Second order geometry of spacelike surfaces in de Sitter 5-space

Masaki Kasedou, Ana Claudia Nabarro and Maria Aparecida Soares Ruas

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Abstract

The de Sitter space is known as a Lorentz space with positive constant curvature in the Minkowski space. A surface with a Riemannian metric is called a spacelike surface. In this work we investigate properties of the second fundamental geometry of spacelike surfaces in de Sitter space S_1^5 by using the action of $GL(2,\mathbb{R}) \times SO(1,2)$ on the system of conics defined by the second fundamental form. The main results are the classification of the second fundamental mapping and the description of the possible configurations of the LMN-ellipse. This ellipse gives information on the lightlike binormal directions and consequently about their associated asymptotic directions.

Key words: spacelike surface, de Sitter 5-space, second order geometry, asymptotic directions, lightlike binormal directions.
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1 Introduction

We investigate properties of the second fundamental form, with respect to lightlike normals, of spacelike surfaces in de Sitter 5-space. In the Euclidean case, Little in [9] described elements of the local second order geometry of surfaces, such as asymptotic directions and inflection points, in terms of invariants of a classical object: the curvature ellipse. The classification of the second fundamental form also appears as an useful device in the investigation of the singularities of the 5-web of the asymptotic lines of surfaces in $\mathbb{R}^5$, given by Romero-Fuster, Ruas and Tari in [12]. For surfaces in Minkowski space $\mathbb{R}^4_1$, Izumiya, Pei and Romero-Fuster introduce the concept of curvature ellipse in [4] and study geometric properties of spacelike surfaces in Minkowski 4-space in terms of properties of this ellipse. In [3], they study properties of the curvature ellipse in spacelike surfaces in Minkowski $(n+1)$-space, $\mathbb{R}^{n+1}_1$, $n \geq 2$ (using isothermal coordinates), and by using this setting they obtain geometric characterization for a spacelike surface M to be contained in hyperbolic n-space (de Sitter n-space or n-dimensional lightcone), $n = 3, 4$. This characterization is given in terms of the umbilicity of theses surfaces with respect to some normal field.

Izumiya, Romero-Fuster and Pei study spacelike surfaces in hyperbolic 4-space (see [5]). For other recent results on spacelike surfaces in de Sitter space see [6], [7] and [8].

Bayard and Sánchez-Bringas in [1], study the complete invariants of a quadratic map $\mathbb{R}^2 \longrightarrow \mathbb{R}^2_1$, under the actions of $SO(2)^+ \times SO(1,1)^+$, where $SO(2)^+$ and $SO(1,1)^+$ are the connected components of the identity of the Euclidean and Lorentzian groups of $\mathbb{R}^2$ and $\mathbb{R}^2_1$ respectively. The invariants of these actions have been applied in the classification of the configurations of curvature ellipses of spacelike surfaces in Minkowski 4-space.

We continue in this paper the investigation of properties of the second order geometry in the case of spacelike surfaces in de Sitter space S^5_1 by using the actions $SO(1,2)$ and $GL(2,\mathbb{R})$ on the system of conics defined by the second fundamental form. We define an ellipse in affine space that we call $\underline{LMN\text{-ellipse}}$. This ellipse gives informations on the lightlike binormal directions and consequently about their associated asymptotic directions. The main results are the classification of the second fundamental mapping and the description of the possible configurations of the $\underline{LMN\text{-ellipse}}$.

The paper is organized as follows. In section 2, we give the basic concepts on extrinsic geometry of spacelike submanifolds in the de Sitter n-space in $\mathbb{R}^{n+1}_1$. In section 3, we introduce some invariants (which are preserved by Lorentzian transformations) of the second order geometry of the spacelike surface in the de Sitter 5-space. In sections 4 and 5, we obtain the normal forms of the matrix α of the second fundamental form, under the action of $GL(2,\mathbb{R}) \times SO(1,2)$, when the rank of α is 1 or 2, respectively. We also study the number of lightlike binormal and their associated asymptotic directions in each case. In section 6, we obtain normal forms of α on 4 parameters, when rank $\alpha = 3$. In section 7, we obtain the equation of asymptotic directions at a non lightlike inflection and non conic point. This is a fourth order equation, then at each non conic and non lightlike inflection point there are at most 4 asymptotic directions associated to the lightlike binormals directions. In section 8, when rank $\alpha = 3$ we study the number of lightlike binormal directions and consequently the number of their associated asymptotic directions. We give some examples of spacelike surfaces and lightlike binormals and their associated asymptotic directions. Section 9 contains the appendix A, in which we define trigonometric polynomials representing the equation of the binormal directions and discuss their solutions.

2 Preliminaries

In this section we review basic notions of spacelike surfaces in de Sitter space. Let $\mathbb{R}^{n+1} = \{\mathbf{x} = (x_0, \ldots, x_n) \mid x_i \in \mathbb{R}\ (i = 0, \ldots, n)\}$ be an $(n+1)$-dimensional vector space. For any vectors $\mathbf{x} = (x_0, \ldots, x_n)$ and $\mathbf{y} = (y_0, \ldots, y_n)$ in $\mathbb{R}^{n+1}$, the *pseudo scalar product* of $\mathbf{x}$ and $\mathbf{y}$ is defined by $\langle \mathbf{x}, \mathbf{y} \rangle = -x_0y_0 + \sum_{i=1}^n x_iy_i$. We call $(\mathbb{R}^{n+1}, \langle , \rangle)$ a *Minkowski* $(n+1)$*-space* and write $\mathbb{R}^{n+1}_1$ instead of $(\mathbb{R}^{n+1}, \langle , \rangle)$. A vector $\mathbf{x} \in \mathbb{R}^{n+1}_1 \setminus \{\mathbf{0}\}$ is *spacelike*, *timelike* or *lightlike* if $\langle \mathbf{x}, \mathbf{x} \rangle$ is positive, negative or equal to zero, respectively. The norm of the vector $\mathbf{x}$ is defined by $||\mathbf{x}|| = \sqrt{|\langle \mathbf{x}, \mathbf{x} \rangle|}$.

We define the *de Sitter n-space* by $S^n_1 = \{\mathbf{x} \in \mathbb{R}^{n+1}_1 \mid \langle \mathbf{x}, \mathbf{x} \rangle = 1\}$. A *Lightcone* in $\mathbb{R}^{n+1}_1$ is the set $LC^* = \{\mathbf{x} \in \mathbb{R}^{n+1}_1 \setminus \{\mathbf{0}\} \mid \langle \mathbf{x}, \mathbf{x} \rangle = 0\}$. Let $U \subset \mathbb{R}^2$ be an open subset and

$\mathbf{X} : U \longrightarrow \mathbb{R}^{n+1}_1$ an embedding map. We write $M = \mathbf{X}(U)$. We say that M is a spacelike surface if any tangent vector $\mathbf{v} \neq 0$ at each point on $\mathbf{X}(U)$ is always spacelike.

Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{n}^T_0, \mathbf{n}^S_1 \ldots, \mathbf{n}^S_{n-2}\}$ be an orthonormal frame in the Minkowski space $\mathbb{R}^{n+1}_1$, where $\mathbf{e}_1, \mathbf{e}_2$ generates the tangent space on M at each point and $\mathbf{n}^T_0$ and $\mathbf{n}^S_1 \ldots, \mathbf{n}^S_{n-2}$ are respectively a timelike normal section and spacelike normal sections.

The *first fundamental form* on M is given by $ds^2 = \sum_{i,j=1}^2 g_{ij} du_i du_j$, where $g_{ij} = \langle \mathbf{X}_{u_i}, \mathbf{X}_{u_j} \rangle$. For $i, j = 1, 2$ and $k = 0, \ldots, n-2$, we denote h^k_{ij} as

$$h^k_{ij}(u) = \langle \mathbf{X}_{u_i u_j}(u), \mathbf{n}^{T\,(or\,S)}_k(u) \rangle = -\langle \mathbf{X}_{u_i}(u), \mathbf{n}^{T\,(or\,S)}_{k,u_j}(u) \rangle.$$

We call h^k_{ij} the coefficients of the *second fundamental forms.*

We now review the concept of curvature ellipse of a spacelike surface in Minkowski space studied in [3]. Let $M = \mathbf{X}(U)$ be the spacelike surface in Minkowski $(n+1)$-space. We write $u_0 \in U$ and $p_0 = \mathbf{X}(u_0)$. We assume that $\langle \mathbf{X}_{u_i}(u_0), \mathbf{X}_{u_j}(u_0) \rangle = \delta_{ij}$ for $i, j = 1, 2$, where δ_{ij} is the Kronecker delta. Let $\vartheta \in S^1$ and write the tangent direction $\mathbf{v}_\vartheta = \cos\vartheta \mathbf{X}_{u_1}(u_0) + \sin\vartheta \mathbf{X}_{u_2}(u_0)$ on the spacelike surface. We consider a spacelike curve $\gamma(t)$ on M parametrized by unit speed, such that $\gamma(t_0) = p_0$ and $(\partial\gamma/\partial t)(t_0)$ is parallel to $\mathbf{v}_\vartheta$. The normal curvature vector $\eta(u_0, \vartheta)$ in the ϑ direction is defined as the normal component of the second order derivative $(\partial^2\gamma/\partial t^2)(t_0)$. We have

$$\eta(u_0, \vartheta) = \mathcal{H}_\mathbf{X} + \mathcal{B}_\mathbf{X} \cos 2\vartheta + \mathcal{C}_\mathbf{X} \sin 2\vartheta,$$

where

$$\begin{aligned}
\mathcal{H}_\mathbf{X} &= -\frac{1}{2}(h^0_{11} + h^0_{22})\mathbf{n}^T_0 + \frac{1}{2}\sum_{k=1}^{n-2}(h^k_{11} + h^k_{22})\mathbf{n}^S_k, \\
\mathcal{B}_\mathbf{X} &= -\frac{1}{2}(h^0_{11} - h^0_{22})\mathbf{n}^T_0 + \frac{1}{2}\sum_{k=1}^{n-2}(h^k_{11} - h^k_{22})\mathbf{n}^S_k, \\
\mathcal{C}_\mathbf{X} &= -h^0_{12}\mathbf{n}^T_0 + \sum_{k=1}^{n-2} h^k_{12}\mathbf{n}^S_k.
\end{aligned}$$

The vector $\eta(u_0, \vartheta)$ is independent on the choice of the curve $\gamma(t)$. As ϑ varies from 0 to 2π, $\eta(u_0, \vartheta)$ traces an ellipse in the normal space, which we call curvature ellipse at p_0. The curvature ellipse is determined by the coefficients of the second fundamental form with respect to normal directions $\mathbf{n}^T_0, \mathbf{n}^S_i$.

Using the same notation we also denote by $\mathbf{X} : U \longrightarrow S^5_1$ a parametrization of spacelike surface in de Sitter 5-space. For any point $p \in \mathbf{X}(U) = M$, each tangent vector $v \in T_pM \setminus \{\mathbf{0}\}$ is spacelike and orthogonal to the position vector p. In this case, we consider the second fundamental form in the direction $\vartheta \in S^1$ as the projection of the normal curvature vector into $N_pM \subset T_pS^5_1$.

Let $\mathbf{n}^T_0, \mathbf{n}^S_1, \mathbf{n}^S_2 : U \longrightarrow NM$ be a timelike normal section and spacelike normal sections of M. Let $S^4_+ := \{\mathbf{v} = (1, v_1, \ldots, v_5) \in \mathbb{R}^6_1 \mid \langle \mathbf{v}, \mathbf{v} \rangle = 0\}$ be a sphere in the lightcone. We define a map $\mathbf{e} : U \times S^1 \longrightarrow LC^{*5}$ as

$$\mathbf{e}(u, \theta) = \mathbf{n}^T_0(u) + \cos\theta \mathbf{n}^S_1(u) + \sin\theta \mathbf{n}^S_2(u).$$

The normalized lightlike direction $\bar{\mathbf{e}}: U \times S^1 \longrightarrow S^4_+$ is given by $\bar{\mathbf{e}}(u,\theta) = \mathbf{e}(u,\theta)/e_0(u,\theta)$, where $e_0(u,\theta)$ is a first component of $\mathbf{e}(u,\theta)$.

The coefficients of the second fundamental form of the embedding of the surface into S^5_1 with respect to the basis $\{\mathbf{n}^T_0, \mathbf{n}^S_1, \mathbf{n}^S_2\}$ are given by

$$\begin{aligned} a_0 &= \langle \mathbf{X}_{u_1u_1}, \mathbf{n}^T_0\rangle,\ a_1 = \langle \mathbf{X}_{u_1u_2}, \mathbf{n}^T_0\rangle,\ a_2 = \langle \mathbf{X}_{u_2u_2}, \mathbf{n}^T_0\rangle, \\ b_0 &= \langle \mathbf{X}_{u_1u_1}, \mathbf{n}^S_1\rangle,\ b_1 = \langle \mathbf{X}_{u_1u_2}, \mathbf{n}^S_1\rangle,\ b_2 = \langle \mathbf{X}_{u_2u_2}, \mathbf{n}^S_1\rangle, \\ c_0 &= \langle \mathbf{X}_{u_1u_1}, \mathbf{n}^S_2\rangle,\ c_1 = \langle \mathbf{X}_{u_1u_2}, \mathbf{n}^S_2\rangle,\ c_2 = \langle \mathbf{X}_{u_2u_2}, \mathbf{n}^S_2\rangle. \end{aligned} \tag{1}$$

The second fundamental form with respect to a lightlike normal $\mathbf{e}(u,\theta)$, $I\!I_{\mathbf{e}(u,\theta)} : T_pM \to \mathbb{R}$, is given by $I\!I_{\mathbf{e}(u,\theta)}(du_1, du_2) = \sum_{i,j=1}^{2}\langle \mathbf{X}_{u_iu_j}, \mathbf{e}(u,\theta)\rangle du_i du_j$. Let $I\!I_{\mathbf{e}(u,\theta)}(du_1, du_2) = L(u,\theta)du_1^2 + 2M(u,\theta)du_1du_2 + N(u,\theta)du_2^2$, then the we have

$$(L, M, N)(u,\theta) = (a_0 + b_0\cos\theta + c_0\sin\theta, a_1 + b_1\cos\theta + c_1\sin\theta, a_2 + b_2\cos\theta + c_2\sin\theta).$$

We call $I\!I_{\mathbf{e}(u,\theta)} = \begin{pmatrix} L & M \\ M & N \end{pmatrix}$ the matrix of the second fundamental form in the direction $\mathbf{e}(u,\theta)$. We remark that if we take the normalized lightlike normal direction $\bar{\mathbf{e}}(u,\theta)$, the corresponding second fundamental form is expressed by $I\!I_{\bar{\mathbf{e}}(u,\theta)}$. We will use this last one but still denote by $I\!I_{\mathbf{e}(u,\theta)}$.

The discriminant of the binary differential equation (briefly, BDE) $I\!I_{\mathbf{e}(u,\theta)} = 0$ is given by

$$\Delta(u,\theta) = M(u,\theta)^2 - L(u,\theta)N(u,\theta).$$

The sign of the discriminant Δ is invariant under change of parametrization of the surface. We say that a direction $\theta \in S^1$ is a *lightlike binormal* at $p = \mathbf{X}(u)$ if $\Delta(u,\theta) = 0$, that is $\text{rank}(I\!I_{\mathbf{e}(u,\theta)}) \leq 1$. In this case we call a kernel direction of the second fundamental form $I\!I_{\mathbf{e}(u,\theta)}$ by *asymptotic direction* associated to the lightlike binormal direction. Then for each θ such that $\Delta(u,\theta) = 0$, the associated asymptotic directions are the solutions of $I\!I_{\mathbf{e}(u,\theta)}(du_1, du_2) = 0$, that is,

$$(du_1\ du_2)\begin{pmatrix} L & M \\ M & N \end{pmatrix}\begin{pmatrix} du_1 \\ du_2 \end{pmatrix} = 0.$$

We also say that the point $p = \mathbf{X}(u)$ is a *lightlike inflection* point if $(L, M, N)(u,\theta) = \mathbf{0}$ for some $\theta \in S^1$, in this case any direction is asymptotic (rank $(I\!I_{\mathbf{e}(u,\theta)}) = 0$).

To explain the relations of the geometry of the asymptotic directions on spacelike surface, let $\mathcal{C} = \{(l,m,n) \mid m^2 - ln = 0\}$ be a cone and $E(u,\theta) = \mathbf{A} + \cos\theta\mathbf{B} + \sin\theta\mathbf{C}$ be the ellipse in the 3-space $\mathbb{R}^3 = \{(l,m,n)\}$, where $\mathbf{A} = (a_0, a_1, a_2)^t$, $\mathbf{B} = (b_0, b_1, b_2)^t$ and $\mathbf{C} = (c_0, c_1, c_2)^t$. We call this ellipse as LMN-ellipse. The center $\mathbf{A}$ of the ellipse varies with the choice of normal frames. We say that the vector (l,m,n) is *elliptic*, *hyperbolic* or *parabolic* respectively if its discriminant $m^2 - ln$ is negative, positive or equal to zero. Equivalently (l,m,n) is, respectively, inside, outside or on the cone $\mathcal{C}$. We remark that the determinant of the second fundamental form is $LN - M^2$. Properties of lightlike binormal directions and lightlike inflection points are related to the configuration of the cone and the LMN-ellipse. We have the following caracterizations.

(1) There is no lightlike binormal $\theta \in S^1$ if and only if $E(u,\theta) \notin \mathcal{C}$, $\forall \theta \in S^1$ $(\mathrm{rank}(I\!I_{\mathbf{e}(u,\theta)}) = 2, \forall \theta \in S^1)$.

(2) $\theta \in S^1$ is a lightlike binormal direction if and only if $E(u,\theta) \in \mathcal{C}$ $(\mathrm{rank}(I\!I_{\mathbf{e}(u,\theta)}) \leq 1)$.

(3) p is a lightlike inflection point if and only if the LMN-ellipse goes through the origin $(\mathrm{rank}(I\!I_{\mathbf{e}(u,\theta)}) = 0)$.

We intend to classify the configurations of the LMN-ellipse in further sections to investigate the geometry of the asymptotic directions, associated to the lightlike binormals, on spacelike surfaces in de Sitter 5-space.

We can consider the one-parameter family of the height functions on M, $H(u,\theta) = \langle \mathbf{X}(u), e(u,\theta)\rangle$. If θ is a lightlike binormal direction at the point p then the height function h_θ, obtained when we fix the parameter θ, has a singularity more degenerate than Morse at p. Moreover any direction in the kernel of the hessian of h_θ at p is an **associated** asymptotic direction (see [5]).

From now on, for short, we omit the word "lightlike" in the expressions "lightlike binormal direction" and "lightlike inflection points".

3 Invariants associated to the second fundamental form

The aim of this section is to introduce some invariants of the second order geometry of the spacelike surface in the de Sitter 5-space. First we define invariants of these surfaces, which are preserved by the Lorentzian transformations, that is, by the following actions (1) and (2).

(1) Change of parametrization, $\widetilde{\psi} \in \mathrm{Diff}(U)$: $\mathbf{X}'(u) = \mathbf{X} \circ \widetilde{\psi}(u)$,

(2) Change of the normal vector fields, $\widetilde{\Phi} : U \longrightarrow SO(1,2)$: $(\mathbf{n}'^T_0, \mathbf{n}'^S_1, \mathbf{n}'^S_2)^t = \widetilde{\Phi}\,(\mathbf{n}^T_0, \mathbf{n}^S_1, \mathbf{n}^S_2)^t$,

where $SO(1,2) = \{\Phi \mid \langle \Phi\mathbf{x}, \Phi\mathbf{y}\rangle = \langle \mathbf{x}, \mathbf{y}\rangle \text{ for } \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^3_1\}$ is the Lorentzian transformation group on Minkowski 3-space. As usual $GL(2,\mathbb{R})$ will denote the general linear group on $\mathbb{R}^2$. Let α and k be defined as

$$\alpha(u) := \begin{pmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ c_0 & c_1 & c_2 \end{pmatrix}, \quad k(u) := -\begin{vmatrix} a_0 & a_1 \\ a_1 & a_2 \end{vmatrix} + \begin{vmatrix} b_0 & b_1 \\ b_1 & b_2 \end{vmatrix} + \begin{vmatrix} c_0 & c_1 \\ c_1 & c_2 \end{vmatrix}.$$

The families $\widetilde{\Phi}$ and $\widetilde{\psi}$ of elements respectively in $SO(1,2)$ and $GL(2,\mathbb{R})$ are parametrized by u. That is, for any $u \in U$, $\Phi := \widetilde{\Phi}(u) \in SO(1,2)$ and $\psi := \widetilde{\psi}(u) \in GL(2,\mathbb{R})$. By computation, we have the following table.

Actions	$\alpha(u)$	$k(u)$	$\Delta(u,\theta)$
(1) $\psi \in GL(2,\mathbb{R})$	$\alpha'(u) = \alpha\Psi(u)$	$(\det\psi)^2 k(u)$	$\Delta'(u,\theta) = (\det\psi)^2\Delta(u,\theta)$
(2) $\Phi \in SO(1,2)$	$\alpha'(u) = \Phi\alpha(u)$	$k(u)$	$\Delta'(u,\theta) = \Delta(u,\theta'(\theta))$, where $\theta'(\theta)$ is a diffeomorphism on S^1

where,

$$\psi = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix} \text{ and } \Psi = \begin{pmatrix} \psi_{11}^2 & \psi_{11}\psi_{12} & \psi_{12}^2 \\ 2\psi_{11}\psi_{21} & \psi_{11}\psi_{22} + \psi_{12}\psi_{21} & 2\psi_{12}\psi_{22} \\ \psi_{21}^2 & \psi_{21}\psi_{22} & \psi_{22}^2 \end{pmatrix}.$$

It follows that $\det \Psi = (\det \psi)^3$. Let $(L, M, N) = (a_0 + b_0 \cos\theta + c_0 \sin\theta, a_1 + b_1 \cos\theta + c_1 \sin\theta, a_2 + b_2 \cos\theta + c_2 \sin\theta)$, then we have another expression of the action (1):

$$\begin{pmatrix} L' & M' \\ M' & N' \end{pmatrix} = D_u \psi^t \begin{pmatrix} L & M \\ M & N \end{pmatrix} D_u \psi,$$

that we can also write by $\psi^t \begin{pmatrix} L & M \\ M & N \end{pmatrix} \psi$.

The action (2) is generated by the following actions

$$\Phi_{1,t} = \begin{pmatrix} \cosh t & \sinh t & 0 \\ \sinh t & \cosh t & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \Phi_{2,\theta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}, \Phi_{3,\pm} = \begin{pmatrix} \pm_1 1 & 0 & 0 \\ 0 & \pm_2 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\pm_1$ and $\pm_2$ are not necessarily to be the same signs.

For each $u \in U$, let $I(u)$ be the first fundamental form of the spacelike surface $\mathbf{X}(U)$, then $k(u)/\det I(u)$ and $\operatorname{rank} \alpha(u)$ are invariant under the actions (1) and (2). The maximal and minimal values of $\Delta(u, \theta)/\det I(u)$ are also invariants.

The following proposition follows from the above discussion.

Proposition 3.1. The signs of $k(u)$ and $\Delta(u, \theta)$, and $\operatorname{rank} \alpha(u)$ are invariant under the actions (1) and (2).

We now fix a point $u \in U$ and concentrate to classify the configurations of LMN-ellipses. To do this we study the action of $G = GL(2, \mathbb{R}) \times SO(1, 2)$ on the system of conics defined by α.

We may distinguish the configuration of the LMN-ellipse by the multiplicity of the solutions of the equation $\Delta(u, \theta) = 0$, $0 \leq \theta \leq 2\pi$. We fix $u \in U$ and let $\Delta(u, \theta) \not\equiv 0$ and $\theta_1, \dots, \theta_k$ be distinct solutions of $\Delta(u, \theta) = 0$ with multiplicities $m_1, \dots, m_k$, where m_i is a number such that

$$\Delta(u, \theta) = \frac{\partial}{\partial\theta} \Delta(u, \theta_i) = \cdots = \frac{\partial^{m_i}}{\partial\theta^{m_i}} \Delta(u, \theta_i) = 0 \text{ and } \frac{\partial^{m_i+1}}{\partial\theta^{m_i+1}} \Delta(u, \theta_i) \neq 0.$$

In this case, we say that p is of $((m_1+1) + \cdots + (m_k+1))$-*type.* We say that α is of type $((m_1+1)+\cdots+(m_k+1))$-*type* if at this point we have both, k binormal and k asymptotic directions, with same multiplicities $m_1, \cdots, m_k$. In Section 8, we show that this always happens when rank of α is 3. If there is no solution we say that α is 0-type then there is no asymptotic direction. We also say that a point p is of $\sharp S^1$-*type* if $\Delta(u, \theta) \equiv 0$. We call p an *inflection* point of M if the LMN-ellipse goes through the origin for some θ, and in this case all directions are asymptotic. If not we say that $p = \mathbf{X}(u)$ is *non-inflection* point. We say that α is of *$\sharp S^1$-type* if all lightlike normal directions are binormals and all tangent directions are asymptotic directions. We will see, in Section 4, that if the ellipse degenerates into a segment (or a point) outside the origin and on the cone then all the directions θ are binormals ($\Delta(u,\theta) \equiv 0$) but there is only one associated asymptotic direction.

Let $\mathbf{X}(u)$ be a spacelike surface in de Sitter 5-space and α the matrix of coefficients of the second fundamental form at $p = \mathbf{X}(u)$ with respect to the lightlike normal direction $e(u,\theta)$. Following [10], we say that a point $p \in \mathbf{X}(U)$ is of type $\mathcal{M}_i$, $i = 1, 2, 3$ if rank of α is i. We now introduce some denominations for vectors and planes which are in the image of the second fundamental form in the cases rank $\alpha = 1$ or 2.

(1) If $\operatorname{rank} \alpha = 1$, then there exists a vector $\mathbf{x} = (x_0, x_1, x_2)$ such that $\langle \mathbf{x} \rangle = \langle \mathbf{A}, \mathbf{B}, \mathbf{C} \rangle$, where $\langle \mathbf{v_1}, \cdots, \mathbf{v_n} \rangle$ denotes the vector space generated by the vectors $\mathbf{v}_i$, $i = 1, \cdots, n$. Let $\bar{D}_1 = x_0 x_2 - x_1^2$. The sign of $\bar{D}_1$ is invariant under the action of G. Therefore we have three cases $\bar{D}_1 > 0$ ($\mathcal{M}_1$-*elliptic case*), $\bar{D}_1 < 0$ ($\mathcal{M}_1$-*hyperbolic case*), $\bar{D}_1 = 0$ ($\mathcal{M}_1$-*parabolic case*). Respectively, the vector $\mathbf{x}$ is elliptic, hyperbolic or parabolic. Equivalently, the vector $\mathbf{x}$ is inside, outside or on the cone $\mathcal{C}$.

(2) If $\operatorname{rank} \alpha = 2$, then there are two vectors $\mathbf{x} = (x_0, x_1, x_2)$ and $\mathbf{y} = (y_0, y_1, y_2)$ such that $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{A}, \mathbf{B}, \mathbf{C} \rangle$. We write $\bar{D}_2$ by

$$\bar{D}_2 = \det \begin{pmatrix} x_0 & 2x_1 & x_2 & 0 \\ y_0 & 2y_1 & y_2 & 0 \\ 0 & x_0 & 2x_1 & x_2 \\ 0 & y_0 & 2y_1 & y_2 \end{pmatrix}.$$

The sign of $\bar{D}_2$ is invariant under the action of G. Therefore we have the cases (see [9, 11]): $\bar{D}_2 > 0$ ($\mathcal{M}_2$-*elliptic case*), $\bar{D}_2 = 0$ ($\mathcal{M}_2$-*parabolic case*) and $\bar{D}_2 < 0$ ($\mathcal{M}_2$-*hyperbolic case*). Equivalently, the plane $\langle \mathbf{x}, \mathbf{y} \rangle$ is elliptic, parabolic or hyperbolic. The following geometric conditions hold: a plane by the origin is elliptic if and only if it intercepts the cone $\mathcal{C}$ only at the origin, it is hyperbolic if and only if it is transversal to the cone and it is parabolic if and only if it is tangent to $\mathcal{C}$ (see [10]).

We assume that $\operatorname{rank} \alpha \geq 1$. Changing coordinates if necessary, we can assume that $\mathbf{A} = (a_0, a_1, a_2)$ does not vanish.

Let (l, m, n) be a vector in the three space $\{(L, M, N)\}$. Let $\alpha = (\mathbf{A}, \mathbf{B}, \mathbf{C})^t$ be the second fundamental form at $u \in U$. We say that α is *A-elliptic* or *A-hyperbolic* respectively if p is not an inflection point and the vector A is always elliptic or always hyperbolic under the action of G. We also say that α is *A-parabolic* if p is not an inflection point and α is neither A-elliptic nor A-hyperbolic. We define the open and closed elliptic discs, called α-discs, by

$$D_\alpha = \{\mathbf{A} + p\,\mathbf{B} + q\,\mathbf{C} \mid 0 \leq p^2 + q^2 < 1\}, \text{ and } \overline{D_\alpha} = \{\mathbf{A} + p\,\mathbf{B} + q\,\mathbf{C} \mid 0 \leq p^2 + q^2 \leq 1\}.$$

These are elliptic discs of center $\mathbf{A}$, in the plane generated by the vectors $\mathbf{B}$ and $\mathbf{C}$. They are regions bounded by the LMN-ellipse. There is a possibility that D_α and $\overline{D_\alpha}$ degenerate to a segment or a point. By using these concepts we have the following lemmas that are important to classify α.

Lemma 3.2. Let D_α be the open α-disc defined as above. For any element $W \in D_\alpha$, there exist a transformation Φ such that W is parallel to $\mathbf{A}'$, where $\alpha' = (\mathbf{A}', \mathbf{B}', \mathbf{C}')^t := \Phi\alpha$. Therefore,

(1) If $W \neq \mathbf{0}$ then $W = rA'$ for some $r \in \mathbb{R}^*$.

(2) If $W = \mathbf{0}$ then $\mathbf{A}' = \mathbf{0}$.

Proof. Suppose that $W = \mathbf{A} + p\,\mathbf{B} + q\,\mathbf{C}$. Then there exists $(\theta, t) \in S^1 \times \mathbb{R}$ such that $(\cos\theta, \sin\theta) = (p/\sqrt{p^2+q^2}, q/\sqrt{p^2+q^2})$ and $\sinh t/\cosh t = \sqrt{p^2+q^2}$. We may write $\Phi = \Phi_{1,t} \circ \Phi_{2,\theta}$. Therefore,

$$A' = \cosh t\,\mathbf{A} + \sinh t(\cos\theta\,\mathbf{B} + \sin\theta\,\mathbf{C}) = \cosh t(\mathbf{A} + p\,\mathbf{B} + q\,\mathbf{C}).$$

This completes the proof. □

Lemma 3.3. (Inertial property for the closed α-discs) Let α and α' be equivalent under the action of G and $W' = A' + p'\,B' + q'\,C' \in \overline{D_{\alpha'}}$ the image of $W = \mathbf{A} + p\,\mathbf{B} + q\,\mathbf{C} \in \overline{D_\alpha}$ by this action. Then there exists a homeomorphism Θ, where $(p', q') = \Theta(p, q)$, of the closed disc $D^2 = \{(p,q) \mid p^2 + q^2 \leq 1\}$ such that for any (p,q), W and W' have same sign and same rank. That is, $\operatorname{sgn}(\det W) = \operatorname{sgn}(\det W') \in \{0, \pm 1\}$ and $\operatorname{rank} W = \operatorname{rank} W'$.

Proof. It is sufficient to check that for each action $\Phi_{1,t}, \Phi_{2,\theta}, \Phi_{3,\pm} \in SO(1,2)$ and $\psi \in \mathrm{Diff}_u(U)$, there exists Θ that satisfies the statement. We omit the case of action $\Phi_{3,\pm} \in SO(1,2)$. Given $\psi \in \mathrm{Diff}_u(U)$, we may put $\Theta = \mathrm{id}_{D^2}$, then we obtain $W' = D_u\psi^t\, W\, D_u\psi$ with a regular matrix $D_u\psi$. In this case, rank and index of W do not change.

Next, given $\Phi_{1,t} \in SO(1,2)$, since $\cosh t - p\sinh t$ is positive, we may define a map $\Theta_t : \{1\} \times D^2 \longrightarrow \{1\} \times D^2$ by

$$\Theta_t\,(1,p,q) \;=\; \frac{1}{\cosh t - p\sinh t}\,(1,p,q)\begin{pmatrix} \cosh t & -\sinh t & 0 \\ -\sinh t & \cosh t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $\Theta_t \circ \Theta_{-t} = \Theta_{-t} \circ \Theta_t = \mathrm{id}_{\{1\}\times D^2}$, therefore Θ_t is a homeomorphism. We now put $(1, p', q') = \Theta_t(1, p, q)$, then we obtain $W = (\cosh t - p\sinh t)W'$. Therefore we have, $\det W' = (\cosh t - p\ \sinh t)^2 \det W$ and $\operatorname{rank} W' = \operatorname{rank} W$.

Finally, given $\Phi_{2,\theta} \in SO(1,2)$, we may put $(p', q') = (p\cos\theta + q\sin\theta, -p\sin\theta + q\cos\theta)$, then

$$\begin{aligned} W' = (1, p', q')\begin{pmatrix} \mathbf{A}' \\ \mathbf{B}' \\ \mathbf{C}' \end{pmatrix} &= (1, p', q')\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \end{pmatrix} \\ &= (1, p, q)\begin{pmatrix} \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \end{pmatrix} = W. \end{aligned}$$

Therefore, the determinant and the rank of W, W' coincide respectively. This completes the proof. □

The proof of the next result is analogous to the proof of Lemma 3.3.

Lemma 3.4. If rank of α is 2 then the type of the plane $\langle \mathbf{A}, \mathbf{B}, \mathbf{C} \rangle$ is an invariant.

Using the last results we have the following proposition.

Proposition 3.5. Let D_α be an open α-disc defined as above, then we have:

(1) α is A-hyperbolic (or A-elliptic) if and only if any element of D_α is contained in the hyperbolic region (or elliptic region respectively).

(2) If $\operatorname{rank} \alpha = 2$ and α is $\mathcal{M}_2$-parabolic then α is A-parabolic or u is inflection point if and only if D_α includes some parabolic point.

(3) If $\operatorname{rank} \alpha = 2$ and $\langle \mathbf{A}, \mathbf{B}, \mathbf{C} \rangle$ is $\mathcal{M}_2$-hyperbolic, then α is A-parabolic or u is inflection point if and only if D_α includes some elliptic, parabolic and hyperbolic points.

(4) If $\operatorname{rank} \alpha = 2$ and $\langle \mathbf{A}, \mathbf{B}, \mathbf{C} \rangle$ is $\mathcal{M}_2$-elliptic, then α is A-parabolic or u is inflection point if and only if $\overline{D_\alpha}$ includes some parabolic point p. If u is an inflection point, then p is at the boundary of the disc and if α is A-parabolic then p is in the open disc.

(5) If $\operatorname{rank} \alpha = 3$, we have no inflection point. Therefore α is A-parabolic if and only if D_α includes some elliptic, parabolic and hyperbolic points.

Proof. By applying the above lemmas, we have the results. □

4 Rank one case

We consider the rank one case and obtain the normal forms of α. Using these normal forms, in this case, it is easy to study the number of binormals and asymptotic directions. It can occur finite number of asymptotic directions with infinite number of binormals or vice-versa; no asymptotic and no binormal directions; or infinite number of binormals associated to an unique asymptotic direction.

Proposition 4.1. If $\operatorname{rank} \alpha = 1$, there are the following equivalence classes.
(a) $\mathcal{M}_1$-elliptic: α is equivalent to some $\alpha_{1,e}$,
(b) $\mathcal{M}_1$-parabolic: α is equivalent to some $\alpha_{1,p}$,
(c) $\mathcal{M}_1$-hyperbolic: α is equivalent to some $\alpha_{1,h}$,
where

$$\alpha_{1,e} = \begin{pmatrix} c & 0 & c \\ s & 0 & s \\ 0 & 0 & 0 \end{pmatrix}, \ \alpha_{1,p} = \begin{pmatrix} c & 0 & 0 \\ s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \alpha_{1,h} = \begin{pmatrix} 0 & c & 0 \\ 0 & s & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2}$$

with $(c, s) = (1, 0)$, $(0, 1)$, $(1, 1)$.

Proof. Suppose that rank $\alpha = 1$, then by acting $\Phi_{1,t}, \Phi_{2,\theta} \in SO(2,1)$, we may assume that $C = \mathbf{0}$. Let D be a (3×1)-matrix and $\gamma \in S^1$ with $A = \cos\gamma D$ and $B = \sin\gamma D$. Next, we consider respectively three cases $||\mathbf{B}|| < ||\mathbf{A}||$, $||\mathbf{B}|| > ||\mathbf{A}||$ and $||\mathbf{B}|| = ||\mathbf{A}||$. (i) In the case that $|\cos\gamma| > |\sin\gamma|$, by acting $\Phi_{1,t}$ with $\tanh t = -\sin\gamma/\cos\gamma$, we obtain $\mathbf{B} = \mathbf{0}$ and we may replace $\mathbf{A} = (\cos\gamma\cosh t + \sin\gamma\sinh t)\mathbf{D}$ by $\mathbf{D}$. (ii) $|\cos\gamma| < |\sin\gamma|$. Acting $\Phi_{1,t}$ with $1/(\tanh t) = -\sin\gamma/\cos\gamma$, then we have $\mathbf{A} = \mathbf{0}$ and we can take $\mathbf{B} = \mathbf{D}$ by the same reason. (iii) In case that $|\cos\gamma| = |\sin\gamma|$, by replacing the sign of the matrix $\mathbf{B}$, then we can take $\mathbf{A} = \mathbf{B} = \mathbf{D}$. Therefore by following these steps we obtain the following three cases:

$\mathcal{M}_1$**-elliptic case**: If $\det\mathbf{D} > 0$, then there exists a regular matrix S such that $S^tDS = \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. By acting $\Phi_{1,t}, \Phi_{2,\theta} \in SO(2,1)$ we may change sign of the matrix α. Therefore we have the following form $\alpha_{1,e}$ as in (2) where $(c,s) = (1,0)$, $(0,1)$, $(1,1)$.

$\mathcal{M}_1$**-parabolic case**: Since $\det\mathbf{D} = 0$, then there exists a regular matrix S such that $S^tDS = \pm\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Removing the sign of the matrix α, we get $\alpha_{1,p}$ as in (2), where $(c,s) = (1,0)$, $(0,1)$, $(1,1)$.

$\mathcal{M}_1$**-hyperbolic case**: Since $\det D < 0$, then there exists a regular matrix S such that $S^t\mathbf{D}S = \pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Removing the sign of the matrix α, we get $\alpha_{1,h}$, where $(c,s) = (1,0)$, $(0,1)$, $(1,1)$.

□

Theorem 4.2. The number of asymptotic directions (AD) and type of binormal directions (BD), for each normal form of Proposition 4.1, are in table 1.

$\mathcal{M}_1$-type		AD	BD
$\mathcal{M}_1$-elliptic or $\mathcal{M}_1$-hyperbolic	$(c,s) = (1,0)$	0	0-type
	$(c,s) = (0,1)$	inflection	(2+2)-type
	$(c,s) = (1,1)$	inflection	4-type
$\mathcal{M}_1$-parabolic	$(c,s) = (1,0)$	1	$\sharp S^1$-type
	$(c,s) = (0,1)$	inflection	$\sharp S^1$-type
	$(c,s) = (1,1)$	inflection	trans-$\sharp S^1$-type

Table 1: $\mathcal{M}_1$-case

Proof. $\mathcal{M}_1$**-elliptic case**: In this case, the second fundamental form is given by $(L,M,N)(u,\theta) = (c + s\cos\theta, 0, c + s\cos\theta)$, and $\Delta(u,\theta) = -(c + s\cos\theta)^2$. Then for each θ such that $\Delta(u,\theta) = 0$ we have $(L,M,N) = \mathbf{0}$ so $p = \mathbf{X}(u)$ is an inflection point. (i) If $(c,s) = (1,0)$ then $\Delta(u,\theta) < 0$ and there are no asymptotic directions

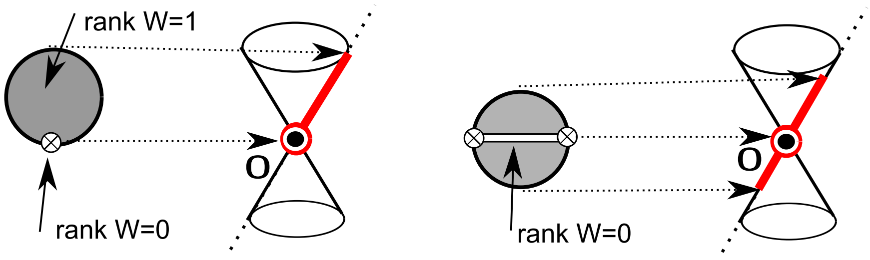

Figure 1: Topological type of ellipse in M_1-parabolic inflection type.

(M_{1e} *non-inflection* 0-*type*). (ii) If $(c, s) = (0, 1)$ then $\Delta(u, \theta) = 0$ if $\theta = \pi/2, 3\pi/2$ with $\frac{\partial}{\partial\theta}\Delta(u, \theta) = 0$ for both θ's (M_{1e} *inflection* $(2 + 2)$-*type*). (iii) If $(c, s) = (1, 1)$ then $\Delta(u, \theta) = 0$ when $\theta = \pi$ with $\frac{\partial^i}{\partial\theta^i}\Delta(u, \theta) = 0$ for $i = 1, \cdots, 3$ (M_{1e} *inflection* 4-*type*).

$\mathcal{M}_1$**-parabolic case**: In this case $(L, M, N)(u, \theta) = (c+s\cos\theta, 0, 0)$ and $\Delta(u, \theta) \equiv 0$ ($\sharp S^1$-type). (i) If $(c, s) = (1, 0)$ then for any θ, $II_{\mathbf{e}(u,\theta)}(du_1, du_2) = du_1^2 = 0$ gives only one asymptotic direction (M_{1p} *non-inflection* $\sharp S^1$-*type*). (ii) If $(c, s) = (0, 1)$ then $(L, M, N) = \mathbf{0}$ if $\theta = \pi/2,\ 3\pi/2$, that is α is M_{1p} *inflection* $\sharp S^1$-*type*. (iii) If $(c, s) = (1, 1)$ then $(L, M, N) = \mathbf{0}$ if $\theta = \pi$ and α is also inflection type (M_{1p} *inflection trans*-$\sharp S^1$-*type*). Let $W = \mathbf{A} + p\mathbf{B} + q\mathbf{C}$ and $p^2 + q^2 \leq 1$. We may distinguish (ii) from (iii) by the topological type of α-disc $\{(p, q)/p^2 + q^2 \leq 1\}$ (see Figure 1).

$\mathcal{M}_1$**-hyperbolic case**: In this case, we obtain $(L, M, N)(u, \theta) = (0, c+s\cos\theta, 0)$ and $\Delta(u, \theta) = (c + s\cos\theta)^2$. (i) If $(c, s) = (1, 0)$ then there are no asymptotic direction (M_{1h} *non-inflection* 0-*type*). (ii) If $(c, s) = (0, 1)$ then $\Delta(u, \theta) = 0$ if $\theta = \pi/2,\ 3\pi/2$ and this is an inflection point (M_{1h} *inflection* $(2 + 2)$-*type*). (iii) If $(c, s) = (1, 1)$ then $\Delta(u, \theta) = 0$ if $\theta = \pi$ (M_{1h} *inflection* 4-*type*).

□

Remark 4.3. The $GL(2, \mathbb{R}) \times SO(1, 2)$ action does not always preserve the form of the curvature ellipse. When *rank* $\alpha = 1$, at a non inflection point, we see that the degenerate ellipse is a radial segment, or a point, in the same equivalence class.

5 Rank two case

In this section, we classify α when $\operatorname{rank}\alpha = 2$. According to the type of the plane generated by **A**, **B** and **C**, and the type of α given in Section 3, we have the following result.

Proposition 5.1. If $\operatorname{rank}\alpha = 2$, there are the following equivalence classes.
(a) $\mathcal{M}_2$-hyperbolic and A-elliptic: α is equivalent to α_{he},

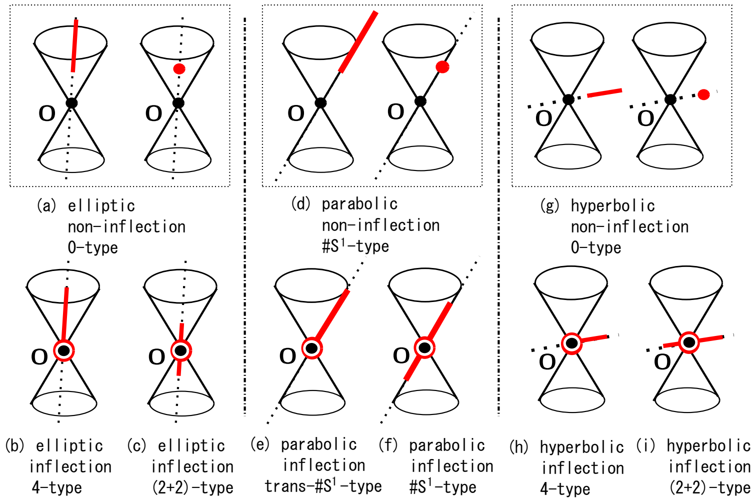

Figure 2: Classification of $\mathcal{M}_1$ case

(b) $\mathcal{M}_2$-hyperbolic and A-parabolic: α is equivalent to α_{hp},
(c) $\mathcal{M}_2$-hyperbolic and A-hyperbolic: α is equivalent to α_{hh},
(d) $\mathcal{M}_2$-parabolic: α is equivalent to α_p,
(e) $\mathcal{M}_2$-elliptic: α is equivalent to α_e.
The normal forms for α are displayed below and must have rank 2.

$$\alpha_{he} = \begin{pmatrix} 1 & 0 & 1 \\ b_0 & 0 & 0 \\ c_0 & 0 & c_2 \end{pmatrix}, \ \alpha_{hp} = \begin{pmatrix} 1 & 0 & 0 \\ b_0 & 0 & 0 \\ c_0 & 0 & 1 \end{pmatrix}, \ \alpha_{hh} = \begin{pmatrix} 1 & 0 & -1 \\ b_0 & 0 & 0 \\ c_0 & 0 & c_2 \end{pmatrix}, \tag{3}$$

$$\alpha_p = \begin{pmatrix} 1 & 1 & 0 \\ b_0 & 0 & 0 \\ c_0 & c_1 & 0 \end{pmatrix}, \ \alpha_e = \begin{pmatrix} 0 & 1 & 0 \\ 0 & b_1 & 0 \\ 1 & c_1 & c_2 \end{pmatrix},$$

where $c_2 < 0$ for α_e.

For the $\mathcal{M}_2$-parabolic case, that is, the plane $\langle \mathbf{A}, \mathbf{B}, \mathbf{C} \rangle$ parabolic, we need the following lemma.

Lemma 5.2. Suppose that $\operatorname{rank} \alpha = 2$ and the plane $\langle \mathbf{A}, \mathbf{B}, \mathbf{C} \rangle$ is parabolic. If $\mathbf{A}$ is parabolic or equal to $\mathbf{0}$, then $\alpha \sim_G \alpha'$, $\alpha' = \begin{pmatrix} \mathbf{A}' \\ \mathbf{B}' \\ \mathbf{C}' \end{pmatrix}$, where $\mathbf{A}'$ is hyperbolic.

Proof. Let $W \in D_\alpha$, $W \neq A$ hyperbolic, then the result follows by Lemma 3.2. □

Proof. (of Proposition 5.1) **$\mathcal{M}_2$-hyperbolic case**: In this case, the plane $\langle \mathbf{A}, \mathbf{B}, \mathbf{C} \rangle$ is hyperbolic, then there are two generators $\mathbf{D}, \mathbf{E}$ such that $\langle \mathbf{D}, \mathbf{E} \rangle = \langle \mathbf{A}, \mathbf{B}, \mathbf{C} \rangle$ and we can assume $\det \mathbf{D} > 0$. We can get the simultaneous diagonal normal forms of D and

E, that is, there is a regular matrix $S \in SO(2)$ such that $S^t\mathbf{D}S = \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and $S^t\mathbf{E}S = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

Therefore, we obtain

$$\alpha' = \begin{pmatrix} a_0' & 0 & a_2' \\ b_0' & 0 & b_2' \\ c_0' & 0 & c_2' \end{pmatrix}.$$

If A is elliptic, parabolic (we assume $a_0' \neq 0$, $a_2' = 0$) or hyperbolic, we first multiply A, B, C respectively by

$$\psi = \begin{pmatrix} 1/\sqrt{|a_0'|} & 0 \\ 0 & 1/\sqrt{|a_2'|} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{|a_0'|} & 0 \\ 0 & 1 \end{pmatrix} \; or \; \begin{pmatrix} 1/\sqrt{|a_0'|} & 0 \\ 0 & 1/\sqrt{|a_2'|} \end{pmatrix},$$

next, we apply $\Phi = \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ to the result, to get $(a_0'', 0, a_2'') = (1, 0, 1)$, $(1, 0, 0)$ or $(1, 0, -1)$ respectively. And, in any case, applying $\Phi_{2,\theta}$, for some appropriate θ we get $b_2'' = 0$. When A is parabolic, we can make another simplification, acting $\psi = \begin{pmatrix} 1 & 0 \\ 0 & 1/\sqrt{|c_2''|} \end{pmatrix}$ and changing sign, using appropriated Φ. Therefore, we obtain α_{he}, α_{hp} and α_{hh} as in (3), where we denote a_i'', b_i'', c_i'' by a_i, b_i, c_i to simplify notation.

$\mathcal{M}_2$**-parabolic case**: In this case the plane $\langle \mathbf{A}, \mathbf{B}, \mathbf{C} \rangle$ is parabolic, then there are hyperbolic and parabolic vectors $\mathbf{D}, \mathbf{E}$ (i.e. $\det \mathbf{D} < 0$ and $\det \mathbf{E} = 0$) such that $\langle \mathbf{D}, \mathbf{E} \rangle = \langle \mathbf{A}, \mathbf{B}, \mathbf{C} \rangle$. Set $S \in GL(2)$ such that $\mathbf{E}' = S^t\mathbf{E}S = \pm\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $S^tDS = D'$. In this case, for any $\gamma, \delta \in \mathbb{R}$,

$$\det(\gamma\mathbf{E}' + \delta\mathbf{D}') = (\det S)^2 \det(\gamma\mathbf{E} + \delta\mathbf{D}) \leq 0,$$

and zero if and only if $\delta = 0$. This means that the plane $\langle \mathbf{D}', \mathbf{E}' \rangle$ is also parabolic, then it contains only one parabolic direction, which is $E' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Moreover, it follows that it must contain $\mathbf{D}'' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. As $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are linear combinations of $\mathbf{D}'', \mathbf{E}'$, we have

$$\alpha' = \begin{pmatrix} a_0' & a_1' & 0 \\ b_0' & b_1' & 0 \\ c_0' & c_1' & 0 \end{pmatrix}.$$

Taking $\Phi_{2,\theta} \circ \alpha'$, for apropriate θ, we can get $b_1' = 0$.

By Lemma 5.2, we may assume that $\mathbf{A}$ is hyperbolic, then it is not hard to show that α' is equivalent to α_p as in (3).

$\mathcal{M}_2$**-elliptic case**: Finally we consider the case when the plane $\langle \mathbf{A}, \mathbf{B}, \mathbf{C} \rangle$ is elliptic. We may assume that $\mathbf{A} \neq 0$ and that there is a regular matrix S such that $\mathbf{A}' = S^t \mathbf{A} S = \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Therefore,

$$\alpha' = \begin{pmatrix} 0 & 1 & 0 \\ b'_0 & b'_1 & b'_2 \\ c'_0 & c'_1 & c'_2 \end{pmatrix}.$$

Since $\operatorname{rank} \alpha = 2$, then the rank of the submatrix $\begin{pmatrix} b'_0 & b'_2 \\ c'_0 & c'_2 \end{pmatrix}$ is equal to 0. Acting $\Phi_{2,\theta}$ (rotation), we obtain $(b'_0, b'_2) = (0, 0)$.

Since $\langle \mathbf{A}, \mathbf{C} \rangle$ is an elliptic plane and does not contain parabolic vectors, then $c'_0 c'_2 < 0$. Taking $\psi = \begin{pmatrix} 1/\sqrt{|c'_0|} & 0 \\ 0 & 1/\sqrt{|c'_2|} \end{pmatrix}$ and changing the sign of c'_0, we obtain α_e as (3), with $c_2 = c''_2 < 0$. □

To complete the classification, we can now apply Proposition 3.5. Using the normal forms, it is easy to study the number of binormals and asymptotic directions. Observe that if $E(u, \theta)$ is a segment and the line passing through it does not contain the origin, then $\Delta(u, \theta) \equiv 0$ does not hold. Therefore, in this case, we have a finite number of binormals.

Theorem 5.3. The number of asymptotic directions (AD) and the types of lightlike binormal directions (BD), for each normal form of Proposition 5.1, are in tables 2 to 5.

A-elliptic	A-hyperbolic	AD	BD
(a) ${b_0}^2 + {c_0}^2 < 1$ and $\lvert c_2\rvert < 1$	(d) ${b_0}^2 + {c_0}^2 < 1$ and $\lvert c_2\rvert < 1$	0	0-type
(b) ${b_0}^2 + {c_0}^2 = 1$ and $\lvert c_2\rvert < 1$	(e) ${b_0}^2 + {c_0}^2 = 1$ and $\lvert c_2\rvert < 1$ or ${b_0}^2 + {c_0}^2 < 1$ and $\lvert c_2\rvert = 1$	1	2-type
(c) ${b_0}^2 + {c_0}^2 = 1$ and $\lvert c_2\rvert = 1$	(f) ${b_0}^2 + {c_0}^2 = 1$ and $\lvert c_2\rvert = 1$	2	(2+2)-type

Table 2: $\mathcal{M}_2$-hyperbolic

A-parabolic	AD	BD
(g) ${b_0}^2 + {c_0}^2 < 1$	1	(1+1)-type
(h) ${b_0}^2 + {c_0}^2 = 1$ and $b_0 \neq \pm 1$	2	(2+1+1)-type
(i) ${b_0}^2 + {c_0}^2 > 1$ and $b_0 \neq \pm 1$	2	(1+1+1+1)-type
(j) $b_0 = \pm 1$ and $c_0 \neq 0$	inflection	(2+1+1)-type
(k) $b_0 = \pm 1$ and $c_0 = 0$	inflection	(3+1)-type

Table 3: $\mathcal{M}_2$-hyperbolic

A-hyperbolic: $\|c_1\| \leq 1$, A-pababolic: $\|c_1\| > 1$	AD	BD
(a) $\|c_1\| < 1$	0	0-type
(b) $\|c_1\| = 1$ and $c_0 \neq c_1$	1	4-type
(d) $\|c_1\| = 1$, $c_0 = c_1$ $b_0 \neq 0$	inflection	4-type
(c) $\|c_1\| > 1$, $(c_1 - c_0)^2 \neq {b_0}^2({c_1}^2 - 1)$	1	(2+2)-type
(e) $\|c_1\| > 1$, $(c_1 - c_0)^2 = {b_0}^2({c_1}^2 - 1)$	inflection	(2+2)-type

Table 4: $\mathcal{M}_2$-parabolic

A-hyperbolic	A-parabolic	AD	BD
(a) $\|b_1\| < 1$	(b) $\|b_1\| > 1$	0	0-type
(c) $\|b_1\| = 1$	——	inflection	2-type

Table 5: $\mathcal{M}_2$-elliptic

Proof. **$\mathcal{M}_2$-hyperbolic case.** The number of binormals is always finite then in the non-inflection case the asymptotic directions are of same type as the binormal directions (these types are defined in Section 3). In fact, observe that in this case, it follows from the normal forms in Proposition 5.1, that for each θ, the associated asymptotic direction is given by $II_{\mathbf{e}(u,\theta)}(du_1, du_2) = L(u,\theta)du_1^2 + N(u,\theta)du_2^2 = 0$ with $LN \leq 0$, and we prove that $\Delta = LN = 0$ has a multiple solution only in the inflection case.

In the next calculations we use the normal forms given in Proposition 5.1 to write (L, M, N) and Δ.

$\mathcal{M}_2$-hyperbolic and A-elliptic: $\Delta(u,\theta) \leq 0$ for all θ. In this case $(L, M, N) = (1+b_0\cos\theta+c_0\sin\theta, 0, 1+c_2\sin\theta)$ and $\Delta(u,\theta) = -(1+b_0\cos\theta+c_0\sin\theta)(1+c_2\sin\theta)$, where $b_0 \neq 0$ and $c_2 \neq c_0$. Since $\Delta(u,\theta) \leq 0$, then ${b_0}^2 + {c_0}^2 \leq 1$ and $|c_2| \leq 1$. Therefore, we have the following possibilities:

(a) α is $\mathcal{M}_{2he}$ *0-type* if and only if ${b_0}^2 + {c_0}^2 < 1$ and $|c_2| < 1$, figure 3(a).

(b) α is $\mathcal{M}_{2he}$ *2-type* if and only if "${b_0}^2 + {c_0}^2 = 1$ and $|c_2| < 1$" or "${b_0}^2 + {c_0}^2 < 1$ and $|c_2| = 1$" (that is the asymptotic direction is given respectively by $L(u,\theta)du_1^2 = 0$ or $N(u,\theta)du_2^2 = 0$), figure 3(b).

(c) α is $\mathcal{M}_{2he}$ *(2+2)-type* if and only if ${b_0}^2 + {c_0}^2 = 1$ and $|c_2| = 1$ (that is the asymptotic direction is given respectively by $L(u,\theta_1)du_1^2 = 0$ or $N(u,\theta_2)du_2^2 = 0$), figure 3(c).

$\mathcal{M}_2$-hyperbolic and A-hyperbolic: $\Delta \geq 0$ for all θ then $(L, M, N) = (1 + b_0\cos\theta + c_0\sin\theta, 0, -1 + c_2\sin\theta)$ where $b_0 \neq 0$ or $c_0 \neq -c_2$. Therefore, we can classify as follows:

(a) α is $\mathcal{M}_{2hh}$ *0-type* if and only if ${b_0}^2 + {c_0}^2 < 1$ and $|c_2| < 1$, figure 4(d).

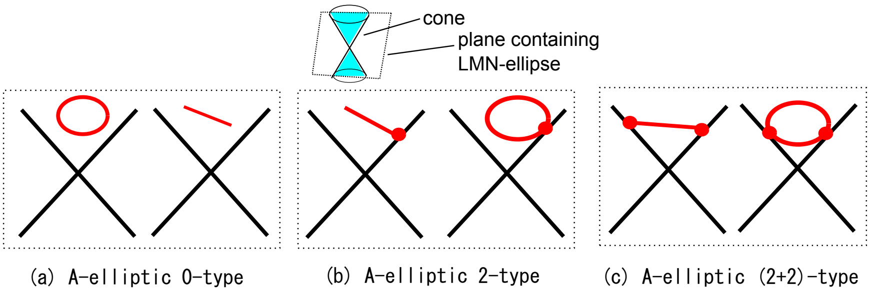

Figure 3: Classification for $\mathcal{M}_2$-hyperbolic A-elliptic case

(b) α is $\mathcal{M}_{2hh}$ *2-type* if and only if "${b_0}^2 + {c_0}^2 = 1$ and $|c_2| < 1$" or "${b_0}^2 + {c_0}^2 < 1$ and $|c_2| = 1$", figure 4(e).

(c) α is $\mathcal{M}_{2hh}$ *(2+2)-type* if and only if ${b_0}^2 + {c_0}^2 = 1$ and $|c_2| = 1$, figure 4(f).

$\mathcal{M}_2$-hyperbolic and A-parabolic: In this case $(L, M, N) = (1+b_0\cos\theta+c_0\sin\theta, 0, \sin\theta)$ and $\Delta(u, \theta) = -\sin\theta(1 + b_0\cos\theta + c_0\sin\theta)$.

Observe that a point p is an inflection point if and only if $(L, M, N) = \mathbf{0}$ for some $\theta \in S^1$, that is $b_0 = \pm 1$. Suppose that $b_0 \neq \pm 1$, that is the equations $L(u, \theta) = 0$ and $N(u, \theta) = 0$ have different solutions.

(a) If ${b_0}^2+{c_0}^2 < 1$ then there are two solutions $\theta = 0, \pi$ ($\Delta(u, \theta) = 0$ is (1+1)-type) and one double asymptotic direction for each binormal. Figure 5(g).

(b) If ${b_0}^2 + {c_0}^2 = 1$ then there are three solutions ($\Delta(u, \theta) = 0$ is (2+1+1)-type) and two asymptotic directions. In this case, the ellipse is tangent to the cone in two different ways, figure 5(h).

(c) If ${b_0}^2 + {c_0}^2 > 1$ then there are four solutions ($\Delta(u, \theta) = 0$ is (1+1+1+1)-type) and two asymptotic directions, figure 5(i).

If $b_0 = \pm 1$ in α_{hp} then the equations $L(u, \theta) = 0$ and $N(u, \theta) = 0$ have a common solution.

(d) If $c_0 \neq 0$ then there are three solutions and $\Delta(u, \theta) = 0$ is (2+1+1)-type, figure 6(j). (We call p an $\mathcal{M}_{2hp}$ *inflection (2+1+1)-type*.)

(e) If $c_0 = 0$ then there are two solutions $\theta = 0, \pi$ and $\Delta(u, \theta) = 0$ is (3+1)-type. In this case we call p an $\mathcal{M}_{2hp}$ *inflection (3+1)-type*, figure 6(k).

$\mathcal{M}_2$-parabolic: Then $(L, M, N) = (1+b_0\cos\theta+c_0\sin\theta, 1+c_1\sin\theta, 0)$ and $\Delta(u, \theta) = -(1 + c_1\sin\theta)^2$.

(a) If $|c_1| < 1$ (that is α is A-hyperbolic) then there are no solutions ($\mathcal{M}_{2p}$ *0-type*), figure 7(a).

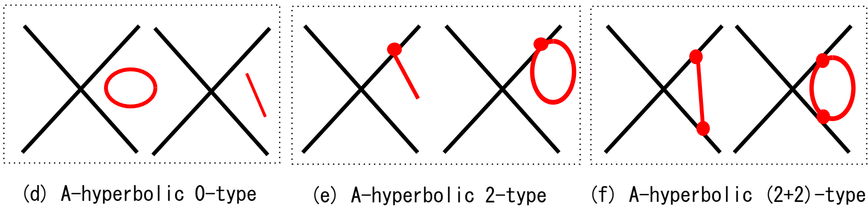

Figure 4: Classification for $\mathcal{M}_2$-hyperbolic A-hyperbolic case

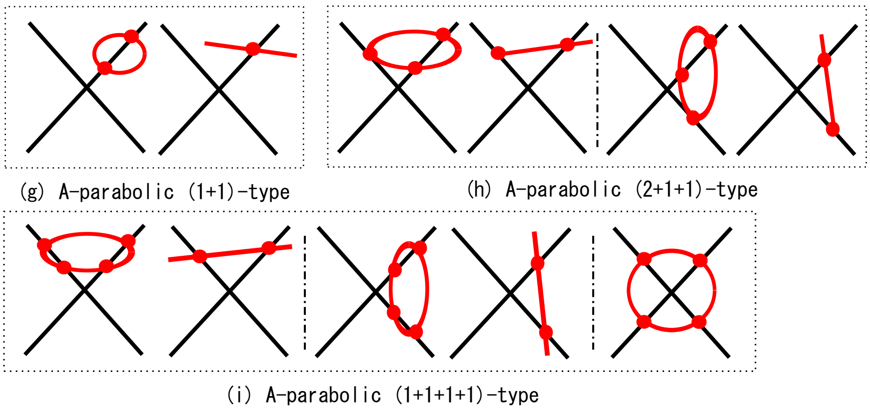

Figure 5: Classification for $\mathcal{M}_2$-hyperbolic A-parabolic case

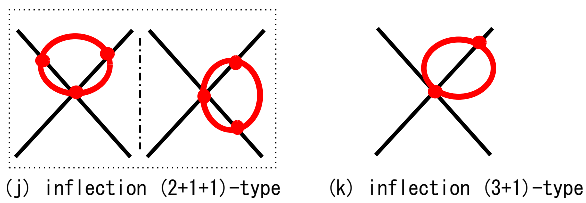

Figure 6: Classification for $\mathcal{M}_2$-hyperbolic inflection case

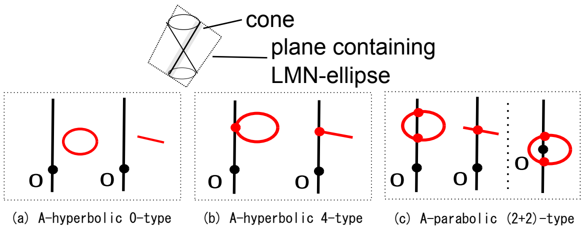

Figure 7: Classification for $\mathcal{M}_2$-parabolic case

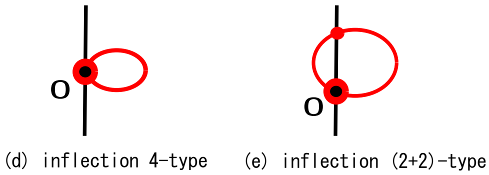

Figure 8: Classification for $\mathcal{M}_2$-parabolic inflection case

(b) If $|c_1| = 1$ (that is α is A-hyperbolic and the ellipse intercepts the cone) then $\Delta(u, \theta) = 0$ has a quartic solution, then we have two cases:

   i. If $c_0 = c_1$ (therefore $b_0 \neq 0$) then p is an inflection point ($\mathcal{M}_{2p}$ *inflection 4-type*), figure 8(d).

   ii. If $c_0 \neq c_1$ then p is a non-inflection point and has one asymptotic direction ($\mathcal{M}_{2p}$ *4-type*), figure 7(b).

(c) If $|c_1| > 1$ (that is α is A-parabolic) then $\Delta(u, \theta) = 0$ has two multiple solutions. By computation, we can classify two cases:

   i. If $(c_1 - c_0)^2 = {b_0}^2({c_1}^2 - 1)$, then one of the solutions of $\Delta(u, \theta) = 0$ satisfies $(L, M, N)(u, \theta) = 0$, therefore p is an inflection point ($\mathcal{M}_{2p}$ *inflection (2+2)-type*), figure 8(e).

   ii. Otherwise, p is a non-inflection point and has one asymptotic direction, ($\mathcal{M}_{2p}$ *(2+2)-type*), figure 7(c).

Finally we consider the **$\mathcal{M}_2$-elliptic case.**

Since $c_2 < 0$, $(L, M, N) = (\sin\theta, 1 + b_1\cos\theta + c_1\sin\theta, c_2\sin\theta)$ and $\Delta(u, \theta) = (1 + b_1\cos\theta + c_1\sin\theta)^2 - c_2\sin^2\theta$ then $\Delta(u, \theta) = 0$ for some θ if and only if $b_1 = \pm 1$. Therefore, we have the following classification.

(a) If $|b_1| < 1$ then there are no solution, and D_α consists of hyperbolic points ($\mathcal{M}_{2eh}$ *0-type*), figure 9(a).

(b) If $|b_1| > 1$ then there are no solution, however D_α includes an origin. ($\mathcal{M}_{2ee}$ *0-type*), figure 9(b).

(c) If $|b_1| = 1$ then the solution θ of $\Delta(u, \theta) = 0$ satisfies $(L, M, N)(u, \theta) = 0$. Therefore, p is an inflection point ($\mathcal{M}_{2e}$ *inflection 2-type*), figure 9(c).

□

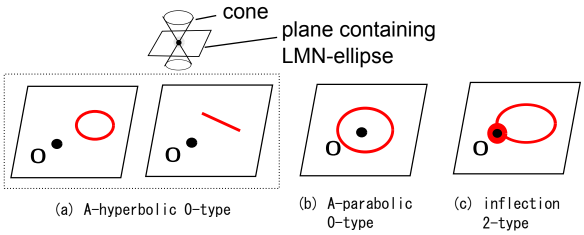

Figure 9: Classification for $\mathcal{M}_2$-elliptic case

6 Rank three case

As the group $G = GL(2) \times SO(1,2)$ has dimension 7 and the space of 3×3 matrices α has dimension 9, the classification of G-orbits in this space has at least 2 parameters, but our aim is to classify as follows. We use similar arguments as in sections 4 and 5, according to the A-elliptic, A-hyperbolic and A-parabolic type of α, to obtain pre-normal forms for rank 3 matrices depending on 4 parameters.

Proposition 6.1. If $\operatorname{rank}\alpha = 3$ we have the following subcases.
(a) $\mathcal{M}_3$ and A-elliptic: α is equivalent to $\alpha^1_{3,e}$ or to $\alpha_{3,e}$,
(b) $\mathcal{M}_3$ and A-hyperbolic: α is equivalent to $\alpha_{3,h}$,
(c) $\mathcal{M}_3$ and A-parabolic: α is equivalent to $\alpha_{3,p}$,
where the normal forms for α are displayed below.

$$\alpha^1_{3,e} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & b_2 \\ c_0 & c_1 & c_2 \end{pmatrix}, \quad \alpha_{3,e} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & b_2 \\ c_0 & c_1 & c_2 \end{pmatrix}, \tag{4}$$
$$\alpha_{3,h} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & b_2 \\ c_0 & c_1 & c_2 \end{pmatrix}, \quad \alpha_{3,p} = \begin{pmatrix} 1 & 0 & 0 \\ b_0 & 0 & b_2 \\ c_0 & 1 & c_2 \end{pmatrix},$$

with $|b_2| \le 1$ for $\alpha^1_{3,e}$ and $\alpha_{3,e}$. In each case, the normal forms must have rank 3.

Proof. First, we fix a point $u_0 \in U$ and find a regular matrix S such that

$$S^T \begin{pmatrix} a_0 & a_1 \\ a_1 & a_2 \end{pmatrix} S = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} \pm 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then we have

$$\alpha' = \begin{pmatrix} a'_0 & 0 & a'_2 \\ b'_0 & b'_1 & b'_2 \\ c'_0 & c'_1 & c'_2 \end{pmatrix},$$

where $(a'_0, a'_2) = (\pm 1, \pm 1)$, $(1,-1)$, $(-1,1)$ or $(\pm 1, 0)$. Taking $\Phi = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, we may change the sign of $(a'_0, 0, a'_2)$ to reduce α to the cases in which $(a'_0, a'_2) = (1,1)$, $(1,-1)$

or $(1,0)$. If $b'_1 \neq 0$, taking $\Phi_{2,\theta}$ with $(\cos\theta, \sin\theta) = (c'_1, -b'_1)/\sqrt{b'^2_1 + c'^2_1}$, we obtain

$$\alpha'' = \begin{pmatrix} 1 & 0 & a'_2 \\ b''_0 & 0 & b''_2 \\ c''_0 & c''_1 & c''_2 \end{pmatrix} \quad \text{where } a'_2 = \pm 1 \text{ or } 0. \tag{5}$$

We shall denote α' and α'' by α to simplify the notation, when it is necessary.

By Proposition 3.5, there are three types for α. For each case, we consider the following simplified form:

(a) $\mathcal{M}_3$ ***A*-elliptic case**:

$$\alpha_{3,e} = \begin{pmatrix} 1 & 0 & 1 \\ b_0 & 0 & b_2 \\ c_0 & c_1 & c_2 \end{pmatrix}, \text{ where } c_1 \neq 0 \text{ and } b_0 \neq b_2.$$

By assumption, $\cosh t\mathbf{A} + \sinh t\mathbf{B}$ must be elliptic for all $t \in \mathbb{R}$, then $|b_0| \leq 1$ and $|b_2| \leq 1$. We get the following cases:
(1a) If $|b_0| = 1$, then we have $\alpha^1_{3,e}$ as in (4) where $c_1 \neq 0$ and $|b_2| < 1$ or $b_2 = -1$.

(1b) If $|b_0| < 1$, we can apply Φ_{1t} and appropriated ψ to obtain $\alpha_{3,e}$ as in (4) where $c_1 \neq 0$ and $|b_2| \leq 1$.

(b) $\mathcal{M}_3$ ***A*-hyperbolic case**: In this case, $a'_2 = -1$ in (5) and there is S such that $S^t\mathbf{A}'S$ give us $\mathbf{A}'' = (0,1,0)$. Then

$$\alpha_{3,h} = \begin{pmatrix} 0 & 1 & 0 \\ b_0 & 0 & b_2 \\ c_0 & c_1 & c_2 \end{pmatrix}, \text{ where } b_0c_2 - b_2c_0 \neq 0,$$

and since rank is 3, b_0 or c_0 does not vanish. Assuming that $b_0 \neq 0$, then by acting $\psi = \begin{pmatrix} 1/\sqrt{|b_0|} & 0 \\ 0 & \sqrt{|b_0|} \end{pmatrix}$ and changing the sign of b_0, we obtain $\alpha_{3,h}$ as in (4) where $c_2 - b_2c_0 \neq 0$.

It is not difficult to verify that we can reduce the case that $b_0 = 0$ and $c_0 \neq 0$ to this one.

(c) $\mathcal{M}_3$ ***A*-parabolic case**: In this case, $a'_2 = 0$ in (5), then by acting $\psi = \begin{pmatrix} 1/\sqrt{|c''_1|} & 0 \\ 0 & 1/\sqrt{|c''_1|} \end{pmatrix}$ and changing the sign of (c''_0, c''_1, c''_2) if necessary, we obtain $\alpha_{3,p}$ as in (4) where $b_2 \neq 0$.

□

On the other hand, the number of the asymptotic directions corresponds to the number of the real solutions of $\Delta(u,\theta) = 0$, which will be discussed in further sections.

In the rank three case, it is more difficult to study the number of asymptotic directions, because we need to analyse the type of solutions of a polynomial of degree at most 4. Then we split this analyse in the next sections.

7 Equation of the asymptotic directions

In this section we obtain the equation of the asymptotic directions associated to the lightlike binormal directions on spacelike surface in de Sitter 5-space in terms of the coefficients of the second fundamental form. The discriminant $\Delta(u,\theta) = M(u,\theta)^2 - L(u,\theta)N(u,\theta)$ can be written as

$$\Delta(u,\theta) = d_1(u) + d_2(u)\cos\theta + d_3(u)\sin\theta + d_4(u)\cos^2\theta + d_5(u)\cos\theta\sin\theta + d_6(u)\sin^2\theta,$$

where $d_1 = a_1^2 - a_0a_2$, $d_2 = 2a_1b_1 - a_0b_2 - a_2b_0$, $d_3 = 2a_1c_1 - a_0c_2 - a_2c_0$, $d_4 = b_1^2 - b_0b_2$, $d_5 = 2b_1c_1 - b_0c_2 - b_2c_0$ and $d_6 = c_1^2 - c_0c_2$. The lightlike direction $\mathbf{e}(u,\theta)$ (more precisely, $\bar{\mathbf{e}}(u,\theta)$) is a binormal if and only if $\Delta(u,\theta) = 0$. This equation can be written as

$$(1 \ \cos\theta \ \sin\theta) \begin{pmatrix} d_1 & d_2/2 & d_3/2 \\ d_2/2 & d_4 & d_5/2 \\ d_3/2 & d_5/2 & d_6 \end{pmatrix} \begin{pmatrix} 1 \\ \cos\theta \\ \sin\theta \end{pmatrix} = 0. \tag{6}$$

If there is some θ which satisfies the above equation (6), then we find the asymptotic directions $(\cos\varphi, \sin\varphi)$ associated to θ by solving the following equations:

$$(\cos\varphi \ \sin\varphi) \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} = 0,$$

where $L(u,\theta)$, $M(u,\theta)$, $N(u,\theta)$ are the coefficients of the second fundamental form with respect to the lightlike direction $\mathbf{e}(u,\theta)$. This equation can be written as

$$\nabla(u,\varphi,\theta) := (\cos^2\varphi \ \ 2\sin\varphi\cos\varphi \ \ \sin^2\varphi) \begin{pmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} \begin{pmatrix} 1 \\ \cos\theta \\ \sin\theta \end{pmatrix} = 0 \tag{7}$$

We write $A(u,\varphi)$, $B(u,\varphi)$ and $C(u,\varphi)$ as

$$(A(u,\varphi) \ B(u,\varphi) \ C(u,\varphi)) \ := \ (\cos^2\varphi \ \ 2\sin\varphi\cos\varphi \ \ \sin^2\varphi) \begin{pmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}.$$

In order to eliminate the parameter θ, we substitute $x = cos(\theta)$, $y = sin(\theta)$. The equation (7) is $A + Bx + Cy = 0$ that gives $x = -(A + Cy)/B$. We now replace x by this expression in Δ. We take the resultant in y of the new $\Delta(\varphi, y)$ and $x^2 + y^2 = 1$ (where the variable x is as above). By using a mathematical software (such as Maxima, Mathematica or Maple), if $B \neq 0$, we obtain the following equation:

$${d_4}^2 C^4 + 2 d_1 d_4 C^4 - {d_2}^2 C^4 + {d_1}^2 C^4 - 2 d_4 d_5 B C^3 - 2 d_1 d_5 B C^3 + 2 d_2 d_3 B C^3 + 2 d_2 d_5 A C^3 - 2 d_3 d_4 A C^3 - 2 d_1 d_3 A C^3 + 2 d_4 d_6 B^2 C^2 + 2 d_1 d_6 B^2 C^2 + {d_5}^2 B^2 C^2 + 2 d_1 d_4 B^2 C^2 - {d_3}^2 B^2 C^2 - {d_2}^2 B^2 C^2 + 2 {d_1}^2 B^2 C^2 - 4 d_2 d_6 A B C^2 + 2 d_2 d_4 A B C^2 - 2 d_1 d_2 A B C^2 + 2 d_4 d_6 A^2 C^2 + 2 d_1 d_6 A^2 C^2 - {d_5}^2 A^2 C^2 - 2 {d_4}^2 A^2 C^2 - 2 d_1 d_4 A^2 C^2 + {d_3}^2 A^2 C^2 + {d_2}^2 A^2 C^2 - 2 d_5 d_6 B^3 C - 2 d_1 d_5 B^3 C + 2 d_2 d_3 B^3 C + 2 d_3 d_6 A B^2 C - 4 d_3 d_4 A B^2 C - 2 d_1 d_3 A B^2 C +$$

$2\,d_5\,d_6\,A^2\,B\,C+2\,d_4\,d_5\,A^2\,B\,C+4\,d_1\,d_5\,A^2\,B\,C-2\,d_3\,d_6\,A^3\,C-2\,d_2\,d_5\,A^3\,C+2\,d_3\,d_4\,A^3\,C+ {d_6}^2\,B^4+2\,d_1\,d_6\,B^4-{d_3}^2\,B^4+{d_1}^2\,B^4-2\,d_2\,d_6\,A\,B^3+2\,d_3\,d_5\,A\,B^3-2\,d_1\,d_2\,A\,B^3-2\,{d_6}^2\,A^2\,B^2+ 2\,d_4\,d_6\,A^2\,B^2-2\,d_1\,d_6\,A^2\,B^2-{d_5}^2\,A^2\,B^2+2\,d_1\,d_4\,A^2\,B^2+{d_3}^2\,A^2\,B^2+{d_2}^2\,A^2\,B^2+2\,d_2\,d_6\,A^3\,B- 2\,d_3\,d_5\,A^3\,B-2\,d_2\,d_4\,A^3\,B+{d_6}^2\,A^4-2\,d_4\,d_6\,A^4+{d_5}^2\,A^4+{d_4}^2\,A^4=0.$

Since A,B,C are expressed by homogeneous trigonometric functions $sin\varphi$ and $\cos\varphi$ of order 2, then the above equation is a trigonometric homogeneous equation of order up to 8. We remark that if $(\sin\varphi,\cos\varphi)$ is one solution of the above equation then $(-\sin\varphi,-\cos\varphi)$ is also solution, so there are at most four asymptotic directions. If we substitute A,B,C and $d_1,\ldots,d_6$ in this equation, then we obtain the following completed square trigonometric homogeneous polynomial: $T^2(u,\varphi)$.

$$T^2(u,\varphi)=(k_4(u)\cos^4\varphi+k_3(u)\cos^3\varphi\sin\varphi+\cdots+k_0(u)\sin^4\varphi)^2=0 \tag{8}$$

with

$$\begin{aligned}
k_4(u) &= |a_2^*|^2-|b_2^*|^2-|c_2^*|^2\\
k_3(u) &= -2(|a_1^*||a_2^*|-|b_1^*||b_2^*|-|c_1^*||c_2^*|)\\
k_2(u) &= |a_1^*|^2+2|a_0^*||a_2^*|-|b_1^*|^2-2|b_0^*||b_2^*|-|c_1^*|^2-2|c_0^*||c_2^*|\\
k_1(u) &= -2(|a_0^*||a_1^*|-|b_0^*||b_1^*|-|c_0^*||c_1^*|)\\
k_0(u) &= |a_0^*|^2-|b_0^*|^2-|c_0^*|^2
\end{aligned}$$

where $|a_i^*|$, $|b_i^*|$ and $|c_i^*|$ are cofactor matrices of α and $rank\ \alpha\geq 2$. We consider, without loss of generality that p is of type M_3. This is not restrictive as M_2-points form a curve on a generic surface M. So the equation obtained at M_3-points is also valid at M_2-points by passing to the limit. Notice that when $rank\ \alpha=1$, $T(u,\varphi)$ always vanishes.

With the previous calculations we prove the next result.

Theorem 7.1. There is at most four asymptotic directions passing through any non inflection point and non conic point, on a generic spacelike surface in S_1^5. These directions are solutions of the implicit differential equation

$$k_0du_2^4+k_1du_1du_2^3+k_2du_1^2du_2^2+k_3du_1^3du_2+k_4du_1^4=0,$$

where the coefficients $k_i, i=0,1,2,3,4$ depend on the coefficients of the second fundamental form, and are given above. If $T(u,\varphi)\equiv 0$ at $p=\mathbf{X}(u)\in M$, then p is an inflection point.

8 The types of asymptotic and binormal directions

We are interested in analyzing the number of lightlike binormal directions and their associated asymptotic directions. To do this it is easier to analyze the LMN-ellipse, as we justify in Section 2. This ellipse is described by using the coefficients of the second fundamental form, using the normal forms of $\alpha(u)$ from Sections 4, 5, 6.

Remember that a direction $\theta \in S^1$ is a *lightlike binormal* at $p = \mathbf{X}(u)$ if $\Delta(u,\theta) = 0$. In this case we call a kernel direction of $I\!I_{\mathbf{e}(u,\theta)}$ by *asymptotic direction* associated to the lightlike binormal. As before, we say simply binormal directions and asymptotic directions. For the cases where α has rank 1 or 2, we analyzed the binormals according to their $(m_1 + m_2 + m_3 + m_4)$-type with $m_1 + \cdots + m_4 = 0, 2, 3, 4$ or S^1-type, in Sections 4 and 5. We have also studied the number of the asymptotic directions, in each case.

If rank $\alpha = 3$, then $T(u,\varphi) = 0$ if and only if φ is an asymptotic direction. According to the number of real solutions of a polynomial of degree at most 4, the asymptotic directions are of the following types: (4), (3+1), (2+2), (2+1+1), (1+1+1+1), (2), (1+1), (0) or of a *conic type* (i.e. the LMN-ellipse $E(u,\theta)$ is on the cone, and all lightlike normal directions are binormals and all tangent directions are asymptotic). If $T(u,\varphi) \equiv 0$, then $p = \mathbf{X}(u)$ is a conic point. It is not an inflection point because $E(u,\theta)$ does not pass through the origin when the rank is 3. We can distinguish the cases by the solution types of the equations $T(u,\varphi) = 0$ or $\Delta(u,\theta) = 0$, because if *rank* $\alpha = 3$, for each binormal θ, there is only one asymptotic direction with same multiplicity.

We now use the results of Appendix A to relate the solutions of $\Delta(u,\theta) = d_1(u) + d_2(u)\cos\theta + d_3(u)\sin\theta + d_4(u)\cos^2\theta + d_5(u)\sin\theta\cos\theta + d_6(u)\sin^2\theta = 0$ with the roots of the trigonometric polynomial $F_2(t)$. Let $\cos\theta = \frac{1-t^2}{1+t^2}$ and $\sin\theta = \frac{2t}{1+t^2}$. Then $\Delta(u,\theta(t)) = \frac{1}{(1+t^2)^2}F_2(t)$, with $t = \tan(\frac{\theta}{2})$, where $F_2(t) = At^4 + Bt^3 + Ct^2 + Dt + E$ with $A = d_1 - d_2 + d_4$, $B = 2d_3 - 2d_5$, $C = 2d_1 - 2d_4 + 4d_6$, $D = 2d_3 + 2d_5$ and $E = d_1 + d_2 + d_4$.

Lemma 8.1. The solutions of $\Delta(u,\theta(t)) = 0$ correspond to the roots of the t-polynomial F_2 adding possibly the solution $\theta = \pi$. More precisely if $\theta = \pi$ is a solution, its multiplicity is equal to $4 - \deg F_2(t)$.

Proof. Use Lemma A.2, of the Appendix A. □

We have the following result.

Proposition 8.2. Consider α with *rank* 3.
(a) The finite number of binormal directions and their multiplicities are given in tables 6, 7, and 8.
(b)The number of asymptotic directions (and their multiplicities) is the same number of binormal directions (and their multiplicites). All the possibilities for the finite number of asymptotic directions are described in tables 6, 7, or 8.
(c) At a conic point all normal lightlike directions are binormals and all tangent directions are asymptotics.

Proof. (a) To discuss the number of solutions of $\Delta(u,\theta) = 0$, as in Lemma 8.1, we change coordinates and in these new coordinates we obtain the associated t-polynomial $F_2(t)$, defined in Appendix A, where $\Delta(u,\theta(t)) = \frac{1}{(1+t^2)^2}F_2(t)$, with $t = \tan(\frac{\theta}{2})$. Since the degree of the t-polynomial $F_2(t)$ is at most four, we have the following classification. For next tables, $\underline{n}$ means the multiplicity of binormal direction $\theta = \pi$ as the solution of $\Delta(u,\theta) = 0$, $*$ means that we do not care about the value and N/A means no applicable.

(1) Suppose that $\deg F_2(t) \leq 2$, then $\theta = \pi$ is binormal direction with multiplicity $4 - \deg F_2(t) \geq 2$, the t-polynomial is written by $F_2(t) = Ct^2 + Dt + E$. If $\deg F_2(t) = 2$, the discriminant is given by $D_{quadratic} = -D^2 + 4CE$. Then the types of binormal directions (BD), and the number of imaginary solutions (NI) of $F_2(t) = 0$ are given in the following table.

BD	NI	$D_{quadric}$	(C, D, E)	deg$F_2(t)$
$\underline{2}$	2	$+$	$C \neq 0$	2
$\underline{2}+1+1$	0	$-$	$C \neq 0$	2
$\underline{2}+2$	0	0	$C \neq 0$	2
$\underline{3}+1$	0	N/A	$C = 0,\ D \neq 0$	1
$\underline{4}$	0	N/A	$C = D = 0,\ E \neq 0$	0
$\sharp S^1$	N/A	N/A	$C = D = E = 0$ i.e. $F_2 \equiv 0$	N/A

Table 6: Types of $F_2(t) = Ct^2 + Dt + E = 0$

(2) Suppose that $\deg F_2(t) = 3$, then we have a cubic equation $F_2(t) = Bt^3 + Ct^2 + Dt + E = 0$ with $B \neq 0$. We have the binormal direction $\theta = \pi$ with multiplicity 1. In this case we have two discriminants:

$$\begin{aligned} D_{cubic,1} &= B^2E^2 + \frac{4}{27}(C^3E + BD^3) - \frac{2}{3}BCDE - \frac{1}{27}C^2D^2, \\ D_{cubic,2} &= 3BD - C^2, \end{aligned}$$

where $D_{cubic,2}$ is used to determine the existence of triple solutions. Then the types of binormal directions (BD), and the number of imaginary solutions (NI) of $F_2(t) = 0$ are given in the following table.

BD	NI	$D_{cubic,1}$	$D_{cubic,2}$
$\underline{1}+1+1+1$	0	$-$	*
$\underline{1}+1$	2	$+$	*
$2+\underline{1}+1$	0	0	$\neq 0$
$3+\underline{1}$	0	0	0

Table 7: Table of types on $F_2(t) = Bt^3 + Ct^2 + Dt + E = 0$ $(B \neq 0)$

(3) Finally we consider the case of the quartic equation $\deg F_2(t) = 4$, $A \neq 0$. In this case $\theta = \pi$ is not a binormal direction. Dividing by A and changing coordinate, we use the simplified form $\bar{F}_2(x) = x^4 + 6Hx^2 + 4Gx + (I - 3H^2)$ where $I = \frac{E}{A} - \frac{BD}{4A^2} + \frac{C^2}{12A^2}$, $G = \frac{D}{4A} - \frac{BC}{8A^2} + \frac{B^3}{32A^3}$ and $H = \frac{C}{6A} - \frac{B^2}{16A^2}$. Then the discriminant of $\bar{F}_2(x) = 0$ is written by $D_{quartic} = I^3 - 27(HI - 4H^3 - G^2)^2$. To apply the classification in [2], the types of binormal directions (BD), and the number of imaginary solutions (NI) of $\bar{F}_2(x) = 0$ are given in the following table. (See also Figure 10.)

No.	BD	NI	Condition
(1)	4	0	$I = G = H = 0$
(2)	3+1	0	$I = G^2 + 4H^3 = 0$ except for $G = H = 0$
(3)	1+1	2	$D_{quartic} < 0$
(4)	1+1+1+1	0	$I > 0,\ D_{quartic} > 0,\ H < -\frac{\sqrt{I}}{2\sqrt{3}}$
(5)	0	4 (distinct)	$I > 0,\ D_{quartic} > 0,\ H > -\frac{\sqrt{I}}{2\sqrt{3}}$
(6)	2+1+1	0	$I > 0,\ D_{quartic} = 0,\ H < -\frac{\sqrt{I}}{2\sqrt{3}}$
(7)	2+2	0	$I > 0,\ D_{quartic} = 0,\ H = -\frac{\sqrt{I}}{2\sqrt{3}}$
(8)	2	2	$I > 0,\ D_{quartic} = 0,\ H > -\frac{\sqrt{I}}{2\sqrt{3}}$ except for case (9)
(9)	0	4 (multiple)	$I > 0,\ (H, G) = (\frac{\sqrt{I}}{2\sqrt{3}}, 0)$ In this case we have $D_{quartic} = 0$ automatically

Table 8: Table of types on $\bar{F}_2(x) = x^4 + 6Hx^2 + 4Gx + (I - 3H^2) = 0$

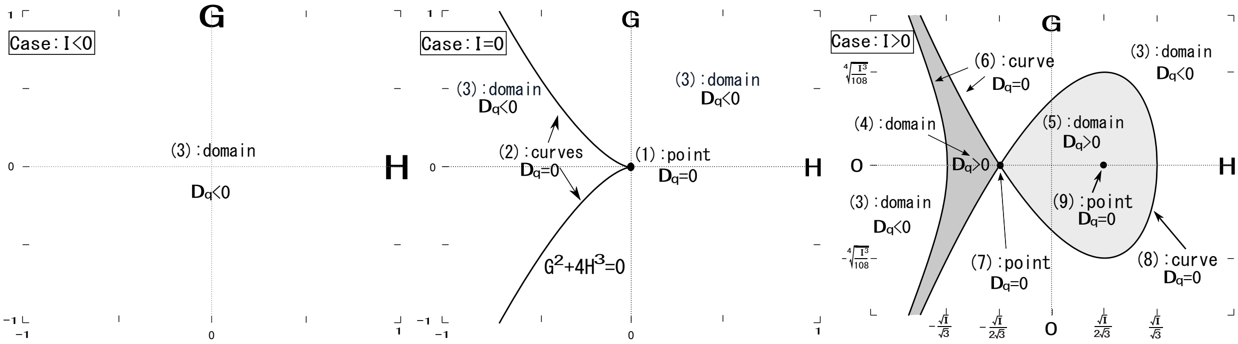

Figure 10: Figures of types of $\bar{F}_2(x) = 0$ ($D_q = D_{quadratic}$ for short)

□

As a consequence, given a pre-normal form of Section 6, with rank 3, to study the associated number of asymptotic directions we can study the equation of binormal diretions by using the conditions given in tables 6, 7, or 8.

Let

$$\alpha(u) := \begin{pmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ c_0 & c_1 & c_2 \end{pmatrix},$$

be any 3×3-matrix. Then, there exists a local embedding $\mathbf{X} : U \subset \mathbb{R}^2 \longrightarrow S^5_1 \subset \mathbb{R}^6_1$ such that α is the matrix of the second fundamental form of $\mathbf{X}$. In fact, we can take $f(u_1, u_2) = a_0 u_1^2 + 2a_1 u_1 u_2 + a_2 u_2^2 + h.o.t$, $g(u_1, u_2) = b_0 u_1^2 + 2b_1 u_1 u_2 + b_2 u_2^2 + h.o.t$ and $h(u_1, u_2) = c_0 u_1^2 + 2c_1 u_1 u_2 + c_2 u_2^2 + h.o.t$, where h.o.t. means higher order terms. Then the Monge form of a spacelike surface in de Sitter 5-space is given by

$$\mathbf{X}(u) = \left(f(u_1, u_2), \sqrt{1 + f^2 - g^2 - h^2 - u_1^2 - u_2^2}, u_1, u_2, g(u_1, u_2), h(u_1, u_2) \right).$$

We give some examples below.

Example 8.3. (1) If $f(u_1,u_2) = \frac{1}{2}u_1^2$, $g(u_1,u_2) = \frac{1}{2}u_2^2$, $h(u_1,u_2) = u_1u_2$ then $\mathbf{A} = (1,0,0)$, $\mathbf{B} = (0,0,1)$, $\mathbf{C} = (0,1,0)$ and the asymptotic directions at $u = (u_1,u_2) = (0,0)$ are given by $du_2^4 - du_1^2du_2^2 - du_1^4 = 0$ (Theorem 7.1), with two simple real solutions.

(2) If $f(u_1,u_2) = \frac{1}{2}(u_1^2+u_2^2)$, $g(u_1,u_2) = \frac{1}{2}(u_1^2-u_2^2)$, $h(u_1,u_2) = u_1u_2$ then at $u = (0,0)$, $\mathbf{A} = (1,0,1)$, $\mathbf{B} = (1,0,-1)$, $\mathbf{C} = (0,1,0)$ and all directions are asymptotic, this is the conic case.

(3) If $f(u_1,u_2) = -\frac{1}{2}u_1^2$, $g(u_1,u_2) = \frac{1}{2}u_1^2 - u_1u_2 + \frac{1}{4}u_2^2$, $h(u_1,u_2) = \frac{1}{200}u_1u_2$ then $\mathbf{A} = (-1,0,0)$, $\mathbf{B} = (0,-1,1/2)$, $\mathbf{C} = (0,1/100,0)$ and the asymptotic directions are given by $\frac{1}{10000}du_2^4 + \frac{1}{5000}du_1du_2^3 - \frac{1}{4}du_1^2du_2^2 - du_1^3du_2 - du_1^4 = 0$. To analyse the type of solutions we obtain the equation of binormal directions $\Delta(u,\theta(t)) = \frac{1}{(1+t^2)^2}F_2(t)$ where $F_2(t) = 1 + (1/25)t^3 - t^2$. Since $A = 0$ and $D_{cubic,1} = -2473/27 < 0$ then we conclude that there are 4 distinct binormal directions, consequently 4 distinct asymptotic directions (Table 7).

Appendix

A Solutions of trigonometric equation

Let

$$f(c,s) := \sum_{j+k=0}^{m} \alpha_{j,k}c^js^k,$$

with $c = \cos(\theta)$ and $s = \sin(\theta)$ a trigonometric polynomial. We say that f is a polynomial of order $m \geq 1$ if the polynomial part of order m is not zero by substituting $c^2 = 1 - s^2$, that is f satisfies

$$f_1 := \sum_{j=0}^{\left[\frac{m}{2}\right]} \alpha_{m-2j,2j} \times (-1)^j \neq 0 \text{ or } f_2 := \sum_{j=0}^{\left[\frac{m-1}{2}\right]} \alpha_{m-1-2j,2j+1} \times (-1)^j \neq 0,$$

where f_1 is the coefficient of s^m and f_2 is the coefficient of cs^{m-1} and they are the only degree m terms after this substitution.

By substituting $c = \frac{1-t^2}{1+t^2}$ and $s = \frac{2t}{1+t^2}$ (then $t = \tan(\theta/2)$) we get $\frac{1}{(1+t^2)^m}F_m(t)$ and removing the denominator we get an associated t-polynomial $F_m(t)$ of f

$$F_m(t) := \sum_{j+k=0}^{m} \alpha_{j,k}(1-t^2)^j(2t)^k(1+t^2)^{m-j-k},$$

with order less or equal to $2m$. We observe that $(1+t^2)$ is not a common factor of $F_m(t)$ if $\alpha_{j,k} \neq 0$ for some coefficients with $j+k = m$ because $m-j-k = 0$. In this case, $F_m(t) = 0$

does not have a solution $\pm i$. We now consider the relations between $f(\cos\theta, \sin\theta) = 0$ and $F_m(t) = 0$, first of all, we prove the following lemma.

Lemma A.1. Let $\bar{f}(\theta) := f(\cos\theta, \sin\theta)$ be a trigonometric polynomial of order m and F_m be an associated t-polynomial of f. For any $1 \le n \le 2m$, the following conditions are equivalent.

(1) $\ell_{2m} = \ell_{2m-1} = \cdots = \ell_{2m-(n-1)} = 0$ and $\ell_{2m-n} \neq 0$, where ℓ_k are coefficients of $F_m(t)$.

$$F_m(t) = \sum_{k=0}^{2m} \ell_k t^k.$$

(2) The germ $\bar{f} : (S^1, \pi) \longrightarrow \mathbb{R}$ is $\mathcal{K}$-equivalent to x^n at $x = 0$, that is

$$\bar{f}(\pi) = \frac{\partial \bar{f}}{\partial\theta}(\pi) = \cdots = \frac{\partial^{n-1}\bar{f}}{\partial\theta^{n-1}}(\pi) = 0 \text{ and } \frac{\partial^n \bar{f}}{\partial\theta^n}(\pi) \neq 0.$$

Proof. Let $g(c,s) = f(-c,s) = \sum_{j,k=1}^m \alpha_{j,k}(-c)^j s^k$. Since $g(c,s) = f(-c,s)$ is a trigonometric polynomial of order m, then its associate t-polynomial is given by

$$\widetilde{G}_m(t) := \sum_{j+k=0}^{m} \alpha_{j,k}(-1+t^2)^j(2t)^k(1+t^2)^{m-j-k}.$$

By computation, $t^{2m} \times \widetilde{G}_m(1/t) = F_m(t)$ and we have $\widetilde{G}_m(t) = \sum_{k=0}^{2m} \ell_{2m-k} t^k$.

Let $\Phi(\theta) = \tan(\theta/2)$ then we have $G_m(t) = (1+t^2)^m \ (\bar{g} \circ \Phi^{-1})(t)$, so that $\bar{g}(\theta) := g(\cos\theta, \sin\theta)$ at $\theta = 0$ is locally $\mathcal{K}$-equivalent to $G_m(t)$ at $t = 0$. This means that (2) is equivalent to the condition that $G_m(t)$ at $t = 0$ is $\mathcal{K}$-equivalent to x^n at $x = 0$. By the coefficients of $G_m(t)$, this condition is also equivalent to the condition (1). This completes the proof. ☐

Now we may conclude the relations between the trigonometric polynomial and its t-polynomial.

Lemma A.2. (Property of the t-polynomial of trigonometric polynomial) Let $f(c,s)$ be a trigonometric polynomial of degree m. If $\deg F_m(t) = 2m-n$ and $F_m(t) = 0$ has k-distinct solutions $t_1, \ldots, t_k \in \mathbb{R}$ and ℓ-distinct solutions $t_{k+1}, \ldots, t_{k+\ell} \in \mathbb{C}\setminus\mathbb{R}$ with the multiplicity $m_1, \ldots, m_{k+\ell}$ (where $m_1 + \cdots + m_{k+\ell} = 2m - n$) then we have:

(1) $t_j \neq \pm i$ for all $j = k+1, \ldots, k+\ell$.

(2) $f(\cos\theta, \sin\theta) = 0$ has $(k+1)$-distinct solutions $\theta_1, \ldots, \theta_k \in S^1$ and $\pi \in S^1$ with the multiplicities $m_1, \ldots, m_k$ and n.

(3) $f(c,s) = 0$ has ℓ-distinct complex solutions $(c_1, s_1), \ldots, (c_\ell, s_\ell) \in \mathbb{C}^2\setminus\mathbb{R}^2$ on $c^2 + s^2 = 1$ with the multiplicities $m_{k+1}, \ldots, m_{k+\ell}$.

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Department of Mathematics
Hokkaido University
Sapporo 060-0810, Japan
E-mail: kasedou@math.sci.hokudai.ac.jp

Departamento de Matemática
Instituto de Ciências Matemáticas e de Computação
Universidade de São Paulo - Campus de São Carlos
Caixa Postal 668
13560-970 São Carlos, SP, Brazil
E-mail: anaclana@icmc.usp.br
E-mail: maasruas@icmc.usp.br