prove the estimate (1.15).

Theorem 5.6. *There exists a certain $\alpha_0\in(0,1)$ such that for any $\alpha\in[\alpha_0,1)$ Theorem 1.3 applies and the family $\{\mathcal{A}_\alpha\}_{\alpha\in[\alpha_0,1)}$, where $\mathcal{A}_\alpha$ is a global attractor for the semigroup of global solutions to* (3.1) *has the property that*

$$\sup_{\alpha\in[\alpha_0,1)}\ \sup_{\left[\begin{smallmatrix}u_{0\alpha}\\ v_{0\alpha}\end{smallmatrix}\right]\in\mathcal{A}_\alpha}\|\left[\begin{smallmatrix}u_{0\alpha}\\ v_{0\alpha}\end{smallmatrix}\right]\|_{X^{\frac{1+\alpha}{4}}\times X^{\frac{-1+\alpha}{4}}}\leqslant R$$

for some positive constant R.

Proof: Choose any $\alpha_0<1$ close enough to 1. Observe, due to part iv) of Theorem 1.3, that $\mathcal{A}_{\alpha_0}$ is bounded in $X^{\frac{1}{2}}\times X$ which, due to part iv) of Lemma 4.1, ensures that $\mathcal{A}_{\alpha_0}$ is bounded in $X^{\frac{1}{2}}_{\lim}\times X_{\lim}$. Using Lemma 5.2 with $\alpha=\alpha_0$ and with $B_{\lim}(r)$ such that it contains $\mathcal{A}_{\alpha_0}$ we obtain that, for some $t_r>0$, $S_{\alpha_0}(t_r)\mathcal{A}_{\alpha_0}$ is contained in a ball of radius R_0 around zero in $X^{\frac{1+\alpha_0}{4}}\times X^{\frac{-1+\alpha_0}{4}}$. Using now that $S_{\alpha_0}(t_r)\mathcal{A}_{\alpha_0}=\mathcal{A}_{\alpha_0}$ we actually obtain that $\mathcal{A}_0$ is contained in a ball of radius R_0 around zero in $X^{\frac{1+\alpha_0}{4}}\times X^{\frac{-1+\alpha_0}{4}}$. Since this argument applies for each $\alpha_0<1$ close enough to 1, we get the result. $\square$

Remark 5.7. *Note that Theorem 5.6 can be proved independently of Lemma 4.1 using that the semigroup associated to* (3.1) *has a Lyapunov functional and that the equilibria are bounded independently of* $\alpha<1$ *close enough to* 1.

Given a metric space V and compact sets B_1, B_2 in V we now denote

$$d_V(B_1,B_2) := \sup_{b_1\in B_1}\inf_{b_2\in B_2} dist_V(b_1,b_2) \tag{5.23}$$

and prove the upper semicontinuity result in (1.16).

Theorem 5.8. *Assume (1.2)-(1.3) and (1.4).*

Then for each $\zeta\in[-1,1)$ *we have that* $\mathcal{A}_\alpha$ *behaves upper semicontinuously with respect to Hausdorff semidistance* $d_{X^{\frac{1+\zeta}{4}}\times X^{\frac{-1+\zeta}{4}}}$ *as* $\alpha\nearrow 1$*, that is,*

$$\lim_{\alpha\nearrow 1} d_{X^{\frac{1+\zeta}{4}}\times X^{\frac{-1+\zeta}{4}}}(\mathcal{A}_\alpha,\mathcal{A}_1)=0,$$

where $\mathcal{A}_\alpha$ *is a global attractor for* (3.1) *as in Theorem 1.3 and* $\mathcal{A}_1$ *given by (5.17) is an attractor for (1.6) in the sense of Theorem 5.5 iii).*

Proof: Suppose that for some $\zeta\in[-1,1)$ we do not have that $\lim_{\alpha\nearrow 1} d_{X^{\frac{1+\zeta}{4}}\times X^{\frac{-1+\zeta}{4}}}(\mathcal{A}_\alpha,\mathcal{A}_1)=0$.

Then there are sequences $\alpha_n\nearrow 1$ (where $\alpha_n>\zeta$) and $\left[\begin{smallmatrix} a_n\\ b_n\end{smallmatrix}\right]\in\mathcal{A}_{\alpha_n}$ such that the sequence $\{\left[\begin{smallmatrix} a_n\\ b_n\end{smallmatrix}\right]\}$ is separated from $\mathcal{A}_1$ in $X^{\frac{1+\zeta}{4}}\times X^{\frac{-1+\zeta}{4}}$. In this latter space each $\left[\begin{smallmatrix} a_n\\ b_n\end{smallmatrix}\right]$ is, due to invariance of $\mathcal{A}_{\alpha_n}$, a limit as $m\to\infty$ of a sequence $\left\{\left[\begin{smallmatrix} u^{\alpha_n}\\ v^{\alpha_n}\end{smallmatrix}\right]\left(t_m^{(n)},\left[\begin{smallmatrix} w_m^{(n)}\\ z_m^{(n)}\end{smallmatrix}\right]\right)\right\}\subset\mathcal{A}_{\alpha_n}$ where $\left\{\left[\begin{smallmatrix} w_m^{(n)}\\ z_m^{(n)}\end{smallmatrix}\right]\right\}\subset\mathcal{A}_{\alpha_n}$ and $t_m^{(n)}\to\infty$. Thus the distance of $\left[\begin{smallmatrix} a_n\\ b_n\end{smallmatrix}\right]$ from some $\left[\begin{smallmatrix} u^{\alpha_n}\\ v^{\alpha_n}\end{smallmatrix}\right]\left(t_{m_n}^{(n)},\left[\begin{smallmatrix} w_{m_n}^{(n)}\\ z_{m_n}^{(n)}\end{smallmatrix}\right]\right)$ is in $X^{\frac{1+\zeta}{4}}\times X^{\frac{-1+\zeta}{4}}$ less than $\frac{1}{n}$, where $\left[\begin{smallmatrix} w_{m_n}^{(n)}\\ z_{m_n}^{(n)}\end{smallmatrix}\right]\in\mathcal{A}_{\alpha_n}$ and we can choose $\{t_{m_n}^{(n)}\}$ increasing to ∞. Note that if $B_\alpha(r)$ denotes a ball in $X^{\frac{1+\alpha}{4}}\times X^{\frac{-1+\alpha}{4}}$ of radius r around zero then, due to Theorem 5.6, $\left[\begin{smallmatrix} w_{m_n}^{(n)}\\ z_{m_n}^{(n)}\end{smallmatrix}\right]\in B_{\alpha_n}(R)$. Hence, due to Corollary 5.3, $\left[\begin{smallmatrix} u^{\alpha_n}\\ v^{\alpha_n}\end{smallmatrix}\right]\left(\frac{t_{m_n}^{(n)}}{2},\left[\begin{smallmatrix} w_{m_n}^{(n)}\\ z_{m_n}^{(n)}\end{smallmatrix}\right]\right)\in B_{\alpha_n}(R_0)$ for almost all n and $\left[\begin{smallmatrix} u^{\alpha_n}\\ v^{\alpha_n}\end{smallmatrix}\right]\left(t_{m_n}^{(n)},\left[\begin{smallmatrix} w_{m_n}^{(n)}\\ z_{m_n}^{(n)}\end{smallmatrix}\right]\right)\in B_{\alpha_n}(R_0)$ for almost all n as well. Given any $\tilde\zeta\in[-1,1)$ there then exists $N_{\tilde\zeta}\in\mathbb{N}$ such that $\left\{\left[\begin{smallmatrix} u^{\alpha_n}\\ v^{\alpha_n}\end{smallmatrix}\right]\left(t_{m_n}^{(n)},\left[\begin{smallmatrix} w_{m_n}^{(n)}\\ z_{m_n}^{(n)}\end{smallmatrix}\right]\right): n\geqslant N_{\tilde\zeta}\right\}$ is bounded in $X^{\frac{1+\tilde\zeta}{4}}\times X^{\frac{-1+\tilde\zeta}{4}}$. There is thus a subsequence of $\left\{\left[\begin{smallmatrix} u^{\alpha_n}\\ v^{\alpha_n}\end{smallmatrix}\right]\left(t_{m_n}^{(n)},\left[\begin{smallmatrix} w_{m_n}^{(n)}\\ z_{m_n}^{(n)}\end{smallmatrix}\right]\right)\right\}$ convergent in $X\times X^{-\frac{1}{2}}$ to a certain $\left[\begin{smallmatrix} w\\ z\end{smallmatrix}\right]$ and, moreover, from each subsequence of this subsequence we can still choose a subsequence convergent now in $X^{\frac{1+\tilde\zeta}{4}}\times X^{\frac{-1+\tilde\zeta}{4}}$ (thus again to $\left[\begin{smallmatrix} w\\ z\end{smallmatrix}\right]$) where $\tilde\zeta$ can be any number from $[-1,1)$. This proves that a subsequence of $\left\{\left[\begin{smallmatrix} u^{\alpha_n}\\ v^{\alpha_n}\end{smallmatrix}\right]\left(t_{m_n}^{(n)},\left[\begin{smallmatrix} w_{m_n}^{(n)}\\ z_{m_n}^{(n)}\end{smallmatrix}\right]\right)\right\}$ which converges to $\left[\begin{smallmatrix} w\\ z\end{smallmatrix}\right]$ in $X\times X^{-\frac{1}{2}}$ actually converges to $\left[\begin{smallmatrix} w\\ z\end{smallmatrix}\right]$ in $X^{\frac{1+\tilde\zeta}{4}}\times X^{\frac{-1+\tilde\zeta}{4}}$ for each $\tilde\zeta\in[-1,1)$. After repeating the argument as in (5.19)-(5.20) we obtain that $\left[\begin{smallmatrix} w\\ z\end{smallmatrix}\right]\in X_{\lim}^{\frac{1}{2}}\times X_{\lim}$. Then, recalling that $\frac{t_{m_n}}{2}\to\infty$ and $\left[\begin{smallmatrix} u^{\alpha_n}\\ v^{\alpha_n}\end{smallmatrix}\right]\left(\frac{t_{m_n}^{(n)}}{2},\left[\begin{smallmatrix} w_{m_n}^{(n)}\\ z_{m_n}^{(n)}\end{smallmatrix}\right]\right)\in B_{\alpha_n}(R_0)$ we obtain that actually $\left[\begin{smallmatrix} w\\ z\end{smallmatrix}\right]\in\mathcal{A}_1$.

Due to the above argument there is thus a subsequence of $\{[\begin{smallmatrix} a_n \\ b_n \end{smallmatrix}]\}$ which converges to $[\begin{smallmatrix} w \\ z \end{smallmatrix}] \in \mathcal{A}_1$ in $X^{\frac{1+\zeta}{4}} \times X^{\frac{-1+\zeta}{4}}$, which now contradicts that $\{[\begin{smallmatrix} a_n \\ b_n \end{smallmatrix}]\}$ is separated from $\mathcal{A}_1$ in $X^{\frac{1+\zeta}{4}} \times X^{\frac{-1+\zeta}{4}}$.

Hence we have that $\lim_{\alpha \nearrow 1} d_{X^{\frac{1+\zeta}{4}} \times X^{\frac{-1+\zeta}{4}}}(\mathcal{A}_\alpha, \mathcal{A}_1) = 0$. □

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(F. D. M. Bezerra) Universidade Federal da Paraíba, Departamento de Matemática, 58051-900 João Pessoa PB, Brazil.
E-mail address: `flank@mat.ufpb.br`

(A. N. Carvalho) Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo-Campus de São Carlos, Caixa Postal 668, 13560-970 São Carlos SP, Brazil.
E-mail address: `andcarva@icmc.usp.br`

(J. W. Cholewa) Institute of Mathematics, Silesian University, 40-007 Katowice, Poland
E-mail address: `jan.cholewa@us.edu.pl`

(M. J. D. Nascimento) Universidade Federal de São Carlos, Departamento de Matemática, 13565-905 São Carlos SP, Brazil.
E-mail address: `marcelo@dm.ufscar.br`