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POLYNOMIAL INTEGRABILITY OF HAMILTONIAN SYSTEMS WITH HOMOGENEOUS POTENTIALS OF DEGREE –K

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POLYNOMIAL INTEGRABILITY OF HAMILTONIAN SYSTEMS WITH HOMOGENEOUS POTENTIALS OF DEGREE -K

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Abstract. In this paper we shall answer positively two open problems proposed by Llibre-Mahdi-Valls in Physica D, 240 (2011). More precisely, we characterize the polynomial integrability of Hamiltonian system with potentials given by the inverse of a homogeneous potential of degree k.

1. Introduction and statement of the main results

The question whether a differential model admits a first integral is of fundamental importance as first integrals give conservation laws for the models and enable us to lower the dimension of the system. Moreover, knowing a sufficient number of first integrals allows to solve the system explicitly. Until the end of 19th century the majority of scientists thought that the equations of classical mechanics were integrable and finding the first integrals was mainly a problem of computation. In fact integrability is a rare phenomena and in general it is very hard to determine whether a given Hamiltonian system is integrable or not.

Denote by $\mathbb{C}^4$ the symplectic linear space with canonical variables $q = (q_1, q_2)$ and $p = (p_1, p_2)$. Consider the Hamiltonian systems defined by the Hamiltonian function

$$H = \frac{1}{2}(p_1^2 + p_2^2) + V(q_1, q_2),$$

where V is a homogeneous function of degree k and its associated Hamiltonian system

$$\begin{aligned} \dot{q}_1 &= p_1, \\ \dot{q}_2 &= p_2, \\ \dot{p}_1 &= -\partial V/\partial q_1, \\ \dot{p}_2 &= -\partial V/\partial q_2 \end{aligned} \tag{1}$$

Let $H = H(q, p)$ and $F = F(q, p)$ be two functions. We define the Poisson bracket $\{H, F\}$ of H and F by

$$\{H, F\} = \sum_{i=1}^{2} \left(\frac{\partial H}{\partial q_i}\frac{\partial F}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial F}{\partial q_i} \right).$$

The functions H and F are in involution if $\{H, F\} = 0$. Moreover, a non-constant function $G = G(q, p)$ is a first integral of system (1) if $\{H, G\} = 0$. System (1) is completely or Liouville integrable if it has additionally to H, a first integral G, which is functionally independent and in involution with H. We recall that H and G are

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functionally independent if their gradients are linearly independent at all points of $\mathbb{C}^4$ except perhaps in a zero Lebesgue set.

Let $PO_2(\mathbb{C})$ be the group of 2 complex matrices A such that $AA^T = \alpha I$, where I is the identity matrix and $\alpha \in \mathbb{C} \setminus \{0\}$. Two potentials $V_1(q)$ and $V_2(q)$ are equivalent if there exists a matrix $A \in PO_2(\mathbb{C})$ such that $V_1(q) = V_2(Aq)$. Guided by this we divide all potentials into equivalent classes, so in what follows a potential denotes a class of equivalent potentials in the above sense. This definition of equivalent potential is motivated by the following simple result (for a proof see [5]): Let V_1 and V_2 be two equivalent potentials. If the Hamiltonian system (1) is integrable with potential V_1 then it is also integrable with V_2.

In the 80's all integrable Hamiltonian systems (1) with homogeneous polynomial potential of degree at most five and having a second polynomial first integral up to degree four in the variables p_1 and p_2 were computed (see [1, 2, 3, 4, 6, 13]). Recently it has been proved that the meromorphic integrability is related to properties of the monodromy group of the differential Galois group of variational equations along a particular solution (see [12] who gave the strongest necessary conditions for the complete integrability of the Hamiltonian systems (1) with a homogeneous potential). These are very powerful techniques that have not been very used till now in the mathematical community. In [10], Maciejewski and Przybylska gave the necessary and sufficient conditions for the complete integrability of Hamiltonian systems (1) with homogeneous polynomial potential of degree 3, using the above mentioned techniques. The case of homogeneous potential of degree 4 was completely characterized by Maciejewski and Przybylska in [11] and Llibre, Mahdi and Valls in [7]. It is well known, see for instance [8], that if the potential V is a homogeneous polynomial or an inverse of a homogeneous polynomial of degree $2, 1, 0$ or -1 then the system is always completely integrable. The case of a homogeneous polynomial of degree -2 and -3 was characterized in [8] and [9], respectively.

There are many results concerning Hamiltonian systems with homogeneous polynomial potentials but very fews is known about potentials not being polynomials.

In this paper we will focus on the characterization of the Hamiltonian system (1) with polynomial potential of degree $-k$, $k \geq 2$, which are completely integrable.

In what follows we say that the Hamiltonian system (1) is not polynomial integrable if it does not admit a polynomial first integral linearly independend of the Hamiltonian H. Our main result characterizes the existence or non existence of polynomial first integrals of the Hamiltonian system (1) with a homogeneous potential of degree $-k$ of the form

$$V = \frac{1}{\sum_{i=0}^{k} a_i q_1^{k-i} q_2^i} = \frac{1}{a_0 q_1^k + a_1 q_1^{k-1} q_2 + ... + a_{k-1} q_1 q_2^{k-1} + a_k q_2^k}, \tag{2}$$

with $a_i \in \mathbb{C},$ for $i = 1, ...k$. More precisely, we prove the following result

Theorem 1. *The Hamiltonian system* (1) *with homogeneous potential* (2) *is polynomially integrable if and only if V is equivalent to one the following potentials with the corresponding additional polynomial first integral G.*

i) $V = 1/(q_2 - iq_1)^k$, $G = p_1 + ip_2$,
ii) $V = 1/(q_2 + iq_1)^k$, $G = p_1 - ip_2$,
iii) $V = 1/(q_2^2 + q_1^2)^{\frac{k}{2}}$, k *even,* $G = q_2 p_1 + q_1 p_2$,
iv) $V = 1/q_1^k$, $G = p_1^2 + 2/q_1^k$.

Theorem 1 is proved in Section 3. In particular, Theorem 1 gives positive answers to two open problems proposed in [9]:

Open problem 1: Consider the Hamiltonian system (1) where $a_{k-1} = 0$ and $k \geq 2$. Show that if $a_1 ... a_{k-2} \neq 0$, then system (1) does not admit polynomial first integral.

Open problem 2: Consider the Hamiltonian system (1) with the potential (2) $k \geq 1$. Prove or disprove that system (1) is completely integrable with an additional polynomial first integral if and only if the potential $V(q)$ is equivalent to one of the following potentials $V = f(q_1)+g(q_2)$, $V = f(aq_1+bq_2)$ or $V = f(q_1^2+q_2^2)$, where $f : \mathbb{C} \to \mathbb{C}$ is any differential function.

2. Preliminary results

In this section we recall some definitions and known results about integrability.

A non–zero point $d \in \mathbb{C}$ is a *Darboux point* of the Hamiltonian system (1) if it is a solution of

$$V'(d) = \gamma d,$$

where $\gamma \in \mathbb{C} \setminus \{0\}$ and $V'(d)$ denotes the gradient of $V(d)$. Observe that, if d is a Darboux point then

$$q(t) = \phi(t)d, \quad p(t) = \dot{\phi}(t)d$$

is a solution of system (1) and $\phi(t)$ satisfies $\ddot{\phi} = \gamma\phi^{k-1}$. So a Darboux point can be considered as a point $[d_1 : d_2]$ of the projective space $\mathbb{CP}^1$.

Setting $z = q_2/q_1$, $q_1 \neq 0$, consider the linear transformation $q \to Aq \in PO_2(\mathbb{C})$. This transformation induces a action of $PO_2(\mathbb{C})$ on the space $\mathbb{CP}^1$ defined in the following way. For each $A \in PO_2(\mathbb{C})$ given by

$$A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}, \quad a^2 + b^2 \neq 0,$$

consider the map $\tau_A : \mathbb{CP}^1 \to \mathbb{CP}^1$ given by $\tau_A([q_1 : q_2]) = [aq_1 + bq_2 : -bq_1 + aq_2]$. Thus in the affine part of $\mathbb{CP}^1$ this action is

$$\tau_A(z) = \frac{az - b}{bz + a}.$$

The next proposition presents a basic property of this action.

Proposition 2. *For $p, \tilde{p} \in \mathbb{CP}^1 \setminus \{[1 : i], [1 : -i]\}$ there exists $A \in PO_2(\mathbb{C})$ such that $\tilde{p} = \tau_A(p)$.*

Given the potential $F = 1/V$, where V is a homogeneous polynomial of degree $k > 0$, we define $v(z) = V(1, z)$, that is a nonzero polynomial of degree not greater than k. Hence we write

$$v(z) = \sum_{i=0}^{k} v_{k-i} z^i.$$

Observe that if V_A is equivalent to V, i.e. $V_A(q) = V(Aq)$ for some $A \in PO_2(\mathbb{C})$ then $v_A(z) = V_A(1, z)$ is given by

$$v_A(z) = (a + bz)^k v(\tau_A(z)).$$

From the above considerations follows that if z^* is a root of the polynomial v then $\tau_A(z^*)$ is a root of the polynomial v_A.

Finally, by the above properties and Proposition 2 we can choose among all equivalent potentials such a representative V, for which the polynomial v has one root in an arbitrary point of $\mathbb{CP}^1 \setminus \{[1:i],[1:-i]\}$. This is always possible except for the cases when all the linear factors of V have the form $(q_2 \pm iq_1)$, i.e., when the potential V has the following form

$$V = \alpha(q_2 - iq_1)^l (q_2 + iq_1)^{k-l}, \quad l = 0, ..., k, \quad \alpha \in \mathbb{C}^*. \tag{3}$$

We call them *exceptional potentials.*

The next result guarantee that we can assume that $v_k = 0$ in (2) wherever V is not a exceptional potential.

Proposition 3. *Assume that a homogeneous potential of degree k is not exceptional, then it is equivalent to $V = q_1\tilde{V}$, where $\tilde{V}$ is a homogeneous polynomial of degree $k-1$.*

As we can normalize a non-zero coefficient of the potential, in an equivalent class of potentials we have at most $k-1$ parameters.

The potential $F = 1/V$, with V not exceptional, it is equivalent to $\tilde{F} = 1/()q_1\tilde{V})$, where $\tilde{V}$ is a homogeneous polynomial of degree $k-1$.

Denote by $\{\mu, \lambda\}$ the two eigenvalues of the Hessian matrix of V evaluated at the Darboux point d

$$V''(d) = \begin{pmatrix} \dfrac{\partial^2 V(d)}{\partial q_1^2} & \dfrac{\partial^2 V(d)}{\partial q_1 \partial q_2} \\ \dfrac{\partial^2 V(d)}{\partial q_2 \partial q_1} & \dfrac{\partial^2 V(d)}{\partial q_2^2} \end{pmatrix}.$$

As V is a homogeneous function of degree k, one of the two eigenvalues, say μ, is equal to $(k-1)$, we call it the *trivial eigenvalue.* The second eigenvalue λ is the *non-trivial eigenvalue.*

Let V be a homogeneous potential of degree k of the Hamiltonian system (1). If system (1) is completely integrable then their eigenvalues are not arbitrary, they are given by the following result, due to Morales and Ramis [12].

Theorem 4 ([12]). *Assume that a Hamiltonian system* (1) *with a homogeneous potential given by* (2) *of degree $k \in \mathbb{Z}\setminus\{0\}$ is completely integrable with holomorphic or meromorphic first integrals, then the non-trivial eigenvalue λ satisfies the conditions given in Table 1, where p is an integer.*

Now setting $z = q_2/q_1$ we have that the potential V of system (1) given in (2) can be written as

$$V = \frac{1}{q_1^k v(z)}, \qquad \text{where } v(z) = a_0 + a_1 z + ... + a_k z^k.$$

Consider the polynomials

$$h(z) = kv(z) - zv'(z) \quad \text{ and } \quad g(z) = (1+z^2)v'(z) - kzv(z),$$

where $v'(z)$ denotes the derivative if $v(z)$ with respect to z.

Next proposition shows how to compute each Darboux points d associated to system (1) with potential V and the non-trivial eigenvalue λ.

Degree	*Eigenvalue* λ	*Degree*	*Eigenvalue* λ
k	$p+p\,(p-1)\,\dfrac{k}{2}$	-3	$\dfrac{25}{24}-\dfrac{1}{24}\left(\dfrac{12}{5}+6p\right)^2$
2	*arbitrary* $z\in\mathbb{C}$	3	$-\dfrac{1}{24}+\dfrac{1}{24}\,(2+6p)^2$
-2	*arbitrary* $z\in\mathbb{C}$	3	$-\dfrac{1}{24}+\dfrac{1}{24}\left(\dfrac{3}{2}+6p\right)^2$
-5	$\dfrac{49}{40}-\dfrac{1}{40}\left(\dfrac{10}{3}+10p\right)^2$	3	$-\dfrac{1}{24}+\dfrac{1}{24}\left(\dfrac{6}{5}+6p\right)^2$
-5	$\dfrac{49}{40}-\dfrac{1}{40}\,(4+10p)^2$	3	$-\dfrac{1}{24}+\dfrac{1}{24}\left(\dfrac{12}{5}+6p\right)^2$
-4	$\dfrac{9}{8}-\dfrac{1}{8}\left(\dfrac{4}{3}+4p\right)^2$	4	$-\dfrac{1}{8}+\dfrac{1}{8}\left(\dfrac{4}{3}+4p\right)^2$
-3	$\dfrac{25}{24}-\dfrac{1}{24}\,(2+6p)^2$	5	$-\dfrac{9}{40}+\dfrac{1}{40}\left(\dfrac{10}{3}+10p\right)^2$
-3	$\dfrac{25}{24}-\dfrac{1}{24}\left(\dfrac{3}{2}+6p\right)^2$	5	$-\dfrac{9}{40}+\dfrac{1}{40}\,(4+10p)^2$
-3	$\dfrac{25}{24}-\dfrac{1}{24}\left(\dfrac{6}{5}+6p\right)^2$	k	$\dfrac{1}{2}\left(\dfrac{k-1}{k}+p\,(p+1)\,k\right).$

TABLE 1. Conditions on the non trivial eigenvalue of Hamiltonian systems

Proposition 5. [9] *Assume that* $g\not\equiv 0$*. The Darboux points* $d=[1:z^*]$ *associated to the potential* (2) *are given by the zeros of* $g(z)=0$ *which are not zeros of* h*. Moreover, the non-trivial eigenvalue of* $V''(d)$ *is given by* $\lambda(z^*)=g'(z^*)/h(z^*)+1$*.*

Another useful result about integrability of Hamiltonian systems (1) is given by the following proposition.

Proposition 6. [9] *Hamiltonian systems* (1) *having a potential* V *given below is completely integrable with the corresponding additional first integral* G*:*

$$\begin{array}{ll} V=f(q_1)+g(q_2), & G=p_1^2+2f(q_1),\\ V=f(aq_1+bq_2), & G=bp_1-aq_2,\\ V=f(q_1^2+q_2^2), & G=q_2p_1-q_1p_2, \end{array}$$

where $f:\mathbb{C}\to\mathbb{C}$ *is any differential function.*

3. PROOF OF THE MAIN THEOREM

In this section we shall prove the main result of this paper, Theorem 1. The proof of this result will be splitted in lemmas. Firstly we consider the expectional potentials. Then we study of the non expectional potentials.

Lemma 7. *Hamiltonian systems* (1) *having a potential* V *given below are completely integrable with the given corresponding additional polynomial first integral* G*:*

i) $V = 1/(q_2 - iq_1)^k$, $G = p_1 + ip_2$,
ii) $V = 1/(q_2 + iq_1)^k$, $G = p_1 - ip_2$,
iii) $V = 1/(q_2^2 + q_1^2)^{\frac{k}{2}}$ *(k even)*, $G = q_2p_1 + q_1p_2$,
iv) $V = 1/q_1^k$, $G = p_1^2 + 2/q_1^k$.

Proof. We obtain the corresponding first integral G from Proposition 6. Then it is a straightforward computation to check that G and H are functionally indepent first integrals. □

Lemma 8. *Hamiltonian systems* (1) *having a potential* (3) *with* $l \neq 0, k$ *or* $l \neq k/2$*, when* k *is even, do not admit an additional polynomial first integral.*

Proof. Consider system (1) with V given by (3) with $l \neq 0, k$ or $l \neq k/2$, when k is even. Then the non–trivial eigenvalue λ of V'' must satisfy Proposition 5 where

$$\begin{aligned} v =& \alpha(z+i)(z-i)^{k-1}, \\ h(z) =& \alpha(z-i)^{k-2}(k+2iz-ikz), \\ g(z) =& -\alpha(k-2)(z-i)^k. \end{aligned}$$

But if $\alpha \neq 0$, the unique solution of $g(z) = 0$ is $z = -i$ and $h(-i) = -2^{k-1}(-i)^k \neq 0$. Moreover $g'(-i)/h(-i) + 1 = k - 1$.

Now consider the sets

$$\begin{aligned} \mathcal{Z}_k^1 &= \left\{ p - p(p-1)\frac{k}{2} : p \text{ is an integer, } k \geq 4 \right\}, \\ \mathcal{Z}_{-5}^1 &= \left\{ \frac{49}{40} - \frac{1}{40}\left(\frac{10}{3} + 10p\right)^2 : p \text{ is an integer} \right\}, \\ \mathcal{Z}_{-5}^2 &= \left\{ \frac{49}{40} - \frac{1}{40}(4+10p)^2 : p \text{ is an integer} \right\}, \\ \mathcal{Z}_{-4}^1 &= \left\{ \frac{9}{8} - \frac{1}{8}\left(\frac{4}{3} + 4p\right)^2 : p \text{ is an integer} \right\}, \\ \mathcal{Z}_k^2 &= \left\{ \frac{1}{2}\left(\frac{k-1}{k} + p(p+1)k\right) : p \text{ is an integer, } k \geq 4 \right\}. \end{aligned}$$

A simple computation shows that $\lambda = k - 1 \notin \{\mathcal{Z}_k^1 \cup \mathcal{Z}_{-5}^1 \cup \mathcal{Z}_{-5}^2 \cup \mathcal{Z}_{-5}^2 \cup \mathcal{Z}_{-4}^1 \cup \mathcal{Z}_k^2\}$. Therefore, from Theorem 4, Hamiltonian systems with potential V given by (3) with $l \neq 0, k$ or $l \neq k/2$, when k is even, do not admit a polynomial first integral. □

Lemma 9. *Hamiltonian system* (1) *with a homogeneous potential of degree* $k \geq 2$ *that is not exceptional does not admit a polynomial first integral, except if* $V = 1/q_1^k$ *whose additional polynomial first integral is* $p_12 + 1/q_1k$.

Proof. If V is homogeneous potential of a Hamiltonian system (1) and it is not exceptional then, it follows from Proposition 3 that there exists $A \in PO_2(\mathbb{C})$, such that V is equivalent to

$$\tilde{V} = \frac{1}{q_1(a_0q_1^{k-1} + a_1q_1^{k-2}q_2 + ... + a_{k-1}q_1q_2^{k-2})}.$$

We shall prove, using induction under the number of non–zero coefficients of $\tilde{V}$, that this potential does not admite a polynomial first integral, except if $a_0 \neq 0$ and $a_1 = a_2 = \ldots = a_{k-1} = 0$.

Firstly, assume that $a_{k-1} \neq 0$. Then, without lost of generality, we can assume $a_{k-1} = 1$. In this case, system (1) is given by

$$
\begin{aligned}
\dot{q_1} = &\ \ p_1\, q_1^2 \left(\textstyle\sum_{i=0}^{k-1} a_i q_1^{k-i-1} q_2^i\right)^2 \\
\dot{q_2} = &\ \ p_2\, q_1^2 \left(\textstyle\sum_{i=0}^{k-1} a_i q_1^{k-1-i} q_2^i\right)^2 \\
\dot{p_1} = &\ \ \textstyle\sum_{i=0}^{k-1} a_i q_1^{k-1-i} q_2^i + q_1 \sum_{i=0}^{k-i-1} a_i q_1^{k-2-i} q_2^i \\
\dot{p_2} = &\ \ q_1 \textstyle\sum_{i=0}^{k-1} i a_i q_1^{k-1-i} q_2^{i-1}.
\end{aligned} \tag{4}
$$

Assuming that system (4) admits a first integral G (functionally independent of H), then it must satisfy

$$
\begin{aligned}
&\frac{\partial G}{\partial p_1}\Big(\sum_{i=0}^{k-1} a_i q_1^{k-1-i} q_2^i + q_1 \sum_{i=0}^{k-i-1} a_i q_1^{k-2-i} q_2^i\Big) + \\
&\qquad + \frac{\partial G}{\partial p_2} q_1 \sum_{i=0}^{k-1} i a_i q_1^{k-1-i} q_2^{i-1} + \left(\frac{\partial G}{\partial q_1} p_1 + \frac{\partial G}{\partial q_2} p_2\right) q_1^2 \left(\sum_{i=0}^{k-1} a_i q_1^{k-i-1} q_2^i\right)^2 = 0.
\end{aligned} \tag{5}
$$

Write $G = \sum_{k=0}^{n} G_j$, where G_j are homogeneous polynomials of degree j and $G_n \neq 0$. We shall prove by induction with respect to k that G_n is constant. The terms of degree $(n+2k)$ in the variables q_1, q_2, p_1 and p_2 in (5) (higher order terms) must satisfy

$$
q_1^2 \left(\frac{\partial G_n}{\partial q_1} p_1 + \frac{\partial G_n}{\partial q_2} p_2\right) \left(\sum_{i=0}^{k-1} a_i q_1^{k-i-1} q_2^i\right)^2 = 0.
$$

So, we have that

$$
G_n(q_1, q_2, p_1, p_2) = \sum_{i+j+2l=0}^{n} b_{ijl}\, p_1^i\, p_2^j\, (q_1 p_2 - q_2 p_1)^l. \tag{6}
$$

Note that the same happens with terms of degree $n + 2k - r$ in (5), for $1 \leq r \leq k+1$. When $r = k+2$ we have

$$
\begin{aligned}
&\left(\frac{\partial G_{n-k-2}}{\partial q_1} p_1 + \frac{\partial G_{n-k-2}}{\partial q_2} p_2\right) q_1^2 \left(\sum_{i=0}^{k-1} a_i q_1^{k-i-1} q_2^i\right)^2 + \\
&\frac{\partial G_n}{\partial p_1}\left(\sum_{i=0}^{k-1} a_i q_1^{k-1-i} q_2^i + q_1 \sum_{i=0}^{k-i-1} a_i q_1^{k-2-i} q_2^i\right) + \frac{\partial G_n}{\partial p_2} q_1 \sum_{i=0}^{k-1} i a_i q_1^{k-1-i} q_2^{i-1} = 0.
\end{aligned} \tag{7}
$$

Evaluating (7) on $q_1 = 0$, we conclude that

$$
\frac{\partial G_n}{\partial p_1}\bigg|_{q_1=0} = 0. \tag{8}
$$

From (6) we get that $\left.\dfrac{\partial G_n}{\partial p_1}\right|_{q_1=0} = -\displaystyle\sum_{i+j+2l=0}^{n} (i+l) b_{ijl}\, p_1^{i+l-1}\, p_2^j\, q_2^l$. Therefore (8) is true if and only if $i=l=0$, which implies that $G_n = G_n(p_2)$.

If $G_n = G_n(p_2)$ in (7) then

$$\text{(9)} \qquad q_1\left(\frac{\partial G_{n-k-2}}{\partial q_1} + \frac{\partial G_{n-k-2}}{\partial q_2} p_2\right)\left(\sum_{i=0}^{k-1} a_i q_1^{k-i-1} q_2^i\right)^2 + \frac{\partial G_n}{\partial p_2}\sum_{i=0}^{k-1} i a_i q_1^{k-1-i} q_2^{i-1} = 0.$$

Evaluating (9) on $q_1 = 0$ we get $q_2^{k-2}(k-1)G_n' = 0$, where G_n' denotes the derivative of G_n with respect to the variable p_2. Therefore G_n is constant, which contradicts the hypotheses of G being a polynomial first integral of system (4). So, in the case, where $a_{k-1} \neq 0$, the potential V does not admit any additional polynomial first integral.

Now suppose that $a_{k-l} = 0$ for $l = 1, 2, .., r-1$ and $a_{k-r} \neq 0$. So, without lost of generality we can make $a_{k-r} = 1$. In this case we have

$$\tilde{V} = \frac{1}{q_1^l \sum_{i=0}^{k-l} a_i q_1^{k-l-i} q_2^i}.$$

Under these conditions system (1) is equivalent to the system below after a reparametrization

$$\text{(10)} \qquad \begin{aligned} \dot{q_1} &= p_1\, q_1^l (\textstyle\sum_{i=0}^{k-l} a_i q_1^{k-l-1} q_2^i)^2 \\ \dot{q_2} &= p_2\, q_1^l (\textstyle\sum_{i=0}^{k-l} a_i q_1^{k-l-i} q_2^i)^2 \\ \dot{p_1} &= l \textstyle\sum_{i=0}^{k-l} a_i q_1^{k-1-i} q_2^i + x \sum_{i=0}^{k-l-i} a_i q_1^{k-l-i-1} q_2^i \\ \dot{p_2} &= q_1 \textstyle\sum_{i=0}^{k-1} i a_i q_1^{k-l-i} q_2^{i-1}. \end{aligned}$$

Assuming that system (10) admits a first integral G (functionally independly of H), then it must satisfy

$$\text{(11)} \qquad \begin{aligned} &\frac{\partial G}{\partial p_1}(\sum_{i=0}^{k-l}(k-i) a_i q_1^{k-l-i} q_2^i + \frac{\partial G}{\partial p_2}\sum_{i=0}^{k-l} i a_i q_1^{k-l-i} q_2^{i-1} + \\ &\qquad + (\frac{\partial G}{\partial q_1} p_1 + \frac{\partial G}{\partial q_2} p_2) q_1^2 (\sum_{i=0}^{k-l} a_i q_1^{k-i-l} q_2^i)^2 = 0. \end{aligned}$$

If $G = \displaystyle\sum_{k=0}^{n} G_j$, where G_j are homogeneous polynomials of degree j and $G_n \neq 0$ the righer order terms in (11) are terms of degree $(n + 2k - 2l + 2)$ in the variables q_1, q_2, p_1, p_2. So we have the following partial differential equation

$$q_1^2\left(\frac{\partial G_n}{\partial q_1} p_1 + \frac{\partial G_n}{\partial q_2} p_2\right)\left(\sum_{i=0}^{k-l} a_i q_1^{k-i-l} q_2^i\right)^2 = 0,$$

which implies that

$$\text{(12)} \qquad G_n(q_1, q_2, p_1, p_2) = \sum_{i+j+2l=0}^{n} d_{ijl}\, p_1^i\, p_2^j\, (q_1 p_2 - q_2 p_1)^l.$$

The same happens with the terms of degree $n+2k-2l+2-r$ of (4), for $1 \le r \le k-l+2$. When $r = k-l+3$ we have the following partial differential equation

$$q_1^{l+1}\left(\frac{\partial G_{n+k-l-1}}{\partial q_1}p_1 + \frac{\partial G_{n+k-l-1}}{\partial q_2}p_2\right) q_1^2 \left(\sum_{i=0}^{k-l} a_i q_1^{k-i-l} q_2^i\right)^2 + \frac{\partial G_n}{\partial p_1}\left(\sum_{i=0}^{k-l}(k-i)a_i q_1^{k-l-i}q_2^i + q_1\frac{\partial G_n}{\partial p_2} q_1 \sum_{i=0}^{k-l} i a_i q_1^{k-l-i} q_2^{i-1}\right) = 0. \tag{13}$$

Evaluating (13) on $q_1 = 0$, we conclude that

$$\left.\frac{\partial G_n}{\partial p_1}\right|_{q_1=0} = 0. \tag{14}$$

But from (12) we get that $\left.\frac{\partial G_n}{\partial p_1}\right|_{q_1=0} = -\sum_{i+j+2l=0}^{n}(i+l)b_{ijl}p_1^{i+l-1}p_2^j q_2^l$. So (14) happens if only if $i = l = 0$, which implies that $G_n = G_n(p_2)$.

If $G_n = G_n(p_2)$ in (13) we get that

$$q_1\left(\frac{\partial G_{n+k-l-1}}{\partial q_1} + \frac{\partial G_{n+k-l-1}}{\partial q_2}p_2\right)\left(\sum_{i=0}^{k-l} a_i q_1^{k-i-l}q_2^i\right)^2 + \frac{\partial G_n}{\partial p_2}\sum_{i=0}^{k-l} i a_i q_1^{k-l-i}q_2^{i-1} = 0. \tag{15}$$

Evaluating (15) in $q_1 = 0$ we obtain that $q_2^{k-l-1}(k-l)G_n' = 0$, where G_n' denotes the derivative of G_n with respect to the variable p_2. Therefore G_n is constant, which contradicts the hypotheses of G being a first integral. So, in the case, where $a_{k-r} \neq 0$ and $a_{k-i} = 0$ for each $i = 1, 2, ..., k-1$ the potential V does not admit an additional polynomial first integral.

We observe that if $r = k$ then $a_0 \neq 0$ is the unique non–zero coefficient of V, i.e. $V = 1/q_1^k$. This potential admits an additional first integral $G = p_1^2 + 1/q_1^k$, as considered in item *iv)* of Lemma 7. □

Proof of Theorem 1. The result follows from Lemmas 7, 8 and 9. □

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