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ON THE CONTINUATION OF SOLUTIONS OF NONAUTONOMOUS SEMILINEAR PARABOLIC PROBLEMS

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Abstract. In this paper we study evolutionary problems which fall into a class of nonautonomous parabolic equations with critical exponents in a scale of Banach spaces E_σ, $\sigma \in [0, 1+\mu)$. We consider a suitable notion of $E_{1+\varepsilon}$-solution and describe continuation properties of the solution. This concerns both a situation when the solution can be continued as $E_{1+\varepsilon}$-solution and a situation when $E_{1+\varepsilon}$-norm of the solution 'blows-up', in which case a piecewise-$E_{1+\varepsilon}$-solution is constructed. This extends the existing results to essentially larger class of parabolic problems.

1. Introduction

In this paper, given a family of unbounded linear operators in the Banach space E_0, $A(t) : D_{E_0} \subset E_0 \longrightarrow E_0$, $t \in \mathbb{R}$, we focus on well posedness of a Cauchy problem of the form

$$\dot{u}(t) + A(t)u(t) = F(t, u(t)),\ t > \tau, \quad u(\tau) = u_\tau, \tag{1.1}$$

where the linear operator appearing on the left hand side in (1.1) essentially depends on the time variable.

Following the pioneering work [26] such problem has been considered by many authors in wide generality and many related results have been obtained (see e.g. the monographs [3, 20, 22, 23, 27, 30] and references therein). Here our main concern will be *critically growing nonlinearities*, that is roughly speaking we will allow $F(t, u(t))$ to exhibit the same order of magnitude as the linear main part operator $A(t)$ (see [9, 13, 15, 29]).

For nonlinearities that behave in a subcritical manner continuation of solutions is satisfactorily described both for autonomous and nonautonomous problems (see e.g. [6]). As observed in [29] this is no longer the case for nonlinearities satisfying a critical growth condition (see also [13, (1.5)-(1.6)]). On the other hand, some previous results concerning continuation properties of solutions of autonomous problems, see [13], cannot be directly applied to (1.1) and require essential modifications. This will be our main goal in the present paper.

To describe our results we start from the following two general assumptions. Conditions sufficient for them in terms of the operators $A(t)$ will be discussed in detail in Section 3.

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Assumption 1.1. *Given a family of Banach spaces* $\{E_\alpha, \alpha \in [0,1+\mu)\}$ *there exists a continuous process* $\{U(t,\tau): \ t \geqslant \tau \in \mathbb{R}\} \subset L(E_0)$ *in* E_0 *such that given* $\tau \in \mathbb{R}$ *and* $u_\tau \in E_0$, *the map* $[\tau,\infty) \ni t \to u(t) = U(t,\tau)u_\tau \in E_0$ *is a classical solution of the linear problem*

$$\dot{u}(t) + A(t)u(t) = 0,\ t > \tau, \quad u(\tau) = u_\tau.$$

Furthermore, given any point $t_0 \in \mathbb{R}$ *there is a time interval* $I \subset \mathbb{R}$ *centered at* t_0 *such that for any* $1+\mu > \sigma \geqslant \zeta \geqslant 0$ *a constant* $M > 0$ *exists for which*

$$\|U(t,\tau)\|_{L(E_\zeta,E_\sigma)} \leqslant M(t-\tau)^{\zeta-\sigma},\ t,\tau \in I,\ t > \tau. \tag{1.2}$$

Assumption 1.2. *Given* $t_0 \in \mathbb{R}$ *there is also a time interval* $I \subset \mathbb{R}$ *centered at* t_0 *such that whenever* $1+\mu > \zeta > \sigma \geqslant 0,\ 1 \geqslant \zeta - \sigma > 0$ *a constant* $M > 0$ *exists for which*

$$\|U(t,\tau) - Id\|_{L(E_\zeta,E_\sigma)} \leqslant M(t-\tau)^{\zeta-\sigma},\ t,\tau \in I,\ t > \tau. \tag{1.3}$$

Concerning the right hand side in (1.1), we will assume that F belongs to a class of maps $\mathcal{L}(\varepsilon,\rho,\gamma(\varepsilon),\eta,C_\eta)$ satisfying a suitable Lipschitz condition relative to $\{E_\alpha, \alpha \in [0,1+\mu)\}$. Note that any such F falls in particular into the class of ε-regular maps considered in [9].

Definition 1.3. *We say that a continuous function* $F: \mathbb{R} \times E_{1+\varepsilon} \to E_{\gamma(\varepsilon)}$ *is of the class* $\mathcal{L}(\varepsilon,\rho,\gamma(\varepsilon),\eta,C_\eta)$ *of Lipschitz maps relative to* $\{E_\alpha, \alpha \in [0,1+\mu)\}$ *with constants* $\rho > 1$, $0 < \varepsilon < \min\{\frac{1}{\rho},\mu\}$, $\gamma(\varepsilon) \in [\rho\varepsilon, 1)$, $\eta > 0$ *and* $C_\eta > 0$ *if and only if for any bounded time interval* $I \subset \mathbb{R}$ *there exists* $c > 0$ *such that for each* $v, w \in E_{1+\varepsilon}$, $t \in I$ *we have*

$$\|F(t,v) - F(t,w)\|_{E_{\gamma(\varepsilon)}} \leqslant c\|v-w\|_{E_{1+\varepsilon}}(\eta\|v\|^{\rho-1}_{E_{1+\varepsilon}} + \eta\|w\|^{\rho-1}_{E_{1+\varepsilon}} + C_\eta) \tag{1.4}$$

and

$$\|F(t,v)\|_{E_{\gamma(\varepsilon)}} \leqslant c(\eta\|v\|^{\rho}_{E_{1+\varepsilon}} + C_\eta). \tag{1.5}$$

We single out for special attention the case when in (1.4)-(1.5) one has $\gamma(\varepsilon) = \rho\varepsilon$ and not $\gamma(\varepsilon) \in (\rho\varepsilon, 1)$ as it exhibits criticality of F relative to (E_1, E_0) (see [9]).

Definition 1.4. *In the case when for a certain* $\eta > 0$ (1.4)-(1.5) *hold with* $\gamma(\varepsilon) \in (\rho\varepsilon, 1)$ *we say that* F *is* subcritical. *When for a certain* $\eta > 0$ (1.4)-(1.5) *hold with* $\gamma(\varepsilon) = \rho\varepsilon$ *but not with* $\gamma(\varepsilon) \in (\rho\varepsilon, 1)$, F *is called* critical *and* ρ *is called a* critical exponent. *In the case when* F *is critical and* (1.4)-(1.5) *hold with any* $\eta > 0$ *we say that* F *is an* almost critical *map.*

We will consider the following notion of solution (see [15]; also [9, 13]).

Definition 1.5. *Given* F *of the class* $\mathcal{L}(\varepsilon,\rho,\gamma(\varepsilon),\eta,C_\eta)$ *relative to* $\{E_\alpha, \alpha \in [0,1+\mu)\}$, $\tau > 0$ *and* $u_\tau \in E_0$ *we say that* $u: [\tau,T] \to E_0 \cup E_{1+\varepsilon}$ *is a mild* $E_{1+\varepsilon}$*-solution (*$E_{1+\varepsilon}$*-solution for short) of* (1.1) *on the interval* $[\tau,T]$ *if and only if* $u \in L^\infty_{loc}((\tau,T],E_{1+\varepsilon})$, *there exists the limit* $\lim_{t\to\tau^+}(t-\tau)^\varepsilon\|u(t)\|_{E_{1+\varepsilon}} = 0$, $u(\tau) = u_\tau$ *and for* $t \in (\tau,T]$ *we have*

$$u(t) = U(t,\tau)u_\tau + \int_\tau^t U(t,s)F(s,u(s))ds. \tag{1.6}$$

If, given $a \in (\tau,\infty]$, u *is an* $E_{1+\varepsilon}$*-solution of* (1.1) *on* $[\tau,T]$ *for any* $T \in (\tau,a)$, *then we say that* u *is an* $E_{1+\varepsilon}$*-solution on the interval* $[\tau,a)$.

With these assumptions $E_{1+\varepsilon}$-solution will be unique and Hölder continuous away from τ.

Theorem 1.6. *Suppose that Assumptions 1.1, 1.2 hold, F is of the class $\mathcal{L}(\varepsilon,\rho,\gamma(\varepsilon),\eta,C_\eta)$ relative to $\{E_\alpha,\alpha\in[0,1+\mu)\}$, $\tau\in\mathbb{R}$ and $u_\tau\in E_0$.*

Then there exists at most one $E_{1+\varepsilon}$-solution $u=u(\cdot,\tau,u_\tau)$ of (1.1) on $[\tau,T]$ and $u\in C^\nu_{loc}((\tau,T],E_{1+\theta})$ for any $0<\theta<\min\{\gamma(\varepsilon),\mu\}$, $0<\nu<\nu^*=\min\{\gamma(\varepsilon),\mu\}-\theta$.

To describe a set of initial data for which (1.1) has the unique $E_{1+\varepsilon}$-solution we will consider a linear subspace $\mathfrak{E}^\tau_\varepsilon$ of E_0

$$\mathfrak{E}^\tau_\varepsilon=\{\varphi\in E_0:\ \text{there exists}\ \lim_{t\to\tau^+}(t-\tau)^\varepsilon\|U(t,\tau)\varphi\|_{E_{1+\varepsilon}}=0\}. \tag{1.7}$$

We also define, for some $\delta>0$,

$$\|\varphi\|^{\mathfrak{E}^\tau_\varepsilon}_\delta=\sup_{t\in(\tau,\tau+\delta]}(t-\tau)^\varepsilon\|U(t,\tau)\varphi\|_{E_{1+\varepsilon}},\ \varphi\in\mathfrak{E}^\tau_\varepsilon \tag{1.8}$$

and

$$B^\delta_{\mathfrak{E}^\tau_\varepsilon}(w_0,r)=\{\varphi\in\mathfrak{E}^\tau_\varepsilon:\|\varphi-w_0\|^{\mathfrak{E}^\tau_\varepsilon}_\delta<r\},\ w_0\in\mathfrak{E}^\tau_\varepsilon. \tag{1.9}$$

With the above set-up we first state the local well posedness result, which complements earlier consideration of [9, Theorem 1], [13, Theorem 2.1] and [15, Theorem 3.1]. In what follows $B(a,b)=\int_0^1 s^{a-1}(1-s)^{b-1}ds$, $a,b>0$, denotes Euler's Beta function and

$$B_{\varepsilon,\rho}:=\max\{B(1-\rho\varepsilon,\gamma(\varepsilon)-\varepsilon),B(\gamma(\varepsilon)-\varepsilon,1-\varepsilon)\}. \tag{1.10}$$

Theorem 1.7. *Suppose that Assumptions 1.1, 1.2 are satisfied and that F is of the class $\mathcal{L}(\varepsilon,\rho,\gamma(\varepsilon),\eta,C_\eta)$ relative to $\{E_\alpha,\alpha\in[0,1+\mu)\}$.*

Then,

i) given $t_0\in\mathbb{R}$, $w_0\in\mathfrak{E}^\tau_\varepsilon$ and given τ in a certain interval $\mathcal{J}\subset\mathbb{R}$ centered at t_0 there exist $\bar\delta_0\in(0,1]$ and $\bar r_0=\frac{1}{4(8c\eta MB_{\varepsilon,\rho})^{\frac{1}{\rho-1}}}$, where $M=M(1+\varepsilon,\gamma(\varepsilon),\mathcal{J})$, $B_{\varepsilon,\rho}$ are as in (1.2), (1.10), such that for any initial condition u_τ satisfying

$$u_\tau\in B^{\delta_0}_{\mathfrak{E}^\tau_\varepsilon}(w_0,r) \tag{1.11}$$

with

$$\delta_0\in(0,\bar\delta_0]\quad\text{and}\quad r\in(0,\bar r_0] \tag{1.12}$$

there exists the unique $E_{1+\varepsilon}$-solution $u=u(\cdot,\tau,u_\tau)$ of (1.1) on $[\tau,\tau+\delta_0]$.

Furthermore,

ii) when F is subcritical or F is almost critical, the time of existence δ_0 can be chosen uniformly with respect to initial condition $u_\tau\in B_{\mathfrak{E}^\tau_\varepsilon}(w_0,r)$ for arbitrarily large r,

iii) for any $0\leqslant\theta<\min\{\gamma(\varepsilon),\mu\}$ we have

$$\lim_{t\to\tau^+}(t-\tau)^\theta\|u(t,\tau,u_\tau)\|_{E_{1+\theta}}=0,\ u_\tau\in B^{\delta_0}_{\mathfrak{E}^\tau_\varepsilon}(w_0,r)\cap\mathfrak{E}^\tau_\theta, \tag{1.13}$$

and

$$\begin{aligned}&\sup_{t\in[\tau,\tau+\delta_0]}(t-\tau)^\theta\|u(t,\tau,u^1_\tau)-u(t,\tau,u^2_\tau)\|_{E_{1+\theta}}\\&\qquad\leqslant C(\theta)\big(\|u^1_\tau-u^2_\tau\|^{\mathfrak{E}^\tau_\theta}_{\delta_0}+\|u^1_\tau-u^2_\tau\|^{\mathfrak{E}^\tau_\epsilon}_{\delta_0}\big),\ u^1_\tau,u^2_\tau\in B^{\delta_0}_{\mathfrak{E}^\tau_\varepsilon}(w_0,r)\cap\mathfrak{E}^\tau_\theta,\end{aligned} \tag{1.14}$$

iv) also, whenever $0\leqslant\theta<\min\{\gamma(\varepsilon),\mu\}$ and $u_\tau\in B^{\delta_0}_{\mathfrak{E}^\tau_\varepsilon}(w_0,r)$, we have

$$\lim_{t\to\tau^+}(t-\tau)^\theta\|U(t,\tau)u_\tau-u_\tau\|_{E_{1+\theta}}=0\ \textit{implies}\ \lim_{t\to\tau^+}(t-\tau)^\theta\|u(t,\tau,u_\tau)-u_\tau\|_{E_{1+\theta}}=0. \tag{1.15}$$

Although we have not used so far embedding properties it is typical for applications that

(1.16) $\qquad E_\beta$ is densely embedded in E_α whenever $0 \leqslant \alpha \leqslant \beta < 1 + \mu$.

Remark 1.8. *i) Under Assumption 1.1 and* (1.16), $E_1 \subset \mathfrak{E}^\tau_\varepsilon$, $\|\cdot\|_\delta^{\mathfrak{E}^\tau_\varepsilon}$ *is the norm in* $\mathfrak{E}^\tau_\varepsilon$ *and* $B^\delta_{\mathfrak{E}^\tau_\varepsilon}(w_0, r)$ *contains a ball in* E_1 *centered at* w_0 *of radius* $\frac{r}{M}$.
ii) If (1.16) *holds, then Theorem 1.7 can be applied with* $w_0 \in E_1$ *and the time of existence* δ_0 *can be then chosen uniformly with respect to* $\tau \in \mathcal{J}$.

Remark 1.9. *With* (1.16) *and assumptions of Theorem 1.6, if* $u = u(\cdot, \tau, u_\tau)$ *is* $E_{1+\varepsilon}$*-solution of* (1.1) *as in this theorem then*
i) $u \in C([\tau, \tau + \delta_0], E_\alpha) \cap C^\nu_{loc}((\tau, \tau + \delta_0], E_{1+\theta})$ *whenever* $u_\tau \in E_\alpha$, $\alpha \in [0, 1]$, $0 < \theta < \min\{\gamma(\varepsilon), \mu\}$, $0 < \nu < \min\{\gamma(\varepsilon), \mu\} - \theta$,
ii) $u \in C([\tau, \tau + \delta_0], E_{1+\varepsilon}) \cap C^\nu_{loc}((\tau, \tau + \delta_0], E_{1+\theta})$ *whenever* $u_\tau \in E_{1+\varepsilon}$,
iii) $u(t, \tau, u_\tau)$ *is continuous in* E_1 *with respect to* $(t, u_\tau) \in [\tau, \tau + \delta_0] \times E_1$.

Given $\tau \in \mathbb{R}$ and $u_\tau \in \mathfrak{E}^\tau_\varepsilon$ we next define

$$I(u_\tau) := \{T \in (\tau, \infty) : \text{there exists the unique } E_{1+\varepsilon}\text{-solution of (1.1) on } [\tau, T]\}.$$

Under the assumptions of Theorem 1.7 $I(u_\tau)$ is nonempty, in which case we denote

(1.17) $$T_{u_\tau} := \sup I(u_\tau)$$

and call $[\tau, T_{u_\tau})$ the *maximal interval of existence* of $E_{1+\varepsilon}$-solution.

Since in applications E_1 often plays a role of a space in which (1.1) is expected to define a continuous process we now state the theorem which involves characterization of the maximal time of existence of $E_{1+\varepsilon}$-solution in terms of E_1-norm even in a certain critical case. This is significant for applications as any 'better' estimate may be often impossible to find.

Theorem 1.10. *Suppose that Assumption 1.1, 1.2 hold,* F *is of the class* $\mathcal{L}(\varepsilon, \rho, \gamma(\varepsilon), \eta, C_\eta)$ *relative to* $\{E_\alpha, \alpha \in [0, 1 + \mu)\}$, $\tau \in \mathbb{R}$, $u_\tau \in \mathfrak{E}^\tau_\varepsilon$ *and* $u = u(\cdot, \tau, u_\tau)$ *is* $E_{1+\varepsilon}$*-solution of* (1.1) *on a maximal interval of existence* $[\tau, T_{u_\tau})$*. Assume also* (1.16).
i) If F *is subcritical or* F *is almost critical, then*

(1.18) $$T_{u_\tau} < \infty \quad \textit{implies} \quad \limsup_{t \to T^-_{u_\tau}} \|u(t, \tau, u_\tau)\|_{E_1} = \infty.$$

ii) In either case when F *is subcritical, almost critical, or* F *is critical,* $T_{u_\tau} < \infty$ *implies that there does not exist even one sequence* $t_n \to T^-_{u_\tau}$*, for which* $\{u(t_n, \tau, u_\tau)\}$ *converges in* E_1*; in particular the map* $[\tau, T_{u_\tau}) \ni t \to u(t) \in E_1$ *cannot be uniformly continuous.*

Note that in Theorem 1.10 for F subcritical, almost critical, or critical, we have that

(1.19) $$T_{u_\tau} < \infty \quad \text{implies} \quad \limsup_{t \to T^-_{u_\tau}} \|u(t)\|_{E_{1+\varepsilon}} = \infty.$$

However, the $E_{1+\varepsilon}$-estimate may not be easy to find in applications.

It is next reasonable to generalize the notion of $E_{1+\varepsilon}$-solution and investigate a possibility of continuing $E_{1+\varepsilon}$-solution even though its $E_{1+\varepsilon}$-norm may blow up.

Definition 1.11. *Suppose that* F *is of the class* $\mathcal{L}(\varepsilon, \rho, \gamma(\varepsilon), \eta, C_\eta)$ *relative to* $\{E_\alpha, \alpha \in [0, 1 + \mu)\}$, $\tau > 0$, $v_0 \in \mathfrak{E}^\tau_\varepsilon$ *and* $I_\tau \subset \mathbb{R}$ *is an interval of the form* $[\tau, a)$ *or* $[\tau, \infty)$.

We say that $\mathcal{U} : I_\tau \to E_0 \cup E_{1+\varepsilon}$ *is a* piecewise-$E_{1+\varepsilon}$-solution *of* (1.1) *on* I_τ *if and only if* $\mathcal{U}(\tau) = u_\tau$ *and, for each* $T \in I_\tau \setminus \{\tau\}$*, there exist a number* $N_T \in \mathbb{N}$ *and a partition* $\tau = \tau_0 < \tau_1 < ... < \tau_{N_T} < T = \tau_{N_T+1}$ *of* $[\tau, T]$ *such that*

$$\|\mathcal{U}(t) - \mathcal{U}(\tau_{i-1})\|_{E_0} \overset{t\to\tau_{i-1}^-}{\longrightarrow} 0, \quad i = 2, ..., N_T + 1, \tag{1.20}$$

$$\limsup_{t\to\tau_{i-1}^-} \|\mathcal{U}(t)\|_{E_{1+\varepsilon}} = \infty \ , \quad i = 2, ..., N_T + 1, \tag{1.21}$$

$$\mathcal{U} \in L^\infty_{loc}((\tau_{i-1}, \tau_i), E_{1+\varepsilon}), \quad i = 1, ..., N_T + 1, \tag{1.22}$$

$$(t - \tau_{i-1})^\varepsilon \|\mathcal{U}(t)\|_{E_{1+\varepsilon}} \overset{t\to\tau_{i-1}^+}{\longrightarrow} 0, \quad i = 1, ..., N_T + 1, \tag{1.23}$$

$$\mathcal{U}(\tau_{i-1}) = u_{\tau_{i-1}}, \quad i = 1, ..., N_T + 1, \tag{1.24}$$

$$\mathcal{U}(t) = U(t, \tau_{i-1})u_{\tau_{i-1}} + \int_{\tau_{i-1}}^t U(t, s)F(s, \mathcal{U}(s))ds, \ t \in (\tau_{i-1}, \tau_i), \ i = 1, ..., N_T + 1. \tag{1.25}$$

If the interval $I_\tau = [\tau, a)$ *is finite,* $\mathcal{U} : [\tau, a) \to E_0 \cup E_{1+\varepsilon}$ *is a piecewise-*$E_{1+\varepsilon}$*-solution of* (1.1) *on* $I_\tau = [\tau, a)$ *and a is a limit of a strictly increasing sequence* $\{\tau_i, i \in \mathbb{N}\}$ *of times such that* $\limsup_{t\to\tau_i^-} \|\mathcal{U}(t)\|_{E_{1+\varepsilon}} = \infty$*, then a is called an accumulation time of singular times.*

Below we describe when $E_{1+\varepsilon}$-solution can be continued as a piecewise-$E_{1+\varepsilon}$-solution.

Theorem 1.12. *Suppose that Assumptions 1.1, 1.2 are satisfied and F is of the class* $\mathcal{L}(\varepsilon, \rho, \gamma(\varepsilon), \eta, C_\eta)$ *relative to* $\{E_\alpha, \alpha \in [0, 1 + \mu)\}$*. Suppose additionally that* (1.16) *holds,* E_1 *is reflexive and, given any* $\tau \in \mathbb{R}$, $u_\tau \in \mathfrak{E}^\tau_\varepsilon$,

$$\sup_{t\in[\tau,T)} \|u(t)\|_{E_1} < \infty \tag{1.26}$$

whenever $T \in (\tau, \infty)$ *and* $E_{1+\varepsilon}$*-solution* $u = u(\cdot, \tau, u_\tau)$ *of* (4.19) *exists for all* $t \in [\tau, T)$*.*

Finally suppose that

$$\begin{aligned} &\text{when } \tau \in \mathbb{R},\ u_\tau \in E_1 \text{ and } T_{u_\tau} < \infty, \text{ the map } [\tau, T_{u_\tau}) \ni t \longrightarrow u(t) \in E_0, \\ &\text{where } u = u(\cdot, \tau, u_\tau) \text{ is } E_{1+\varepsilon}\text{-solution of (1.1), is uniformly continuous.} \end{aligned} \tag{1.27}$$

Under these assumptions, given $\tau \in \mathbb{R}$, $u_\tau \in E_1$ *and having the unique* $E_{1+\varepsilon}$*-solution* $u = u(\cdot, \tau, T_{u_\tau})$ *of* (1.1) *for which* $T_{u_\tau} < \infty$*, there exist* $a \in (T_{u_\tau}, \infty]$ *and the unique extension* $\mathcal{U} : [\tau, a) \to E_1$ *of u such that* $\mathcal{U}$ *is a piecewise-*$E_{1+\varepsilon}$*-solution of* (1.1) *on* $[\tau, a)$ *and either* $a = \infty$ *or a is an accumulation time of singular times.*

The proofs of the above results will be given in Section 2. In Section 3 we discuss sufficient conditions for Assumptions 1.1, 1.2 in terms of $A(t)$. In Section 4 we show how the abstract results work in applications.

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2. Proofs of abstract results

2.1. **Uniqueness and Hölder continuity of $E_{1+\varepsilon}$-solution: proof of Theorem 1.6.** Given $-\infty < \tau < T < \infty$ we define

$$\mathfrak{M}_\tau^T := \{\psi \in L^\infty_{loc}((\tau,T],E_{1+\varepsilon}) : \lim_{t\to\tau^+}(t-\tau)^\varepsilon \|\psi(t)\|_{E_{1+\varepsilon}} = 0\}. \tag{2.1}$$

Theorem 1.6 will be a consequence of the following two lemmas.

Lemma 2.1. *Suppose that Assumptions 1.1, 1.2 hold, F is of the class $\mathcal{L}(\varepsilon,\rho,\gamma(\varepsilon),\eta,C_\eta)$ relative to $\{E_\alpha, \alpha\in[0,1+\mu)\}$, $u\in\mathfrak{M}_\tau^T$ and (1.6) is valid for $t\in(\tau,T]$ with some $u_\tau\in E_0$.*
Then $u\in C^\nu([\delta,T],E_{1+\theta})$ for any $\delta\in(\tau,T)$, $\nu\in(0,\nu_*)$ and $\nu_*=\min\{\gamma(\varepsilon),\mu\}-\theta>0$.

Proof: Due to Assumptions 1.1, 1.2, given a bounded time interval $[-T,T]\subset\mathbb{R}$ and any $0\leqslant\zeta\leqslant\sigma<1+\mu$, one can choose a positive constant M for which we have

$$\|U(t,\tau)\|_{L(E_\zeta,E_\sigma)} \leqslant M(t-\tau)^{-(\sigma-\zeta)},\ T\geqslant t>\tau\geqslant -T, \tag{2.2}$$

and, if $1\geqslant\sigma-\zeta\geqslant 0$,

$$\|U(t,\tau)-Id\|_{L(E_\sigma,E_\zeta)} \leqslant M(t-\tau)^{\sigma-\zeta},\ T\geqslant t>\tau\geqslant -T. \tag{2.3}$$

On the other hand, since $u\in\mathfrak{M}_\tau^T$, for any $\delta>\tau$ close enough to τ we have

$$\|u(t)\|_{E_{1+\varepsilon}} \leqslant (t-\tau)^{-\varepsilon},\ t\in(\tau,\delta), \tag{2.4}$$

in which case letting $\tilde c = c(\eta+C_\eta)$ we deduce from (1.5) that

$$\|F(s,u(s))\|_{E_{\gamma(\varepsilon)}} \leqslant \tilde c((s-\tau)^{-\varepsilon\rho}+1),\ t\in(\tau,\delta). \tag{2.5}$$

Not loosing generality we will assume that $\delta>\tau$ is close enough to τ and (2.5) holds. Since $u\in\mathfrak{M}_\tau^T$ and $\delta>\tau$ then $\|u\|_{L^\infty((\delta,T),E_{1+\varepsilon})}\leqslant c_\delta$ and by (1.5) we conclude that

$$m_\delta := \|F(t,u(t))\|_{L^\infty((\delta,T),E_{\gamma(\varepsilon)})} < \infty. \tag{2.6}$$

For $\tau<\delta\leqslant t\leqslant t+h\leqslant T$ from the variation of constants formula we infer that

$$\begin{aligned}\|u(t+h)-u(t)\|_{E_{1+\theta}} &\leqslant \|(U(t+h,\tau)-U(t,\tau))u_\tau\|_{E_{1+\theta}}\\ &+\int_t^{t+h}\|U(t+h,s)F(s,u(s))\|_{E_{1+\theta}}ds+\int_\delta^t\|(U(t+h,s)-U(t,s))F(s,u(s))\|_{E_{1+\theta}}ds\\ &+\int_\tau^\delta\|(U(t+h,s)-U(t,s))F(s,u(s))\|_{E_{1+\theta}}ds =: J_1+J_2+J_3+J_4.\end{aligned}$$

Choosing arbitrary

$$\hat\varepsilon\in(\theta,\mu)\cap(\theta,\gamma(\varepsilon)) \tag{2.7}$$

and applying (2.2)-(2.3) we get for $J_1=\|(U(t+h,t)-Id)U(t,\tau)u_\tau\|_{E_{1+\theta}}$

$$\begin{aligned}J_1 &\leqslant \|U(t+h,t)-Id\|_{L(E_{1+\hat\varepsilon},E_{1+\theta})}\|U(t,\tau)\|_{L(E_0,E_{1+\hat\varepsilon})}\|u_\tau\|_{E_0}\\ &\leqslant M^2h^{\hat\varepsilon-\theta}(t-\tau)^{-1-\hat\varepsilon}\|u_\tau\|_{E_0} \leqslant M^2h^{\hat\varepsilon-\theta}(\delta-\tau)^{-1-\hat\varepsilon}\|u_\tau\|_{E_0}.\end{aligned} \tag{2.8}$$

Using (2.2), (2.6), (2.7) we also obtain

$$
\begin{aligned}
J_2 &= \int_t^{t+h} \|U(t+h,s)\|_{L(E_{\gamma(\varepsilon)},E_{1+\theta})} \|F(s,u(s))\|_{E_{\gamma(\varepsilon)}} ds \\
&\leqslant M m_\delta \int_t^{t+h} (t+h-s)^{\gamma(\varepsilon)-\theta-1} ds \leqslant M m_\delta (\gamma(\varepsilon)-\theta)^{-1} (T-\tau)^{\gamma(\varepsilon)-\hat{\varepsilon}} h^{\hat{\varepsilon}-\theta}.
\end{aligned}
\tag{2.9}
$$

On the other hand, by (2.3), (2.6) and (2.7) we have

$$
\begin{aligned}
J_3 &\leqslant \int_\delta^t \|U(t+h,t)-Id\|_{L(E_{1+\hat{\varepsilon}},E_{1+\theta})} \|U(t,s)\|_{L(E_{\gamma(\varepsilon)},E_{1+\hat{\varepsilon}})} \|F(s,u(s))\|_{E_{\gamma(\varepsilon)}} ds \\
&\leqslant M^2 m_\delta \int_\delta^t h^{\hat{\varepsilon}-\theta} (t-s)^{\gamma(\varepsilon)-1-\hat{\varepsilon}} ds \leqslant M^2 (\gamma(\varepsilon)-\hat{\varepsilon})^{-1} m_\delta h^{\hat{\varepsilon}-\theta} (t-\delta)^{\gamma(\varepsilon)-\hat{\varepsilon}} \\
&\leqslant M^2 (\gamma(\varepsilon)-\hat{\varepsilon})^{-1} m_\delta h^{\hat{\varepsilon}-\theta} (T-\tau)^{\gamma(\varepsilon)-\hat{\varepsilon}},
\end{aligned}
\tag{2.10}
$$

whereas due to (2.5) we get

$$
\begin{aligned}
J_4 &\leqslant \int_\tau^\delta \|U(t+h,t)-Id\|_{L(E_{1+\hat{\varepsilon}},E_{1+\theta})} \|U(t,s)\|_{L(E_{\gamma(\varepsilon)},E_{1+\hat{\varepsilon}})} \|F(s,u(s))\|_{E_{\gamma(\varepsilon)}} ds \\
&\leqslant \tilde{c} M^2 \int_\tau^\delta h^{\hat{\varepsilon}-\theta} (t-s)^{\gamma(\varepsilon)-1-\hat{\varepsilon}} ((s-\tau)^{-\varepsilon\rho}+1) ds \\
&\leqslant \tilde{c} M^2 h^{\hat{\varepsilon}-\theta} \left(\int_\tau^t (t-s)^{\gamma(\varepsilon)-1-\hat{\varepsilon}} (s-\tau)^{-\varepsilon\rho} ds + \int_\tau^t (t-s)^{\gamma(\varepsilon)-1-\hat{\varepsilon}} ds \right) \\
&\leqslant \tilde{c} M^2 h^{\hat{\varepsilon}-\theta} \left(B(\gamma(\varepsilon)-\hat{\varepsilon}, 1-\varepsilon\rho) \frac{(T_{u_\tau}-\tau)^{\gamma(\varepsilon)-\hat{\varepsilon}}}{(\delta-\tau)^{\varepsilon\rho}} + (\gamma(\varepsilon)-\hat{\varepsilon})^{-1} (T_{u_\tau}-\tau)^{\gamma(\varepsilon)-\hat{\varepsilon}} \right).
\end{aligned}
\tag{2.11}
$$

As a consequence of the above estimates for any $\delta > \tau$ close enough to τ there exists $\bar{c} > 0$ such that for each $\tau < \delta \leqslant t \leqslant t+h \leqslant T$ we have $\|u(t+h)-u(t)\|_{E_{1+\theta}} \leqslant \bar{c} h^{\hat{\varepsilon}-\theta}$. Recalling that $\hat{\varepsilon}$ could be any number satisfying (2.7) we get the result. □

Lemma 2.2. *If $\varphi, \tilde{\varphi} \in \mathfrak{M}_\tau^T$, $u_\tau \in E_0$ and (1.6) is valid in $(\tau, T]$ both for $u = \varphi$ and $u = \tilde{\varphi}$, then $\varphi, \tilde{\varphi}$ are identical on $(\tau, T]$.*

Proof: By assumption we have

$$
\begin{aligned}
\|\varphi(t)-\tilde{\varphi}(t)\|_{E_{1+\varepsilon}} &\leqslant c C_\eta M \int_\tau^t (t-s)^{\gamma(\varepsilon)-1-\varepsilon} \|\varphi(s)-\tilde{\varphi}(s)\|_{E_{1+\varepsilon}} ds \\
&\quad + c\eta M \int_\tau^t (t-s)^{\gamma(\varepsilon)-1-\varepsilon} \|\varphi(s)-\tilde{\varphi}(s)\|_{E_{1+\varepsilon}} (\|\varphi(s)\|_{E_{1+\varepsilon}}^{\rho-1} + \|\tilde{\varphi}(s)\|_{E_{1+\varepsilon}}^{\rho-1}) ds, \ t \in (\tau, T].
\end{aligned}
$$

Since $\varphi, \tilde{\varphi} \in \mathfrak{M}_\tau^T$, given $\xi \in (0,1)$, there is a certain $h \in (0, \xi)$ such that

$$
(t-\tau)^\varepsilon \|\varphi(t)\|_{E_{1+\varepsilon}} + (t-\tau)^\varepsilon \|\tilde{\varphi}(t)\|_{E_{1+\varepsilon}} \leqslant \xi, \ t \in (\tau, \tau+h).
$$

Using this and restricting t to the interval $(\tau, \tau+h)$ where $h \in (0, \xi)$ we obtain

$$
\begin{aligned}
&(t-\tau)^\varepsilon \|\varphi(t)-\tilde{\varphi}(t)\|_{E_{1+\varepsilon}} \\
&\qquad \leqslant c C_\eta M B(\gamma(\varepsilon)-\varepsilon, 1-\varepsilon) \xi^{\gamma(\varepsilon)-\varepsilon} \sup_{s\in(\tau,\tau+h)} (s-\tau)^\varepsilon \|\varphi(s)-\tilde{\varphi}(s)\|_{E_{1+\varepsilon}} \\
&\qquad + \xi^{\rho-1+\gamma(\varepsilon)-\varepsilon\rho} 2c\eta M B(1-\varepsilon\rho, \gamma(\varepsilon)-\varepsilon) \sup_{s\in(\tau,\tau+h)} (s-\tau)^\varepsilon \|\varphi(s)-\tilde{\varphi}(s)\|_{E_{1+\varepsilon}}.
\end{aligned}
$$

We remark that the inequality above will hold true if we replace its left hand side by $\sup_{s\in(\tau,\tau+h)}(s-\tau)^{\varepsilon}\|\varphi(s)-\tilde{\varphi}(s)\|_{E_{1+\varepsilon}}$. On the other hand recalling that $\rho>1$, $\gamma(\varepsilon)\geqslant\rho\varepsilon$ and choosing $\xi>0$ small enough we can ensure that the right hand side above is less than $\frac{1}{2}\sup_{s\in(\tau,\tau+h)}(s-\tau)^{\varepsilon}\|\varphi(s)-\tilde{\varphi}(s)\|_{E_{1+\varepsilon}}$. Consequently $\sup_{s\in(\tau,\tau+h)}(s-\tau)^{\varepsilon}\|\varphi(s)-\tilde{\varphi}(s)\|_{E_{1+\varepsilon}}=0$ and thus $\varphi=\tilde{\varphi}$ in $[\tau,\tau+h]$ for some $h>0$.

Now, if $\tau^*\in(\tau,\tau+h]$ is such that $\varphi(\tau^*)=\tilde{\varphi}(\tau^*)$ then applying the variation of constants formula with the initial time τ^* and with the initial value $\varphi(\tau^*)=\tilde{\varphi}(\tau^*)$ we get

$$\varphi(t)-\tilde{\varphi}(t)=\int_{\tau^*}^{t}U(t,s)(F(s,\varphi(s)-F(s,\tilde{\varphi}(s))ds \quad \text{in} \quad [\tau^*,T].$$

Hence, letting $c^*=\sup_{s\in[\tau^*,T]}(\|\varphi(s)\|_{E_{1+\varepsilon}}^{\rho-1}+\|\tilde{\varphi}(s)\|_{E_{1+\varepsilon}}^{\rho-1})$,

$$\|\varphi(t)-\tilde{\varphi}(t)\|_{E_{1+\varepsilon}}\leqslant cM(C_\eta+\eta c^*)\int_{\tau^*}^{t}(t-s)^{\gamma(\varepsilon)-1-\varepsilon}\|\varphi(s)-\tilde{\varphi}(s)\|_{E_{1+\varepsilon}}ds \quad \text{in} \quad [\tau^*,T]$$

and by singular Gronwall's inequality we conclude that φ and $\tilde{\varphi}$ coincide. □

2.2. **Proof of Theorem 1.7.** Let us fix an interval $I=(t_0-\xi,t_0+\xi)$ around t_0 such that (1.2) holds with $\zeta=\gamma(\varepsilon)$ and $\sigma=1+\varepsilon$. Let us also choose an interval $\mathcal{J}$ centered at t_0 such that $I\setminus\mathcal{J}$ is the union of two intervals of length $l>0$.

We first note that if $\delta^*\in(0,l)$, $\tau\in\mathcal{J}$, $\delta\in(0,1]\cap(0,\delta^*)$, $v\in C((\tau,\tau+\delta],E_{1+\varepsilon})$, $\lambda(v,t):=\sup_{s\in(\tau,t]}\{(s-\tau)^{\varepsilon}\|v(s)\|_{E_{1+\varepsilon}}\}$, $R>0$, $t\in(\tau,\tau+\delta]$ and $\lambda(v,t)\leqslant R$ then, by Assumption 1.1 and (1.5), we have

$$\begin{aligned}\|U(t,s)F(s,v(s))\|_{E_{1+\varepsilon}}&\leqslant\|U(t,s)\|_{L(E_{\gamma(\varepsilon)},E_{1+\varepsilon})}\|F(s,v(s))\|_{E_{\gamma(\varepsilon)}}\\&\leqslant M(t-s)^{-1+\gamma(\varepsilon)-\varepsilon}c(\eta\|v(s)\|_{E_{1+\varepsilon}}^{\rho}+C_\eta)\end{aligned}\tag{2.12}$$

and consequently

$$\begin{aligned}(t-\tau)^{\varepsilon}\|\int_{\tau}^{t}U(t,s)F(s,v(s))ds\|_{E_{1+\varepsilon}}&\leqslant cC_\eta M(t-\tau)^{\varepsilon}\int_{\tau}^{t}(t-s)^{-1+\gamma(\varepsilon)-\varepsilon}ds\\&+c\eta M(t-\tau)^{\varepsilon}\int_{\tau}^{t}(t-s)^{-1+\gamma(\varepsilon)-\varepsilon}(s-\tau)^{-\rho\varepsilon}[(s-\tau)^{\varepsilon}\|v(s)\|_{E_{1+\varepsilon}}]^{\rho}ds\\&\leqslant cMB(1-\rho\varepsilon,\gamma(\varepsilon)-\varepsilon)[C_\eta(t-\tau)^{\gamma(\varepsilon)}+\eta\lambda^{\rho}(v,t)]\leqslant cMB_{\varepsilon,\rho}[C_\eta(t-\tau)^{\gamma(\varepsilon)}+\eta R^{\rho}].\end{aligned}\tag{2.13}$$

Also, if $v,\tilde{v}\in C((\tau,\tau+\delta],E_{1+\varepsilon})$, $t\in(\tau,\tau+\delta]$ and $\lambda(v,t)\leqslant R$, $\lambda(\tilde{v},t)\leqslant R$ then, with a similar usage of Assumption 1.1 and (1.4), we get

$$\begin{aligned}&(t-\tau)^{\varepsilon}\|\int_{\tau}^{t}U(t,s)[F(s,v(s))-F(s,\tilde{v}(s))]ds\|_{E_{1+\varepsilon}}\\&\leqslant cC_\eta M(t-\tau)^{\varepsilon}\int_{\tau}^{t}(t-s)^{-1+\gamma(\varepsilon)-\varepsilon}(s-\tau)^{-\varepsilon}(s-\tau)^{\varepsilon}\|v(s)-\tilde{v}(s)\|_{E_{1+\varepsilon}}ds\\&\quad+c\eta M(t-\tau)^{\varepsilon}\int_{\tau}^{t}(t-s)^{-1+\gamma(\varepsilon)-\varepsilon}(s-\tau)^{-\rho\varepsilon}\left(((s-\tau)^{\varepsilon}\|v(s)\|_{E_{1+\varepsilon}})^{\rho-1}\right.\\&\qquad\left.+((s-\tau)^{\varepsilon}\|\tilde{v}(s)\|_{E_{1+\varepsilon}})^{\rho-1}\right)(s-\tau)^{\varepsilon}\|v(s)-\tilde{v}(s)\|_{E_{1+\varepsilon}}ds.\end{aligned}$$

Hence, letting

$$\Gamma_{\varepsilon}(t):=cMB_{\varepsilon,\rho}[C_\eta(t-\tau)^{\gamma(\varepsilon)-\varepsilon}+2\eta R^{\rho-1}],\tag{2.14}$$

we conclude that

$$(2.15)\qquad \begin{aligned}(t-\tau)^{\varepsilon}\|\int_{\tau}^{t} U(t,s)[F(s,v(s))-F(s,\tilde{v}(s))]ds\|_{E_{1+\varepsilon}}\\ \leqslant \Gamma_{\varepsilon}(t)\sup_{s\in(\tau,t]}\{(s-\tau)^{\varepsilon}\|v(s)-\tilde{v}(s)\|_{E_{1+\varepsilon}}\}.\end{aligned}$$

We now choose $R_0 \geqslant R > 0$ and $\delta \in (0,1] \cap (0,\delta^*)$ such that

$$(2.16)\qquad c\eta M B_{\varepsilon,\rho} R_0^{\rho-1} = \frac{1}{8} \quad \text{and} \quad cC_{\eta} M B_{\varepsilon,\rho}\delta^{\gamma(\varepsilon)-\varepsilon} = \min\{\frac{R}{8},\frac{1}{4}\}.$$

We also set $r := \frac{R}{4} \leqslant \frac{R_0}{4} = \frac{1}{4(8c\eta M B_{\varepsilon,\rho})^{\frac{1}{\rho-1}}}$ and, since $\lim_{t\to\tau^+}\|(t-\tau)^{\varepsilon}U(t,\tau)w_0\|_{E_{1+\varepsilon}} = 0$ we choose $\bar{\delta}_0 \in (0,\delta]$ such that

$$(2.17)\qquad \|(t-\tau)^{\varepsilon}U(t,\tau)w_0\|_{E_{1+\varepsilon}} \leqslant \frac{R}{2}, \qquad \tau < t \leqslant \tau + \bar{\delta}_0.$$

For any fixed $\delta_0 \in (0,\bar{\delta}_0]$, $u_\tau \in B^{\delta_0}_{\mathfrak{E}^{\tau}_{\varepsilon}}(w_0,r)$ let us consider the set

$$(2.18)\qquad K(R,\tau) = \big\{v \in C((\tau,\tau+\delta_0],E_{1+\varepsilon}),\ \sup_{t\in(\tau,\tau+\delta_0]}\{(t-\tau)^{\varepsilon}\|v(t)\|_{E_{1+\varepsilon}}\} \leqslant R\big\}$$

and define $d(v,\tilde{v}) = \sup_{t\in(\tau,\tau+\delta_0]}\{(t-\tau)^{\varepsilon}\|v(t)-\tilde{v}(t)\|_{E_{1+\varepsilon}}\}$ for $v,\tilde{v} \in K(R,\tau)$. Note that d is a metric in $K(R,\tau)$ and that $(K(R,\tau),d)$ is a complete metric space.

We will next consider the map

$$(\mathcal{T}v)(t) = U(t,\tau)u_\tau + \int_{\tau}^{t} U(t,s)F(s,v(s))ds,\ v \in K(R,\tau),\ t \in (\tau,\tau+\delta_0].$$

Adapting Lemma 2.1 one can see that $\mathcal{T}v \in C((\tau,\tau+\delta_0],E_{1+\varepsilon})$ for $v \in K(R,\tau)$.

It then follows from (1.11), (2.13) and (2.16)-(2.17) that

$$\begin{aligned}\|(t-\tau)^{\varepsilon}(\mathcal{T}v)(t)\|_{E_{1+\varepsilon}} &\leqslant (t-\tau)^{\varepsilon}\|U(t,\tau)u_\tau + \int_{\tau}^{t} U(t,s)F(s,v(s))ds\|_{E_{1+\varepsilon}}\\ &\leqslant \|(t-\tau)^{\varepsilon}U(t,\tau)u_\tau\|_{E_{1+\varepsilon}} + cM(t-\tau)^{\varepsilon}\int_{\tau}^{t}(t-s)^{-1+\gamma(\varepsilon)-\varepsilon}(\eta\|v(s)\|^{\rho}_{E_{1+\varepsilon}} + C_\eta)ds\\ &\leqslant \|(t-\tau)^{\varepsilon}U(t,\tau)(u_\tau-\omega_0)\|_{E_{1+\varepsilon}} + \|(t-\tau)^{\varepsilon}U(t,\tau)\omega_0\|_{E_{1+\varepsilon}}\\ &\qquad + cM\eta B_{\varepsilon,\rho}R^{\rho} + cMC_\eta B_{\varepsilon,\rho}\delta_0^{\gamma(\varepsilon)}\\ &\leqslant r + \|(t-\tau)^{\varepsilon}U(t,\tau)\omega_0\|_{E_{1+\varepsilon}} + cM\eta B_{\varepsilon,\rho}R^{\rho} + cMC_\eta B_{\varepsilon,\rho}\delta_0^{\gamma(\varepsilon)-\varepsilon}\delta_0^{\varepsilon} \leqslant R,\end{aligned}$$

which yields that $\mathcal{T}$ takes $K(R,\tau)$ into $K(R,\tau)$. On the other hand, applying (2.14)-(2.16), we get $d(\mathcal{T}v_1,\mathcal{T}v_2) \leqslant \frac{1}{2}d(v_1,v_2)$.

Consequently, due to the Banach fixed point theorem, we infer that $\mathcal{T}$ has the unique fixed point $u = u(\cdot,\tau,u_\tau)$ in $K(R,\tau)$ and we now show that $\lim_{t\to\tau^+}\|(t-\tau)^{\varepsilon}u(t)\|_{E_{1+\varepsilon}} = 0$.

Adapting (2.13), we have for each $t \in (\tau,\tau+\delta_0]$ and the above fixed point u

$$(t-\tau)^{\varepsilon}\|u(t)\|_{E_{1+\varepsilon}} \leqslant (t-\tau)^{\varepsilon}\|U(t,\tau)u_\tau\|_{E_{1+\varepsilon}} + cMB_{\varepsilon,\rho}[C_\eta(t-\tau)^{\gamma(\varepsilon)} + \eta R^{\rho-1}\lambda(u,t)],$$

where by assumption, given any $\xi > 0$, we can choose $h \in (0,\xi)$ such that for $t \in (\tau,\tau+h)$ we have $(t-\tau)^{\varepsilon}\|U(t,\tau)u_\tau\|_{E_{1+\varepsilon}} < \xi$. Hence, we get

$$(t-\tau)^{\varepsilon}\|u(t)\|_{E_{1+\varepsilon}} \leqslant \xi + cMB_{\varepsilon,\rho}[C_\eta\xi^{\gamma(\varepsilon)} + \eta R^{\rho-1}\lambda(u,t)],\ t \in (\tau,\tau+h).$$

Since the right hand side above is a nondecreasing function of t, we obtain

$$\lambda(u,t) \leqslant \xi + cMB_{\varepsilon,\rho}[C_\eta \xi^{\gamma(\varepsilon)} + \eta R^{\rho-1}\lambda(u,t)],\ t \in (\tau, \tau+h),$$

and, via (2.16), $\frac{7}{8}\lambda(u,t) \leqslant \xi + cMB_{\varepsilon,\rho}C_\eta \xi^{\gamma(\varepsilon)},\ t \in (\tau,\tau+h)$. This yields

$$\lambda(u,t) = \sup_{s\in(\tau,t]} \{(s-\tau)^\varepsilon \|u(s)\|_{E_{1+\varepsilon}}\} \to 0 \ \text{ as } \ t \to \tau^+, \tag{2.19}$$

which ensures that $(t-\tau)^\varepsilon u(t) \to 0$ in $E_{1+\varepsilon}$ as $t \to \tau^+$.

Finally, letting $u(\tau) = u_\tau$, we extend the fixed point $u = u(\cdot,\tau,u_\tau)$ constructed above to the interval $[\tau, \tau+\delta_0]$ and obtain $E_{1+\varepsilon}$ solution of (1.1). Since the uniqueness follows from Theorem 2.2, part i) of Theorem 1.7 is proved.

Part ii) now follows from Corollary 2.3 below (see also Remark 2.4).

Corollary 2.3. *Suppose that Assumptions 1.1, 1.2 hold,* $F : \mathbb{R}\times E_{1+\varepsilon} \to E_{\gamma(\varepsilon)}$ *is continuous and there exist constants* $\rho > 1$, $0 < \varepsilon < \min\{\frac{1}{\rho}, \mu\}$, $\gamma(\varepsilon) \in [\rho\varepsilon, 1)$ *such that for each* $\eta > 0$ *there is a certain* $C_\eta > 0$ *and, moreover, for any bounded time interval* $I \subset \mathbb{R}$ *there exists some* $c > 0$ *for which we have*

$$\|F(t,v) - F(t,w)\|_{E_{\gamma(\varepsilon)}} \leqslant c\|v-w\|_{E_{1+\varepsilon}}(\eta\|v\|^{\rho-1}_{E_{1+\varepsilon}} + \eta\|w\|^{\rho-1}_{E_{1+\varepsilon}} + C_\eta),\ v,w \in E_{1+\varepsilon},\ t \in I,$$

and

$$\|F(t,v)\|_{E_{\gamma(\varepsilon)}} \leqslant c(\eta\|v\|^{\rho}_{E_{1+\varepsilon}} + C_\eta),\ v,w \in E_{1+\varepsilon},\ t \in I.$$

Then, given any $t_0 \in \mathbb{R}$, τ *in a certain interval* $\mathcal{J} \subset \mathbb{R}$ *centered at* t_0 *and given any* $r_0 > 0$, *there exists* $\delta_0 > 0$ *such that for any initial condition* $u_\tau \in B^{\delta_0}_{\mathfrak{E}^\tau_\varepsilon}(0, r_0)$ *there exists the unique* $E_{1+\varepsilon}$*-solution* $u = u(\cdot,\tau,u_\tau)$ *of* (1.1) *on* $[\tau,\tau+\delta_0]$.

Proof: Letting $w_0 = 0$ and coming back to the proof of Theorem 1.7 observe that given any $r_0 > 0$ one can now choose $\eta > 0$ such that r in (1.12) satisfies $r > r_0$. Proceeding as in the proof of Theorem 1.7 we obtain for any $u_\tau \in B^{\delta_0}_{\mathfrak{E}^\tau_\varepsilon}(0, r)$ the existence of $E_{1+\varepsilon}$-solution $u = u(\cdot,\tau,u_\tau)$ of (1.1) on $[\tau,\tau+\delta_0]$. □

Remark 2.4. *If* F *is subcritical then not loosing generality one can assume that* $\eta > 0$ *in* (1.4)-(1.5) *can be chosen arbitrarily small. Indeed, given* $\rho > 1$, $\varepsilon \in (0, \frac{1}{\rho})$, $\varepsilon < \mu$, $\gamma(\varepsilon) \in (\rho\varepsilon, 1)$ *we can choose* $\tilde\rho > \rho$ *close enough to* ρ *and we will have* $\varepsilon \in (0, \frac{1}{\tilde\rho})$ *and* $\gamma(\varepsilon) \in (\tilde\rho\varepsilon, 1)$. *Then* $\eta\|w\|^{\rho-1}_{E_{1+\varepsilon}}$ *can be estimated by* $\tilde\eta\|w\|^{\tilde\rho-1}_{E_{1+\varepsilon}} + c_{\tilde\eta,\eta}$, *which yields* (1.4)-(1.5) *with parameters* $\varepsilon, \gamma(\varepsilon)$ *as before,* ρ *replaced by* $\tilde\rho > \rho$ *suitably close to* ρ *and* η *replaced by* $\tilde\eta$, *which we can fix as small as we wish. Parameters* ε, $\gamma(\varepsilon)$ *and* c *in* (1.4)-(1.5) *will remain the same and the only difference will come from the replacement of* C_η *by* $C_\eta + c_{\tilde\eta,\eta}$, *which will not influence the heart of our consideration.*

We now prove conditions (1.13)-(1.15). Using a similar argument as in (2.13), for $\theta \in (0,\gamma(\varepsilon)) \cap (0,\mu)$ and for the unique $E_{1+\varepsilon}$-solution $u = u(\cdot,\tau,u_\tau)$ of (1.1) we get

$$\begin{aligned}(t-\tau)^\theta\|u(t)\|_{E_{1+\theta}} &\leqslant (t-\tau)^\theta\|U(t,\tau)u_\tau\|_{E_{1+\theta}} + (t-\tau)^\theta\int_\tau^t \|U(t,s)F(s,u(s))\|_{E_{1+\theta}}ds\\ &\leqslant (t-\tau)^\theta\|U(t,\tau)u_\tau\|_{E_{1+\theta}} + cMC_\eta(\gamma(\varepsilon)-\theta)^{-1}(t-\tau)^{\gamma(\varepsilon)}\\ &\qquad + \eta cMB(1-\varepsilon\rho,\gamma(\varepsilon)-\theta)(\sup_{\tau<s\leqslant t}\{(s-\tau)^\varepsilon\|u(s)\|_{E_{1+\varepsilon}}\})^\rho.\end{aligned}$$

Recalling that $u_\tau \in B^{\delta_0}_{\mathfrak{E}^\tau_\varepsilon}(w_0, r) \cap \mathfrak{E}^\tau_\theta$ and using (2.19) we conclude that $(t-\tau)^\theta \|u(t)\|_{E_{1+\theta}} \to 0$ as $t \to \tau^+$, which proves (1.13).

For $\theta \in (0, \gamma(\varepsilon)) \cap (0, \mu)$ analogously as in (2.15) we next have

$$
\begin{aligned}
(t-\tau)^\theta \|u(t,\tau,u^1_\tau) - u(t,\tau,u^2_\tau)\|_{E_{1+\theta}} &\leqslant (t-\tau)^\theta \|U(t,\tau)(u^1_\tau - u^2_\tau)\|_{E_{1+\theta}} \\
&\quad + (t-\tau)^\theta \int_\tau^t \|U(t,s)[F(s,u(s,\tau,u^1_\tau)) - F(s,u(s,\tau,u^2_\tau))]\|_{E_{1+\theta}} ds \\
&\leqslant (t-\tau)^\theta \|U(t,\tau)(u^1_\tau - u^2_\tau)\|_{E_{1+\theta}} \\
&\quad + \Gamma_\theta(t) \sup_{\tau < s \leqslant \tau+\delta_0} \{(s-\tau)^\varepsilon \|u(s,\tau,u^1_\tau) - u(s,\tau,u^2_\tau)\|_{E_{1+\varepsilon}}\},
\end{aligned} \tag{2.20}
$$

where

$$\Gamma_\theta(t) = cM(1+\theta, \gamma(\varepsilon), T) \max\{B(\gamma(\varepsilon)-\theta, 1-\varepsilon), B(1-\rho\varepsilon, \gamma(\varepsilon)-\theta)\}[C_\eta (t-\tau)^{\gamma(\varepsilon)-\theta} + 2\eta R^{\rho-1}].$$

Taking $\theta = \varepsilon$ we get

$$
\begin{aligned}
(t-\tau)^\varepsilon \|u(t,\tau,u^1_\tau) - u(t,\tau,u^2_\tau)\|_{E_{1+\varepsilon}} &\leqslant (t-\tau)^\varepsilon \|U(t,\tau)(u^1_\tau - u^2_\tau)\|_{E_{1+\varepsilon}} \\
&\quad + \Gamma_\varepsilon(t) \sup_{\tau < s \leqslant \tau+\delta_0} \{(s-\tau)^\varepsilon \|u(s,\tau,u^1_\tau) - u(s,\tau,u^2_\tau)\|_{E_{1+\varepsilon}}\}.
\end{aligned}
$$

Since, by (2.16), $\Gamma_\varepsilon(\tau+\delta_0) \leqslant \frac{1}{2}$ and $\Gamma_\varepsilon(t)$ is increasing with respect to t we conclude that

$$\sup_{\tau < s \leqslant \tau+\delta_0} \{(s-\tau)^\varepsilon \|u(s,\tau,u^1_\tau) - u(s,\tau,u^2_\tau)\|_{E_{1+\varepsilon}}\} \leqslant 2 \sup_{\tau < s \leqslant \tau+\delta_0} (s-\tau)^\varepsilon \|U(s,\tau)(u^1_\tau - u^2_\tau)\|_{E_{1+\varepsilon}}.$$

Consequently, using the above inequality and (2.20) we get (1.14).

Assuming that $0 \leqslant \theta < \min\{\gamma(\varepsilon), \mu\}$ and $\lim_{t\to\tau^+} (t-\tau)^\theta \|U(t,\tau)u_\tau - u_\tau\|_{E_{1+\theta}} = 0$ we now show that $\lim_{t\to\tau^+} (t-\tau)^\theta \|u(t,\tau,u_\tau) - u_\tau\|_{E_{1+\theta}} = 0$, for which we first use the variation of constants formula and (1.2), (1.5) to get for each $t \in (\tau, \tau+\delta_0]$

$$
\begin{aligned}
&(t-\tau)^\theta \|u(t) - u_\tau\|_{E_{1+\theta}} \\
&\leqslant (t-\tau)^\theta \|U(t,\tau)u_\tau - u_\tau\|_{E_{1+\theta}} + (t-\tau)^\theta \int_\tau^t \|U(t,s)F(s,u(s))\|_{E_{1+\theta}} ds \\
&\leqslant (t-\tau)^\theta \|U(t,\tau)u_\tau - u_\tau\|_{E_{1+\theta}} \\
&\qquad + cM(t-\tau)^\theta \int_\tau^t (t-s)^{\gamma(\varepsilon)-1-\theta} (C_\eta + \eta(s-\tau)^{-\varepsilon\rho} \|(s-\tau)^\varepsilon u(s)\|^\rho_{E_{1+\varepsilon}}) ds \\
&\leqslant (t-\tau)^\theta \|U(t,\tau)u_\tau - u_\tau\|_{E_{1+\theta}} + cC_\eta M B(1-\rho\varepsilon, \gamma(\varepsilon)-\theta)(t-\tau)^{\gamma(\varepsilon)} \\
&\qquad + cM\eta B(1-\rho\varepsilon, \gamma(\varepsilon)-\theta)\lambda^\rho(u,t).
\end{aligned} \tag{2.21}
$$

Thus (1.15) is a consequence of (2.21) and (2.19). The proof of Theorem 1.7 is complete. □

2.3. **Proof of Remark 1.8.** Note first that $E_{1+\varepsilon} \subset \mathfrak{E}^\tau_\varepsilon$ as, whenever $\varphi \in E_{1+\varepsilon}$ by Assumption 1.1 we have $(t-\tau)^\varepsilon \|U(t,\tau)\varphi\|_{E_{1+\varepsilon}} \leqslant M(t-\tau)^\varepsilon \|\varphi\|_{E_{1+\varepsilon}} \to 0$ as $t \to \tau^+$. Now, if $\psi \in E_1$ and $E_{1+\varepsilon} \ni \varphi_n \overset{E_1}{\to} \psi$ then using again Assumption 1.1 we obtain

$$(t-\tau)^\varepsilon \|U(t,\tau)\psi\|_{E_{1+\varepsilon}} \leqslant M\|\psi - \varphi_n\|_{E_1} + (t-\tau)^\varepsilon \|U(t,\tau)\varphi_n\|_{E_{1+\varepsilon}}$$

and for each $\zeta > 0$ we can choose $n \in \mathbb{N}$ and $h_\zeta > 0$ such that the right hand side of the above inequality becomes less than ζ uniformly for $t \in (\tau, \tau+h_\zeta)$. This proves that $E_1 \subset \mathfrak{E}^\tau_\varepsilon$.

By assumption $\|\varphi\|_{\delta}^{\mathfrak{E}_{\varepsilon}^{\tau}}=0$ implies $0=\|U(t,\tau)\varphi\|_{E_0}\to\|\varphi\|_{E_0}$ as $t\to\tau^+$ and we get $\varphi=0$. It then follows easily $\|\cdot\|_{\delta}^{\mathfrak{E}_{\varepsilon}^{\tau}}$ is the norm in $\mathfrak{E}_{\varepsilon}^{\tau}$.

Finally, if $\psi\in\{\phi\in E_1:\|\phi-w_0\|_{E_1}\leqslant\frac{r}{M}\}$ then $\sup_{s\in(\tau,\tau+\delta]}(s-\tau)^{\varepsilon}\|U(s,\tau)(\psi-w_0)\|_{E_{1+\varepsilon}}\leqslant M\|\psi-w_0\|_{E_1}\leqslant r$ and, evidently, $\psi\in B^{\delta}_{\mathfrak{E}_{\varepsilon}^{\tau}}(w_0,r)$, which completes the proof of part i).

We can now apply Theorem 1.7 with $w_0\in E_1$ and ensure that the time of existence δ_0 can be then chosen uniformly in a certain neighborhood of a given point $t_0\in\mathbb{R}$. Actually, following the proof of existential part of Theorem 1.7 it suffices to ensure that the number $\bar\delta_0(R)$ in (2.17) can be chosen uniformly with respect to $\tau\in\mathcal{J}$.

Choose $w_0\in E_1$ and recall that $E_{1+\varepsilon}$ is dense in E_1. Using (1.2) for any $\phi\in E_{1+\varepsilon}$ we have

$$\begin{aligned}\sup_{\tau<t\leqslant\tau+\bar\delta_0}\|(t-\tau)^{\varepsilon}U(t,\tau)w_0\|_{E_{1+\varepsilon}}&\leqslant\sup_{\tau<t\leqslant\tau+\bar\delta_0}\|(t-\tau)^{\varepsilon}U(t,\tau)(w_0-\phi)\|_{E_{1+\varepsilon}}\\&+\sup_{\tau<t\leqslant\tau+\bar\delta_0}\|(t-\tau)^{\varepsilon}U(t,\tau)\phi\|_{E_{1+\varepsilon}}\leqslant M\|w_0-\phi\|_{E_1}+\bar\delta_0^{\varepsilon}M\|\phi\|_{E_{1+\varepsilon}}\ \tau\in\mathcal{J}.\end{aligned}\tag{2.22}$$

Evidently ϕ can be chosen such that $M\|w_0-\phi\|_{E_1}\leqslant\frac{R}{4}$ and $\bar\delta_0$ can be chosen (independently of τ) such that $\bar\delta_0^{\varepsilon}M\|\phi\|_{E_{1+\varepsilon}}\leqslant\frac{R}{4}$ in which case $\sup_{\tau<t\leqslant\tau+\bar\delta_0}\|(t-\tau)^{\varepsilon}U(t,\tau)w_0\|_{E_{1+\varepsilon}}\leqslant\frac{R}{2}$ for any $\tau\in\mathcal{J}$. □

Proof of Remark 1.9. We first prove that

$$\lim_{t\to\tau^+}\|(U(t,\tau)-I)\psi\|_{E_\alpha}=0\ \text{ for }\ \psi\in E_\alpha,\ \alpha\in[0,1+\mu).\tag{2.23}$$

For this observe that $\|(U(t,\tau)-I)\phi\|_{E_\alpha}\leqslant M(t-\tau)^{\beta-\alpha}\|\phi\|_{E_\beta}\to0$ as $t\to\tau^+$ whenever $\phi\in E_\beta$, $0\leqslant\alpha<\beta<1+\mu$. On the other hand, if $E_\beta\ni\phi_n\overset{E_\alpha}{\to}\psi\in E_\alpha$ then

$$\|(U(t,\tau)-I)\psi\|_{E_\alpha}\leqslant(M+1)\|\psi-\phi_n\|_{E_\alpha}+\|(U(t,\tau)-I)\phi_n\|_{E_\alpha}.$$

For $\zeta>0$ one can thus choose $n\in\mathbb{N}$ and $h_\zeta>0$ such that the right hand side of the above inequality will be less than ζ uniformly for $t\in(\tau,\tau+h_\zeta)$, which proves (2.23).

We next infer that

$$\lim_{t\to\tau^+}\|u(t)-U(t,\tau)u_\tau\|_{E_\alpha}=0\ \text{ for }\ \alpha\in[0,1),\ u_\tau\in E_\alpha.\tag{2.24}$$

Indeed, since u is $E_{1+\varepsilon}$-solution, for $\delta>\tau$ close enough to τ and $t\in(\tau,\delta)$ we have $\|u(t)\|_{E_{1+\varepsilon}}\leqslant(t-\tau)^{-\varepsilon}$. Via (1.5) also $\|F(s,u(s)\|_{E_{\gamma(\varepsilon)}}\leqslant\tilde c((s-\tau)^{-\varepsilon\rho}+1)$ for $t\in(\tau,\delta)$. Whenever $\gamma(\varepsilon)\leqslant\alpha<1$ and $t\in(\tau,\delta)$ we can thus estimate $\|u(t)-U(t,\tau)u_\tau\|_{E_\alpha}$ by $\int_\tau^t\|U(t,s)\|_{L(E_{\gamma(\varepsilon)},E_\alpha)}\|F(s,u(s))\|_{E_{\gamma(\varepsilon)}}ds$ and get

$$\begin{aligned}\|u(t)-U(t,\tau)u_\tau\|_{E_\alpha}&\leqslant\int_\tau^t M(t-s)^{\gamma(\varepsilon)-\alpha}\tilde c((s-\tau)^{-\varepsilon\rho}+1)ds\\&\leqslant\tilde cM\left((t-\tau)^{1+\gamma(\varepsilon)-\alpha-\varepsilon\rho}B(1+\gamma(\varepsilon)-\alpha,1-\varepsilon\rho)+(1+\gamma(\varepsilon)-\alpha)^{-1}(t-\tau)^{1+\gamma(\varepsilon)-\alpha}\right),\end{aligned}$$

where the right hand side tends to 0 as $t\to\tau^+$. Connecting (2.23) and (2.24) we get that $\lim_{t\to\tau^+}\|u(t)-u_\tau\|_{E_\alpha}=0$ whenever $u_\tau\in E_\alpha$ and $\alpha\in[0,1)$. By (1.15) and (2.23) the latter is also true for $\alpha=1$ and using Theorem 1.6 we obtain i).

For the proof of ii) note that given $u_\tau\in E_{1+\varepsilon}$ one can actually find a fixed point of $(\mathcal{T}v)(t)=U(t,\tau)u_\tau+\int_\tau^t U(t,s)F(s,v(s))ds$ in a complete metric space

$$\mathcal{K}_\xi(R,\tau)=\{v\in C([\tau,\tau+\xi],E_{1+\varepsilon}):|||v-u_\tau|||\leqslant R\}$$

with some $R>0$, $\xi>0$ and $|||v|||=\sup_{t\in[\tau,\tau+\xi]}\|v(t)\|_{E_{1+\varepsilon}}$. Indeed, given $v\in\mathcal{K}_\xi(R,\tau)$, we have by (2.23), (1.2) and (1.5) that for a suitably small $\xi>0$

$$\|(Tv)(t)-u_\tau\|_{E_{1+\varepsilon}}\leqslant\|U(t,\tau)u_\tau-u_\tau\|_{E_{1+\varepsilon}}+\int_\tau^t\|U(t,s)\|_{L(E_{\gamma(\varepsilon)},E_{1+\varepsilon})}\|F(s,v(s))\|_{E_{\gamma(\varepsilon)}}ds$$

$$\leqslant\|U(t,\tau)u_\tau-u_\tau\|_{E_{1+\varepsilon}}+M\int_\tau^t(t-s)^{\gamma(\varepsilon)-1-\varepsilon}c(\eta\|v(s)\|^\rho_{E_{1+\varepsilon}}+C_\eta)ds$$

$$\leqslant\|U(t,\tau)u_\tau-u_\tau\|_{E_{1+\varepsilon}}+c(\eta R^\rho+C_\eta)M(\gamma(\varepsilon)-\epsilon)^{-1}\xi^{\gamma(\varepsilon)-\epsilon}\leqslant R,\ t\in[\tau,\tau+\xi].$$

Hence $\mathcal{T}$ takes $\mathcal{K}_\xi(R,\tau)$ into itself. On the other hand, (1.2), (1.4) imply for $v,\tilde v\in\mathcal{K}_\xi(R,\tau)$

$$\|(Tv)(t)-(T\tilde v)(t)\|_{E_{1+\varepsilon}}\leqslant\int_\tau^t\|U(t,s)\|_{L(E_{\gamma(\varepsilon)},E_{1+\varepsilon})}\|F(s,v(s))-F(s,\tilde v(s))\|_{E_{\gamma(\varepsilon)}}ds$$

$$\leqslant c(2\eta R^{\rho-1}+C_\eta)\sup_{t\in[\tau,\tau+\xi]}\|v(t)-\tilde v(t)\|_{E_{1+\varepsilon}}M(\gamma(\varepsilon)-\epsilon)^{-1}\xi^{\gamma(\varepsilon)-\epsilon},\ t\in[\tau,\tau+\xi],$$

so that for $\xi>0$ small enough $\mathcal{T}:\mathcal{K}_\xi(R,\tau)\to\mathcal{K}_\xi(R,\tau)$ is a contraction. By uniqueness this ensures that $E_{1+\varepsilon}$-solution of (1.1) can be viewed as a fixed point of $\mathcal{T}$ in $\mathcal{K}_\xi(R,\tau)$ and hence it is right-continuous in $E_{1+\varepsilon}$ at τ. Combining this with Theorem 1.6 we get ii).

Finally, applying ii) and (1.14) with $\theta=0$ we obtain iii). □

2.4. **Proofs of continuation results.** In what follows we prove Theorems 1.10, 1.12.

Proof of part i) in Theorem 1.10 Recalling Remark 1.8 we assume that $T_{u_\tau}<\infty$, $\limsup_{t\to T^-_{u_\tau}}\|u(t,\tau,u_\tau)\|_{E_1}<r^*$ for some $r^*>0$ and for any $n\in\mathbb{N}$ large enough we define $\tau_n:=T_{u_\tau}-\frac{1}{n}$, $u_{\tau_n}:=u(T_{u_\tau}-\frac{1}{n},\tau,u_\tau)$. We then consider the Cauchy problem

$$\dot u(t)+A(t)u(t)=F(t,u(t)),\ t>\tau_n,\quad u(\tau_n)=u_{\tau_n},\tag{2.25}$$

where initial conditions u_{τ_n} belong both to $E_{1+\varepsilon}$ and to a ball $B_{E_1}(0,r^*)$ in E_1 of radius r^*. Also, the initial times τ_n converge to T_{u_τ}.

We then have $\sup_{s\in(\tau_n,\tau_n+\delta]}(s-\tau_n)^\varepsilon\|U(s,\tau_n)u_{\tau_n}\|_{E_{1+\varepsilon}}\leqslant M\|u_{\tau_n}\|_{E_1}\leqslant Mr^*$ and hence $u_{\tau_n}\in B^\delta_{\mathfrak{E}^{\tau_n}_\varepsilon}(0,Mr^*)$.

Due to Theorem 1.7 i)-ii) there is the unique $E_{1+\varepsilon}$-solution of (2.25) on $[\tau_n,\tau_n+\delta_0]$, where δ_0 does not depend on n (see Remark 1.8 ii)). By uniqueness, the latter solution coincides with $u(\cdot,\tau,u_\tau)$ on $[\tau_n,T_{u_\tau}]$ for each n sufficiently large. From this we infer that, by concatenation, $u=u(\cdot,\tau,u_\tau)$ can be continued as an $E_{1+\varepsilon}$-solution of (1.1) onto the interval $[\tau,T_{u_\tau}+\delta_0)$, which contradicts the definition of T_{u_τ}. □

Proof of part ii) in Theorem 1.10 Assume that $T_{u_\tau}<\infty$ and let $\tau_n\to T^-_{u_\tau}$ be such that $u(\tau_n,\tau,u_\tau)\to w_0$ in E_1 as $n\to\infty$. Then $\sup_{s\in(\tau_n,\tau_n+\delta]}(s-\tau_n)^\varepsilon\|U(s,\tau_n)(u_{\tau_n}-w_0)\|_{E_{1+\varepsilon}}\leqslant M\|u_{\tau_n}-w_0\|_{E_1}$. Hence, if r is chosen as in Theorem 1.7 relative to w_0 and $N\in\mathbb{N}$ is such that $\|u_{\tau_n}-w_0\|_{E_1}\leqslant\frac{r}{M}$ for $n\geqslant N$ then $u_{\tau_n}\in B^\delta_{\mathfrak{E}^{\tau_n}_\varepsilon}(\omega_0,r)$ for $n\geqslant N$ and $\delta>0$ close to zero.

Due to Theorem 1.7 (see Remark 1.8 ii)) there is the unique $E_{1+\varepsilon}$-solution of (2.25) on $[\tau_n,\tau_n+\delta_0]$, where δ_0 does not depend on n. Again, by uniqueness, this solution coincides with $u(\cdot,\tau,u_\tau)$ on $[\tau_n,T_{u_\tau}]$ for each n sufficiently large and thus $u=u(\cdot,\tau,u_\tau)$ can be continued as an $E_{1+\varepsilon}$-solution of (1.1) onto $[\tau,T_{u_\tau}+\delta_0)$, which contradicts definition of T_{u_τ}. □

Proof of (1.19) Assume that $T_{u_\tau}<\infty$ and let $\limsup_{t\to T^+_{u_\tau}}\|u(t,\tau,u_\tau)\|_{E_{1+\varepsilon}}<r^*$ for some $r^*>0$. For any $n\in\mathbb{N}$ large enough define $\tau_n:=T_{u_\tau}-\frac{1}{n}$, $u_{\tau_n}:=u(T_{u_\tau}-\frac{1}{n},\tau,u_\tau)$ and consider the Cauchy problem (2.25).

Since u_{τ_n} belongs to a ball $B_{E_{1+\varepsilon}}(0,r^*)$ in $E_{1+\varepsilon}$ of radius $r^*>0$ around zero, then for any $\delta>0$ small enough we have $\sup_{s\in(\tau_n,\tau_n+\delta]}(s-\tau_n)^\varepsilon\|U(s,\tau_n)u_{\tau_n}\|_{E_{1+\varepsilon}}\leqslant\delta^\varepsilon M\|u_{\tau_n}\|_{E_{1+\varepsilon}}\leqslant\delta^\varepsilon Mr^*$. Hence, if $r>0$ is chosen relatively to $w_0=0$ as in Theorem 1.7 and $\delta^\varepsilon\in(0,\frac{r}{r^*M})$, we observe that u_{τ_n} belongs to $B^{\delta}_{\mathfrak{E}^{\tau_n}_\varepsilon}(0,r)$.

As a consequence of Theorem 1.7 (see Remark 1.8 ii)) the problem (2.25) has the unique $E_{1+\varepsilon}$-solution on $[\tau_n,\tau_n+\delta_0]$, where δ_0 does not depend on n. By uniqueness the latter solution coincides with $u(\cdot,\tau,u_\tau)$ on $[\tau_n,T_{u_\tau}]$ for each n large enough and $u=u(\cdot,\tau,u_\tau)$ can be continued as an $E_{1+\varepsilon}$-solution of (1.1) onto $[\tau,T_{u_\tau}+\delta_0)$, which leads to contradiction. $\square$

Proof of Theorem 1.12 By assumption, given $\tau\in\mathbb{R}$ and $u_\tau\in\mathfrak{E}^\tau_\varepsilon$, we obtain from Theorem 1.7 that there exists the unique $E_{1+\varepsilon}$-solution u of (1.1) on the maximal interval of existence $[\tau,T_{u_\tau})$ and we denote $u_0:=u_\tau$, $T_{u_0}:=T_{u_\tau}$.

If $T_{u_0}<\infty$ then, using (1.26)-(1.27) and reflexivity of E_1, we conclude that there exists a certain $u_1\in E_1$ such that $\lim_{t\to T^-_{u_0}}\|u(t,\tau,u_\tau)-u_1\|_{E_0}=0$ and $u(t,\tau,u_\tau)\overset{t\to T^-_{u_0}}{\rightharpoonup}u_1$ weakly in E_1. Thus $u(t,\tau,u_\tau)$ can be extended to a function $\mathcal{U}_0$ defined on $[\tau,T_{u_0}]$ and satisfying $\mathcal{U}_0\in L^\infty_{loc}((\tau,T_{u_0}),E_{1+\varepsilon})$, $\mathcal{U}_0(\tau)=u_\tau=u_0$, $\mathcal{U}_0(t)\overset{E_0}{\to}\mathcal{U}_0(T_{u_0})=u_1\in\mathfrak{E}^{T_{u_0}}_\varepsilon$ as $t\to T^-_{u_0}$ and

$$\mathcal{U}_0(t)=U(t,\tau)u_0+\int_\tau^t U(t,s)F(s,\mathcal{U}_0(s))ds\ \text{ for }\ t\in(\tau,T_{u_0}).$$

By Theorem 1.7 there exists the unique $E_{1+\varepsilon}$-solution $u(\cdot,T_{u_0},u_1)$ of the Cauchy problem

$$\dot u(t)+A(t)u(t)=F(t,u(t)),\ t>T_{u_0},\quad u(T_{u_0})=u_1,$$

which can be continued on the maximal interval of existence $[T_{u_0},T_{u_1})$. Now, if $T_{u_1}<\infty$, repeating the above argument we find $u_2\in E_1$ such that $\lim_{t\to T^-_{u_1}}\|u(t,T_{u_0},u_1)-u_2\|_{E_0}=0$ and $u(t,T_{u_0},u_1)\overset{t\to T^-_{u_1}}{\rightharpoonup}u_2$ weakly in E_1. Thus $u(t,T_{u_0},u_1)$ can be extended to a function $\mathcal{U}_1$ defined on $[T_{u_0},T_{u_1}]$ and satisfying $\mathcal{U}_1\in L^\infty_{loc}((T_{u_0},T_{u_1}),E_{1+\varepsilon})$, $\mathcal{U}_1(T_{u_0})=u_1$, $\mathcal{U}_1(t)\overset{E_0}{\to}\mathcal{U}_1(T_{u_1})=u_2\in\mathfrak{E}^{T_{u_1}}_\varepsilon$ as $t\to T^-_{u_1}$ and

$$\mathcal{U}_1(t)=U(t,T_{u_0})u_1+\int_{T_{u_0}}^t U(t,s)F(s,\mathcal{U}_1(s))ds\ \text{ for }\ t\in(T_{u_0},T_{u_1}).$$

If there is $k\in\mathbb{N}$ such that, proceeding as above, we obtain in a $(k+1)$-th step that $T_{u_k}=\infty$, then function $\mathcal{U}$ defined on $[\tau,\infty)$ by concatenations of $\mathcal{U}_j$, $j=0,\dots,k+1$ is an extension of u to a piecewise-$E_{1+\varepsilon}$-solution on $[\tau,\infty)$.

Otherwise, proceeding inductively we will obtain a sequence of maps $\mathcal{U}_j$ on $[\tau,T_{u_j}]$, $j=0,1,\dots$, and by concatenations we again define a piecewise-$E_{1+\varepsilon}$-solution on $[\tau,a)$, with $a:=\sum_{j=0}^\infty T_{u_j}$. In this latter case it is evident that either $a=\infty$ or, if $a<\infty$, a is accumulation time of singular times $T_j:=\sum_{l=0}^j T_{u_l}$, $j\in\mathbb{N}$.

The above construction ensures that the extension of $E_{1+\varepsilon}$-solution to a piecewise-$E_{1+\varepsilon}$-solution is uniquely defined and hence the proof is complete. $\square$

3. Linear nonautonomous parabolic problems

In what follows X denotes a Banach space. We will discuss sufficient conditions for Assumptions 1.1, 1.2 in terms of $A(t)$.

Definition 3.1. *The family* $\{A(t):\ t\in\mathbb{R}\}$ *of closed operators* $A(t): D_X\subset X\longrightarrow X$*, which are defined on the same dense subset* D_X *of the Banach space* X*, is* locally uniformly sectorial *(of the class* $\mathcal{LUS}(D_X,X)$ *for short) if and only if for each* $t\in\mathbb{R}$ *the complex half plane* $\{\lambda\in\mathbb{C}: Re\lambda\leqslant 0\}$ *is contained in the resolvent set* $\rho(A(t))$ *of* $A(t)$ *and for any bounded time interval* $I\subset\mathbb{R}$ *there exists a certain* $M>0$ *such that*

$$\|(\lambda I-A(t))^{-1}\|_{L(X)}\leqslant\frac{M}{1+|\lambda|},\ Re\lambda\leqslant 0,\ t\in I. \tag{3.1}$$

If $\{A(t):\ t\in\mathbb{R}\}$ is of the class $\mathcal{LUS}(D_X,X)$ then, for each $s\in\mathbb{R}$, $-A(s)$ generates asymptotically decaying C^0 analytic semigroup $\{e^{-A(s)t}: t\geqslant 0\}$ in X. Actually, for a family $\{A(t):\ t\in\mathbb{R}\}$ of the class $\mathcal{LUS}(D_X,X)$ we have that $Re\sigma(A(s))>a>0$ and

$$\|e^{-A(t)s}\|_{L(X)}\leqslant Ce^{-as},\ s\geqslant 0,\quad \|A(t)e^{-A(t)s}\|_{L(X)}\leqslant\frac{C_1}{s}e^{-as},\ s>0,$$

where $a,C,C_1>0$ are independent of $s>0$ and t in bounded time intervals (see [26, §1.1]. Consequently, fractional powers $A^{\alpha}(t)$ can be defined as the inverse of $A^{-\alpha}(t): X\to R(X)$,

$$A^{-\alpha}(t)=\frac{1}{\Gamma(\alpha)}\int_0^{\infty}s^{\alpha-1}e^{-A(t)s}ds,\ \alpha>0. \tag{3.2}$$

Also, one can consider the associated fractional power spaces $X^{\alpha}(t)$, $\alpha\geqslant 0$,

$$X^{\alpha}(t):=D(A^{\alpha}(t))\ \text{ with the norm }\ \|\phi\|_{X^{\alpha}(t)}=\|A^{\alpha}(t)\phi\|_X\ \text{ for } \phi\in X^{\alpha},\ \alpha>0, \tag{3.3}$$

where for $\alpha=0$ we set $A^0(t):=Id$, $X^0(t):=X$. As in [26, §1.9, (1.56)] we then have

$$\|A^{\alpha}(t)e^{-A(t)s}\|_{L(X)}\leqslant c_{\alpha}e^{-as}s^{-\alpha},\ s>0, \tag{3.4}$$

where c_{α} neither depends on $s>0$ nor on t varying on bounded time intervals.

Since $A(t)$ coincides with the inverse of $A^{-1}(t)$, then $X^1(t)$ coincides as a set with D_X for every $t\in\mathbb{R}$. Concerning topologies we have the following result.

Proposition 3.2. *If* $\{A(t):\ t\in\mathbb{R}\}$ *is of the class* $\mathcal{LUS}(D_X,X)$ *and* $I\subset\mathbb{R}$ *is such that*

$$\sup_{t,s\in I}\|A(t)A^{-1}(s)\|_{L(X)}<\infty, \tag{3.5}$$

then $X^1(t)$ *are independent of* t*, except for norms, which are uniformly equivalent on* I*.*

Proof: Evidently, for the graph norms we have

$$\|\phi\|_{X^1(t)}=\|A(t)\phi\|_X=\|A(t)A^{-1}(s)A(s)\phi\|_X\leqslant c\|A(s)\phi\|_X=c\|\phi\|_{X^1(s)},\ t,s\in I.\quad\square$$

If $A(t)$ belongs to a class of operators having locally uniformly bounded purely imaginary powers, that is if $A(t)$ is a positive operator satisfying

$$\exists_{\epsilon>0}\ \sup_{s\in[-\epsilon,\epsilon]}\|A^{is}(t)\|_{L(X)}<\infty, \tag{3.6}$$

then fractional power spaces can be characterized as (see [28], also [5])

$$X^{(1-\theta)\alpha+\theta\beta}=[X^{\alpha}(t),X^{\beta}(t)]_{\theta},\ 0<\theta<1,\ 0\leqslant\alpha<\beta<\infty. \tag{3.7}$$

Remark 3.3. *It is known that* (3.6) *holds in many applications (see* [8, 11, 19, 24, 25, 28]*).*

Definition 3.4. *We will say that the family of positive operators* $\{A(t): t\in\mathbb{R}\}$ *is of the class* $\mathcal{BIP}(X)$ *if and only if, given any* $t\in\mathbb{R}$*,* $A(t)$ *has the property* (3.6).

Corollary 3.5. *If $\{A(t):\ t\in\mathbb{R}\}$ is of the class $\mathcal{LUS}(D_X,X)\cap\mathcal{BIP}(X)$ and $I\subset\mathbb{R}$ is such that* (3.5) *holds then $X^\theta(t)$, $\theta\in[0,1]$, are independent of $t\in I$, except for norms, which are uniformly equivalent on I.*

Following [3], given $\{A(t):\ t\in\mathbb{R}\}$ of the class $\mathcal{LUS}(D_X,X)$, we consider the extrapolated space $X^{-1}(t)$ generated by $(X,A(t))$, where

$$X^{-1}(t)\ \text{ is the completion of } (X,\|A^{-1}(t)\cdot\|_X).$$

We then extend $A(t)$ to a closed operator in $X^{-1}(t)$ (with the same notation).

Whenever $t,s\in\mathbb{R}$ are such that $A^{-1}(s)A(t),A^{-1}(t)A(s):D_X\subset X\to X$ are bounded operators, which happens in particular when the domains of the adjoint operators $A'(t)$ and $A'(s)$ are the same, then $X^{-1}(t)$ coincides with $X^{-1}(s)$ as for some $c_1,c_2>0$ we have

$$c_1\|A^{-1}(s)x\|_X\leqslant\|A^{-1}(t)x\|_X\leqslant c_2\|A^{-1}(s)x\|_X,\ x\in X,$$

(see [4]). This leads to the following counterpart of Proposition 3.2 for extrapolated spaces.

Proposition 3.6. *If $\{A(t):\ t\in\mathbb{R}\}$ is of the class $\mathcal{LUS}(D_X,X)$ and $A^{-1}(t)A(s):D_X\subset X\to X$ are uniformly bounded for $t,s\in I$; i.e. for the closure $\overline{A^{-1}(t)A(s)}$ we have*

$$\sup_{t,s\in I}\|\overline{A^{-1}(t)A(s)}\|_{L(X)}<\infty,\tag{3.8}$$

then $X^{-1}(t)$ are independent of $t\in I$ except for norms which are uniformly equivalent on I.

Due to [3, Proposition V.1.31], if $\{A(t):t\in\mathbb{R}\}$ is of the class $\mathcal{LUS}(D_X,X)$ then (closed extension of) $A(t)$ belongs to a class $Lis(X,X^{-1}(t))$ of linear isomorphisms from X into $X^{-1}(t)$. Furthermore, $\{\lambda\in\mathbb{C}:Re\lambda\leqslant0\}\subset\rho(A(t))$ and given any bounded time interval I

$$\|(\lambda I-A(t))^{-1}\|_{L(X^{-1}(t))}\leqslant\frac{M}{1+|\lambda|},\ Re\lambda\leqslant0,\ t\in I.\tag{3.9}$$

for some $M>0$. Letting $Y(t)=X^{-1}(t)$ and applying (3.2) one can associate with $(Y(t),A(t))$ the fractional power scale $\{Y^\alpha(t):\alpha\geqslant0\}$ and consider, as in [3, p. 266],

$$X^\alpha(t):=Y^{\alpha+1}(t),\ \alpha\in[-1,\infty),\tag{3.10}$$

which is the extrapolated fractional power scale of order 1 generated by $(X,A(t))$.

Corollary 3.7. *If $\{A(t):t\in\mathbb{R}\}$ is of the class $\mathcal{LUS}(D_X,X)\cap\mathcal{BIP}(X)$ and $I\subset\mathbb{R}$ is such that* (3.5), (3.8) *hold, then for each $\theta\in[-1,1]$ spaces $Y^{\theta+1}(t)=X^\theta(t)$ are independent of $t\in I$, except for norms, which are uniformly equivalent on I; that is for every $\theta\in[-1,1]$*

$$\|\phi\|_{X^\theta(t)}\leqslant c\|\phi\|_{X^\theta(s)},\ s,t\in I,$$

for some $c>0$ and every ϕ from the set $X^\theta(t)=X^\theta(s)$.

Given $t_0\in\mathbb{R}$, $\alpha_0\in[0,1)$ and letting $\mu_0:=1-\alpha_0$ we next define

$$E_\alpha:=Y^{\alpha+\alpha_0}(t_0),\quad\|\cdot\|_{E_\alpha}=\|A^{\alpha+\alpha_0}(t_0)\cdot\|_{Y(t_0)},\quad\alpha\in[0,1+\mu_0].\tag{3.11}$$

Lemma 3.8. *Suppose that $\{A(t):t\in\mathbb{R}\}$ is of the class $\mathcal{LUS}(D_X,X)\cap\mathcal{BIP}(X)$, conditions* (3.5), (3.8) *hold on each bounded time interval $I\subset\mathbb{R}$ and $\{E_\alpha,\alpha\in[0,1+\mu_0]\}$ is defined as in* (3.11)*, where μ_0 is a strictly positive number. Then,*
i) $\{A(t):t\in\mathbb{R}\}$ is of the class $\mathcal{LUS}(D_{E_0},E_0)$ with $D_{E_0}=E_1$,

ii) for any bounded time interval $I \subset \mathbb{R}$ and $\sigma \in [0, 1 + \mu_0]$ there exist constants $c, c', c'' > 0$ such for each $t, s \in I$ we have

$$\|\phi\|_{E_\sigma} \leqslant c''\|A^\sigma(t)\phi\|_{E_0} \leqslant c\|A^\sigma(s)\phi\|_{E_0} \leqslant c'\|\phi\|_{E_\sigma},\ \phi \in E_\sigma. \tag{3.12}$$

Proof: Recall that $\{Y^\alpha(t) : \alpha \geqslant 0\}$ is the fractional power scale generated by $(Y(t), A(t))$. Hence (realization of) $A(t)$ can be viewed as a closed densely defined operator in $Y^{\alpha_0}(t)$ with the domain $Y^{\alpha_0+1}(t)$. The resolvent set of $A(t)$ in this setting will still contain $\{\lambda \in \mathbb{C} : Re\lambda \leqslant 0\}$ and for each bounded time interval I there will be a constant $M > 0$ such that

$$\|(\lambda I - A(t))^{-1}\phi\|_{Y^{\alpha_0}(t)} \leqslant \frac{M}{1 + |\lambda|}\|\phi\|_{Y^{\alpha_0}(t)},\ Re\lambda \leqslant 0,\ t \in I,\ \phi \in Y^{\alpha_0}(t). \tag{3.13}$$

Part i) is thus a consequence of Corollary 3.7 and (3.9).

Concerning part ii) we first observe that, due to Corollary 3.7, if $\phi \in E^\sigma$ then ϕ belongs to each of the sets $Y^{\sigma+\alpha_0}(t)$, $Y^{\sigma+\alpha_0}(s)$ as these sets coincide for $t, s \in \mathbb{R}$ and $A^\sigma(t)\phi, A^\sigma(s)\phi$ are the elements of $E^0 = Y^{\alpha_0}(t_0)$. Actually $A^\sigma(t), A^\sigma(s)$ are one-to-one from E_σ onto E_0.

Given a bounded time interval $I \subset \mathbb{R}$ we can thus use equivalence of norms stated in Corollary 3.7 to get, for some constants $\bar{c}, \tilde{c}, \hat{c}$ depending on I but not on $t, s \in I$, that

$$\begin{aligned}\|A^\sigma(t)\phi\|_{E_0} &= \|A^\sigma(t)\phi\|_{Y^{\alpha_0}(t_0)} \leqslant \bar{c}\|A^\sigma(t)\phi\|_{Y^{\alpha_0}(t)} = \bar{c}\|\phi\|_{Y^{\sigma+\alpha_0}(t)} \leqslant \tilde{c}\|\phi\|_{Y^{\sigma+\alpha_0}(s)} \\ &= \tilde{c}\|A^\sigma(s)\phi\|_{Y^{\alpha_0}(s)} \leqslant \hat{c}\|A^\sigma(s)\phi\|_{Y^{\alpha_0}(t_0)} = \hat{c}\|A^\sigma(s)\phi\|_{E_0}\end{aligned}$$

whenever $t, s \in I$. Similarly, using again the equivalence of norms, we also have

$$\|\phi\|_{E_\sigma} = \|\phi\|_{Y^{\sigma+\alpha_0}(t_0)} \leqslant \tilde{c}\|\phi\|_{Y^{\sigma+\alpha_0}(t)} = \tilde{c}\|A^\sigma(t)\phi\|_{Y^{\alpha_0}(t)} \leqslant \hat{c}\|A^\sigma(t)\phi\|_{Y^{\alpha_0}(t_0)} = \hat{c}\|A^\sigma(t)\phi\|_{E_0},$$

$$\|A^\sigma(s)\phi\|_{E_0} = \|A^\sigma(s)\phi\|_{Y^{\alpha_0}(t_0)} \leqslant \tilde{c}\|A^\sigma(s)\phi\|_{Y^{\alpha_0}(s)} = \tilde{c}\|\phi\|_{Y^{\sigma+\alpha_0}(s)} \leqslant \hat{c}\|\phi\|_{Y^{\sigma+\alpha_0}(t_0)} = \hat{c}\|\phi\|_{E_\sigma},$$

which proves ii). □

Corollary 3.9. *Under the assumptions of Lemma 3.8 we have that for any bounded time interval $I \subset \mathbb{R}$ and $\sigma \in [0, 1 + \mu_0]$ there exists a constant $c > 0$ such that*

$$\|A^\sigma(t)A^{-\sigma}(s)\|_{L(E_0)} \leqslant c \ \textit{ for each } \ t, s \in I. \tag{3.14}$$

Proof: It suffices to note that $A^\sigma(t)$, $A^\sigma(s)$ are one-to-one from E_σ onto E_0 and use (3.12). □

We will next assume that $\{A(t) : t \in \mathbb{R}\}$ is of the class $\mathcal{LUS}(D_X, X)$ and, in addition,

$$\exists_{\mu\in(0,1]}\ \forall_{T>0}\ \exists_{C>0}\ \forall_{t,\tau,s\in[-T,T]}\ \|(A(t) - A(\tau))A^{-1}(s)\|_{L(X)} \leqslant C|t - \tau|^\mu. \tag{3.15}$$

Following [3, 15, 20, 23, 26], we will consider in X a nonautonomous linear problem

$$\dot{u}(t) + A(t)u(t) = 0,\ t > \tau, \quad u(\tau) = u_\tau. \tag{3.16}$$

Recall that a continuous function $[\tau, \infty) \ni t \to u(t) \in E_0$ is a *classical solution* of (3.16) if it is continuously differentiable in (τ, ∞), $u(t) \in D_X$ for each $t > \tau$ and u satisfies (3.16). Recall also that a two parameter family $\{U(t, \tau) : (t, \tau) \in \mathbb{R}^2,\ t \geqslant \tau\}$ of maps $U(t, \tau) : E_0 \to E_0$ is a continuous process in E_0 provided that $U(\tau, \tau) = Id$, $U(t, \sigma)U(\sigma, \tau) = U(t, \tau)$ for $t \geqslant \sigma \geqslant \tau \in \mathbb{R}$ and $\{(t, s) \in \mathbb{R}^2 : t \geqslant s\} \times V \ni (t, \tau, v) \mapsto U(t, \tau)v \in E_0$ is a continuous map.

The following result is known (see [15, §2] for the proof).

Proposition 3.10. *If $\{A(t) : t \in \mathbb{R}\}$ is of the class $\mathcal{LUS}(D_X, X)$ and* (3.15) *holds then, there exists a continuous process $\{U(t, \tau) : t \geqslant \tau \in \mathbb{R}\} \subset L(X)$ in X such that given $\tau \in \mathbb{R}$ and $u_\tau \in X$, the map $[\tau, \infty) \ni t \to u(t) = U(t, \tau)u_\tau \in X$ is a classical solution of* (3.16).

To describe smoothing properties of the process we state the following result.

Proposition 3.11. *Under the assumptions of Proposition 3.10 for each bounded time interval $I = [-T, T]$ there is a positive constant N such that with μ as in* (3.15) *we have*

$$\|A^{\sigma}(t)U(t,\tau)A^{-\zeta}(\tau)\|_{L(X)} \leqslant N(t-\tau)^{\zeta-\sigma},\ 0 \leqslant \zeta \leqslant \sigma < 1+\mu,\ -T \leqslant \tau < t \leqslant T. \tag{3.17}$$

For the proof of (3.17) we refer the reader to [26] (see also [15, Theorem 2.2]). To obtain another smoothing property we will need the following *additional assumption*:

$$\forall_{1+\mu>\xi>0}\ \forall_{T>0}\ \exists_{c>0}\ \forall_{t,\tau\in[-T,T]}\ \|A^{\xi}(t)A^{-\xi}(\tau)\|_{L(X)} \leqslant c. \tag{3.18}$$

Proposition 3.12. *If $\{A(t):\ t\in\mathbb{R}\}$ is of the class $\mathcal{LUS}(D_X, X)$ and* (3.15) *holds then*

$$\forall_{T>0}\ \forall_{\substack{1\geqslant\zeta>\sigma\geqslant 0\\ 1>\zeta-\sigma>\delta>0}}\ \exists_{N>0}\ \forall_{-T\leqslant\tau\leqslant t\leqslant T}\ \|A^{\sigma}(t)[U(t,\tau)-Id]A^{-\zeta}(\tau)\|_{L(X)} \leqslant N(t-\tau)^{\delta}. \tag{3.19}$$

Actually, if also (3.18) *is satisfied then*

$$\forall_{T>0}\ \forall_{\substack{1+\mu>\zeta>\sigma\geqslant 0\\ 1>\zeta-\sigma}}\ \exists_{N>0}\ \forall_{-T\leqslant\tau\leqslant t\leqslant T}\ \|A^{\sigma}(t)[U(t,\tau)-Id]A^{-\zeta}(\tau)\|_{L(X)} \leqslant N(t-\tau)^{\zeta-\sigma}. \tag{3.20}$$

Proof: From [26, (1.53)] we infer that

$$U(t,\tau)A^{-\zeta}(\tau) = e^{(t-\tau)A(t)}A^{-\zeta}(\tau) + \int_{\tau}^{t} e^{(t-s)A(t)}[A(s)-A(t)]U(s,\tau)A^{-\zeta}(\tau)ds.$$

We next rewrite $A^{\sigma}(t)[U(t,\tau)-Id]A^{-\zeta}(\tau)$ as a sum J_1+J_2, where

$$J_1 = A^{\sigma}(t)[e^{(t-\tau)A(t)}-Id]A^{-\zeta}(\tau) \ \text{ and }\ J_2 = \int_{\tau}^{t} A^{\sigma}(t)e^{(t-s)A(t)}[A(s)-A(t)]U(s,\tau)A^{-\zeta}(\tau)ds.$$

From (3.15) we obtain that (3.5) holds on any bounded time interval $I\subset\mathbb{R}$. Hence, under the assumptions that $\{A(t):\ t\in\mathbb{R}\}$ is of the class $\mathcal{LUS}(D_X, X)$ and (3.15) holds, one obtains as in [26, §1.9, (1.59)] that

$$\forall_{1\geqslant\zeta>\xi\geqslant 0}\ \forall_{T>0}\ \exists_{c>0}\ \forall_{t,\tau\in[-T,T]}\ \|A^{\xi}(t)A^{-\zeta}(\tau)\|_{L(X)} \leqslant c. \tag{3.21}$$

If $1\geqslant\zeta>\sigma\geqslant 0$ and $0<\delta<\zeta-\sigma<1$ then applying [21, Theorem 1.4.3] we can estimate $\|J_1v\|_X$ for every $v\in X$ by $\frac{1}{\delta}c_{1-\delta}(t-\tau)^{\delta}\|A^{\delta+\sigma}(t)A^{-\zeta}(\tau)v\|_X$, which via (3.21) can be bounded on $[-T,T]$ by $\frac{1}{\delta}cc_{1-\delta}(t-\tau)^{\delta}\|v\|_X$.

If $1+\mu>\zeta>\sigma\geqslant 0$ and (3.18) holds, then choosing $\tilde{\delta}=\zeta-\sigma$ and applying [21, Theorem 1.4.3] we estimate $\|J_1v\|_X$ for each $v\in X$ by $\frac{1}{\delta}c_{1-\tilde{\delta}}(t-\tau)^{\tilde{\delta}}\|A^{\tilde{\delta}+\sigma}(t)A^{-\zeta}(\tau)v\|_X = \frac{1}{\zeta-\sigma}c_{1-\zeta+\sigma}(t-\tau)^{\zeta-\sigma}\|A^{\zeta}(t)A^{-\zeta}(\tau)v\|_X$, which due to (3.18) can be next estimated on $[-T,T]$ by $\frac{1}{\zeta-\sigma}cc_{1-\zeta+\sigma}(t-\tau)^{\zeta-\sigma}\|v\|_X$.

Consequently, not assuming (3.18) we obtain that $\|J_1\|_{L(X)}\leqslant\frac{1}{\delta}cc_{1-\delta}(t-\tau)^{\delta}$, $0<\delta<\zeta-\sigma$, whereas assuming (3.18) $\|J_1\|_{L(X)}\leqslant\frac{1}{\zeta-\sigma}cc_{1-\zeta+\sigma}(t-\tau)^{\zeta-\sigma}$.

The integral J_2 is equal to $\int_{\tau}^{t} A^{\sigma}(t)e^{(t-s)A(t)}[(A(s)-A(t))A^{-1}(s)]A(s)U(s,\tau)A^{-\zeta}(\tau)ds$, where by (3.4), (3.15) we have $\|A^{\sigma}(t)e^{(t-s)A(t)}[(A(s)-A(t))A^{-1}(s)]\|_{L(X)}\leqslant c(t-s)^{-\sigma}(t-s)^{\mu}$. Note that if $0\leqslant\zeta\leqslant 1$ we obtain from (3.17) that $\|A(s)U(s,\tau)A^{-\zeta}(\tau)\|_{L(X)}\leqslant c(s-\tau)^{\zeta-1}$, whereas if $1+\mu>\zeta>1$, then $A(s)U(s,\tau)A^{-\zeta}(\tau)=A(s)U(s,\tau)A^{-1}(\tau)A^{1-\zeta}(\tau)$ and

$$\|A(s)U(s,\tau)A^{-\zeta}(\tau)\|_{L(X)} \leqslant \|A(s)U(s,\tau)A^{-1}(\tau)\|_{L(X)}\|A^{1-\zeta}(\tau)\|_{L(X)} \leqslant c,$$

as in this case $A^{1-\zeta}(\tau)=A^{-(\zeta-1)}(\tau)=\frac{1}{\Gamma(\zeta-1)}\int_0^{\infty}s^{\zeta-2}e^{-A(\tau)s}ds$ is a bounded operator and

$$\|A^{1-\zeta}(\tau)\|_{L(X)} \leqslant \frac{C}{\Gamma(\zeta-1)}\int_0^{\infty}s^{\zeta-2}e^{-as}ds = Ca^{1-\zeta}.$$

Since t, τ vary in a bounded interval we thus infer that for $0 \leqslant \zeta \leqslant 1$

$$\begin{aligned}\|J_2\|_{L(X)} &\leqslant \tilde{c} \int_\tau^t (t-s)^{\mu-\sigma}(s-\tau)^{\zeta-1} ds \leqslant \tilde{c}(t-\tau)^{\mu-\sigma+\zeta} B(1+\mu-\sigma,\zeta) \\ &= \tilde{c}B(1+\mu-\sigma,\zeta)(t-\tau)^{\mu}(t-\tau)^{\zeta-\sigma} \leqslant \bar{c}(t-\tau)^{\zeta-\sigma},\end{aligned}$$

whereas for $1+\mu > \zeta > 1$

$$\|J_2\|_{L(X^\alpha)} \leqslant \tilde{c} \int_\tau^t (t-s)^{\mu-\sigma} ds = \hat{c}(t-\tau)^{1+\mu-\sigma} = \hat{c}(t-\tau)^{1+\mu-\zeta}(t-\tau)^{\zeta-\sigma} \leqslant \bar{c}(t-\tau)^{\zeta-\sigma}.$$

Combining the above estimates we get the result. □

Theorem 3.13. *Suppose that $\{A(t) : t \in \mathbb{R}\}$ is of the class $\mathcal{LUS}(D_X, X) \cap \mathcal{BIP}(X)$, conditions* (3.5), (3.8) *hold on each bounded time interval $I \subset \mathbb{R}$ and $\{E_\alpha, \alpha \in [0, 1+\mu_0]\}$ is defined as in* (3.11)*. Suppose furthermore that*

$$\exists_{\mu \in (0,\mu_0]}\ \forall_{T>0}\ \exists_{C>0}\ \forall_{t,\tau,s \in [-T,T]}\ \|(A(t)-A(\tau))A^{-1}(s)\|_{L(E^0)} \leqslant C|t-\tau|^{\mu}. \tag{3.22}$$

Under these assumptions:
i) there exists a continuous process $\{U(t,\tau) :\ t \geqslant \tau \in \mathbb{R}\} \subset L(E_0)$ defined by (3.16) *in E_0 such that given $\tau \in \mathbb{R}$ and $u_\tau \in E_0$, the map $[\tau,\infty) \ni t \to u(t) = U(t,\tau)u_\tau \in E^0$ is a classical solution of* (3.16)*; furthermore,*
ii) $\|U(t,\tau)\|_{L(E_\zeta,E_\sigma)} \leqslant M(t-\tau)^{\zeta-\sigma}$, $0 \leqslant \zeta \leqslant \sigma < 1+\mu$, $-T \leqslant \tau < t \leqslant T$, and
iii) $\|U(t,\tau)-Id\|_{L(E_\zeta,E_\sigma)} \leqslant M(t-\tau)^{\zeta-\sigma}$, $1+\mu > \zeta > \sigma \geqslant 0$, $1 \geqslant \zeta-\sigma > 0$, $-T \leqslant \tau < t \leqslant T$, where constant M in ii)-iii) can depend on ζ, σ, T but does not depend on $t, \tau \in [-T,T]$.

Proof: Due to Lemma 3.8 we obtain that $\{A(t) : t \in \mathbb{R}\}$ is of the class $\mathcal{LUS}(D_{E_0}, E_0)$ with $D_{E_0} = E_1$. From this and (3.22) we obtain i) applying Proposition 3.10 with $X = E_0$.

From Proposition 3.11 (applied with $X = E_0$), for each bounded time interval $I \subset \mathbb{R}$ there is then a positive constant N such that

$$\|A^{\sigma}(t)U(t,\tau)A^{-\zeta}(\tau)\|_{L(X)} \leqslant N(t-\tau)^{\zeta-\sigma},\ 0 \leqslant \zeta \leqslant \sigma < 1+\mu,\ t \in I. \tag{3.23}$$

Since $A^{\zeta}(\tau)$ is one-to-one from E_σ onto E_0, (3.23) can be rewritten equivalently as

$$\|A^{\sigma}(t)U(t,\tau)\phi\|_{E_0} \leqslant N(t-\tau)^{\zeta-\sigma}\|A^{\zeta}(\tau)\phi\|_{E_0},\ \phi \in E_\zeta, \tag{3.24}$$

and by (3.12) also as $\|[U(t,\tau)\phi\|_{E_\sigma} \leqslant M(t-\tau)^{\zeta-\sigma}\|\phi\|_{E_\zeta},\ \phi \in E_\zeta$, which gives ii).

Finally, by Corollary 3.9 we can use Proposition 3.12 with $X = E_0$ and obtain from (3.20)

$$\forall_{T>0}\ \forall_{\substack{1+\mu>\zeta>\sigma\geqslant 0\\ 1>\zeta-\sigma}}\ \exists_{N>0}\ \forall_{-T\leqslant\tau\leqslant t\leqslant T}\ \|A^{\sigma}(t)[U(t,\tau)-Id]A^{-\zeta}(\tau)\|_{L(E_0)} \leqslant N(t-\tau)^{\zeta-\sigma}. \tag{3.25}$$

Inequality in (3.25) can be rewritten equivalently as

$$\|A^{\sigma}(t)[U(t,\tau)-Id]\phi\|_{E_0} \leqslant N(t-\tau)^{\zeta-\sigma}\|A^{\zeta}(\tau)\phi\|_{E_0},\ \phi \in E_\zeta,$$

and by (3.12) also as $\|[U(t,\tau)-Id]\phi\|_{E_\sigma} \leqslant M(t-\tau)^{\zeta-\sigma}\|\phi\|_{E_\zeta},\ \phi \in E_\zeta$, which gives iii). □

Note that condition (3.22) can be expressed equivalently as in the proposition below.

Proposition 3.14. *Suppose $\{A(t) : t \in \mathbb{R}\}$ is of the class $\mathcal{LUS}(D_X, X) \cap \mathcal{BIP}(X)$,* (3.5), (3.8) *hold on each bounded time interval $I \subset \mathbb{R}$ and $\{E_\alpha, \alpha \in [0, 1+\mu_0]\}$ is as in* (3.11).

Then (3.22) *is equivalent to*

$$A(\cdot) \in C^{\mu}_{loc}(\mathbb{R}, L(E_1, E_0)). \tag{3.26}$$

Proof: For any bounded time interval $I \subset \mathbb{R}$ condition (3.22) implies that

$$\|(A(t) - A(\tau))\phi\|_{E_0} \leqslant C|t-\tau|^{\mu}\|A(s)\phi\|_{E_0} \text{ and } \|A(t)\phi\|_{E_0} \leqslant c\|A(s)\phi\|_{E_0}$$

whenever $t, \tau, s \in I$, $\phi \in D_{E_0}$. Due to (3.12) we then have $\|(A(t) - A(\tau))\phi\|_{E_0} \leqslant \tilde{C}|t - \tau|^{\mu}\|\phi\|_{E_1}$ for $t, \tau \in I$, which proves that $A(\cdot) \in C^{\mu}(I, L(E_1, E_0))$. On the other hand, if $A(\cdot) \in C^{\mu}_{loc}(I, L(E_1, E_0))$ then, given a bounded time interval I, we have that $A(\cdot) \in C^{\mu}(I, L(E_1, E_0))$. Combining this with (3.12) we obtain $\|(A(t) - A(\tau))\psi\|_{E_0} \leqslant C|t - \tau|^{\mu}\|\psi\|_{E_1} \leqslant \tilde{C}|t-\tau|^{\mu}\|A(s)\psi\|_{E_0}$, $t, \tau, s \in I$, $\psi \in E_1$. Letting $\phi = A^{-1}(s)\psi$ we get (3.22). □

Under the assumptions of Theorem 3.13 both Theorems 1.6 and 1.7 apply provided that the required assumption on F holds. In applications we often have some $\nu_0 \in (0,1)$ such that for each bounded time interval $I \subset \mathbb{R}$ and B bounded in $E_{1+\varepsilon}$ there is $c > 0$ such that

$$\|F(t,v) - F(s,w)\|_{E_0} \leqslant c(|t-s|^{\nu_0} + \|v - w\|_{E_{1+\varepsilon}}), \ t, s \in I, \ v, w \in B. \tag{3.27}$$

Then $E_{1+\varepsilon}$-solution will have the properties of a classical solution; see Proposition 3.15.

Proposition 3.15. *Suppose $\{A(t) : t \in \mathbb{R}\}$ is of the class $\mathcal{LUS}(D_X, X) \cap \mathcal{BIP}(X)$,* (3.5), (3.8) *hold on each bounded time interval $I \subset \mathbb{R}$ and $\{E_\alpha, \alpha \in [0, 1 + \mu_0]\}$ is as in* (3.11). *Assume also* (3.22), (3.27) *and that F is of the class $\mathcal{L}(\varepsilon, \rho, \gamma(\varepsilon), \eta, C_\eta)$ relative to $\{E_\alpha, \alpha \in [0, 1 + \mu)\}$.*

Then $\{A(t) : t \in \mathbb{R}\}$ is of the class $\mathcal{LUS}(D_{E_0}, E_0)$ with $D_{E_0} = E_1$ and Theorems 1.6, 1.7 apply. The unique $E_{1+\varepsilon}$-solution, $u = u(\cdot, \tau, u_\tau)$, is of the class $C^1((\tau, \tau+\delta_0], E_0)$, $u(t) \in D_{E_0}$ for $t \in (\tau, \tau + \delta_0]$ and $\dot{u}(t) + A(t)u(t) = F(t, u(t))$ for each $t \in (\tau, \tau + \delta_0]$.

Proof: By Theorem 3.13 we know that Theorems 1.7 and 1.6 apply. Hence there is the unique $E_{1+\varepsilon}$-solution of (1.1), $u = u(\cdot, \tau, u_\tau)$ and $u \in C^{\nu}_{loc}((0, T_{u_\tau}), E_{1+\varepsilon})$ for some $\nu \in (0, 1)$. The latter property and (3.27) yield that $F(\cdot, u(\cdot)) \in C^{\sigma}_{loc}((0, T_{u_\tau}), E_0)$ for $\sigma = \min\{\nu_0, \nu\}$. The result now follows as in [23, §5.7, Theorem 7.1] and [15, §2.3]). □

Remark 3.16. *Under assumptions of Proposition 3.15, following* [30, Theorem 3.10] *and letting $\mathcal{F}^{1,\sigma}(\tau, \tau + \delta_0], E_0)$ as in* [30, p. 5] *we have for $E_{1+\varepsilon}$-solution $u = u(\cdot, \tau, u_\tau)$ of* (1.1)

$$A(\cdot)u(\cdot) \in C((\tau, \tau + \delta_0], E_0), \quad \frac{d}{dt}u(\cdot) \in \mathcal{F}^{1,\sigma}(\tau, \tau + \delta_0], E_0).$$

4. Applications

In what follows we show how the abstract results apply in sample problems.

4.1. **Nonautonomous wave equation with structural damping.** In this example, following [16, 11, 13, 12], we consider the initial boundary value problem of the form:

$$\begin{cases} u_{tt} + \eta(t)(-\Delta)^{\frac{1}{2}}u_t + \nu u_t + (-\Delta)u = f(t,u), \quad t > 0, \quad x \in \Omega, \\ u(0,x) = u_0(x), \ x \in \Omega \quad u_t(0,x) = v_0(x), \ x \in \Omega, \quad u(t,x) = 0, \ t \geqslant 0, \ x \in \partial\Omega, \end{cases} \tag{4.1}$$

where $(u_0, v_0) \in H^1_0(\Omega) \times L^2(\Omega)$ and $-\Delta$ is the negative Dirichlet Laplacian in $L^2(\mathbb{R}^N)$.

Assumption 4.1. *Suppose Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geqslant 3$, $\nu \geqslant 0$ and*

$$\eta \in C^{\mu}_{loc}(\mathbb{R}, (0, \infty)) \ \textit{for some} \ \mu \in (0, 1]. \tag{4.2}$$

We remark that due to (4.2), given any bounded time interval $I \subset \mathbb{R}$, there are constants $\kappa_1, \kappa_2 > 0$, such that $\eta(t) \in [\kappa_1, \kappa_2]$ for each $t \in I$. Letting $v = \dot{u}$ we rewrite (4.1) in the form

$$\frac{d}{dt}\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] + A(t)\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] = F(t, \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]),\ t > 0, \qquad \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]_{t=\tau} = \left[\begin{smallmatrix} u_\tau \\ v_\tau \end{smallmatrix}\right], \tag{4.3}$$

where $A(t)$ and $F(t, \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right])$ can be viewed in matrix form as

$$A(t) = \begin{bmatrix} 0 & -I \\ -\Delta & \eta(t)(-\Delta)^{\frac{1}{2}} + \nu I \end{bmatrix}, \quad F(t, \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]) = \begin{bmatrix} 0 \\ f^e(t,u) \end{bmatrix} \tag{4.4}$$

and f^e denotes a Nemitskiĭ operator associated with f.

We set in this example $X = H^1_0(\Omega) \times L^2(\Omega)$, $D_X = (H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega)$ and, referring to [14, proof of Lemma 1 iii)] and [11, Proposition 1], we conclude that $\{A(t) : t \in \mathbb{R}\}$ is of the class $\mathcal{LUS}(D_X, X) \cap \mathcal{BIP}(X)$. Furthermore, for any $t, s \in \mathbb{R}$ we obtain

$$A(t)A^{-1}(s) = \begin{bmatrix} 0 & -I \\ -\Delta & \eta(t)(-\Delta)^{\frac{1}{2}} + \nu I \end{bmatrix} \begin{bmatrix} \eta(s)(-\Delta)^{-\frac{1}{2}} + \nu(-\Delta)^{-1} & (-\Delta)^{-1} \\ -I & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ (\eta(s)-\eta(t))(-\Delta)^{\frac{1}{2}} & I \end{bmatrix},$$

$$A^{-1}(s)A(t) = \begin{bmatrix} \eta(s)(-\Delta)^{-\frac{1}{2}} + \nu(-\Delta)^{-1} & (-\Delta)^{-1} \\ -I & 0 \end{bmatrix} \begin{bmatrix} 0 & -I \\ -\Delta & \eta(t)(-\Delta)^{\frac{1}{2}} + \nu I \end{bmatrix} = \begin{bmatrix} I & (\eta(s)-\eta(t))(-\Delta)^{-\frac{1}{2}} \\ 0 & I \end{bmatrix}.$$

Consequently, for any bounded time interval $I \subset \mathbb{R}$, we have

$$\sup_{t,s\in I} \|A(t)A^{-1}(s)\|_{L(X)} = \sup_{t,s\in I} \sup_{\|\left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right]\|_X = 1} \| \begin{bmatrix} \phi \\ (\eta(s)-\eta(t))(-\Delta)^{\frac{1}{2}}\phi + \psi \end{bmatrix} \|_X \leqslant (1 + 2\kappa_2), \tag{4.5}$$

$$\sup_{t,s\in I} \|\overline{A^{-1}(s)A(t)}\|_{L(X)} = \sup_{t,s\in I} \sup_{\|\left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right]\|_X = 1} \| \begin{bmatrix} \phi + (\eta(s)-\eta(t))(-\Delta)^{-\frac{1}{2}}\psi \\ \psi \end{bmatrix} \|_X \leqslant (1 + 2\kappa_2), \tag{4.6}$$

which are counterparts of (3.5) and (3.8).

Let $\{Z^\alpha, \alpha \geqslant -1\}$ be the extrapolated fractional power scale generated by $(L^2(\Omega), -\Delta)$. As in (3.11), choosing $\alpha_0 = 0$, we define the spaces E_α, $\alpha \in [0, 2]$. Due to [11, Theorem 2]:

$$E_\alpha := Y^{\alpha+\alpha_0}(t_0) = Z^{\frac{\alpha}{2}} \times Z^{\frac{\alpha-1}{2}}, \qquad \alpha \in [0, 2]. \tag{4.7}$$

By [3], $Z^{-\alpha}(t)$, $\alpha \in (0, 1)$, is viewed as completion of $(L^2(\Omega), \|(-\Delta)^{-\alpha} \cdot \|_{L^2(\Omega)})$.

With this set-up we now prove that

$$A(\cdot) \in C^\mu_{loc}(\mathbb{R}, L(E_1, E_0)) \text{ with } E_1 = H^1_0(\Omega) \times L^2(\Omega) \text{ and } E_0 = L^2(\Omega) \times H^{-1}(\Omega), \tag{4.8}$$

where μ is as in (4.2). Indeed, given $t, s \in [-T, T]$ we immediately have

$$\sup_{\|\left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right]\|_{E_1} = 1} \|[A(t) - A(s)] \left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right] \|_{E_0} = \sup_{\|\left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right]\|_{E_1} = 1} \left\| \begin{bmatrix} 0 \\ [\eta(t)-\eta(s)](-\Delta)^{\frac{1}{2}}\psi \end{bmatrix} \right\|_{E_0}$$

$$= |\eta(t) - \eta(s)| \sup_{\|\left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right]\|_{E_1} = 1} \|(-\Delta)^{\frac{1}{2}}\psi\|_{Z^{-\frac{1}{2}}} \leqslant c|t - s|^\mu.$$

Due to Proposition 3.14, (4.8) is equivalent with (3.22) and we can apply Theorem 3.13.

Proposition 4.2. *Suppose that Assumption 4.1 holds and let* $E_\alpha = Z^{\frac{\alpha}{2}} \times Z^{\frac{\alpha-1}{2}}$ *for* $\alpha \in [0, 1 + \mu)$, *where* μ *is as in* (4.2).

Then there exists a continuous process $\{U(t,\tau) : (t,\tau) \in \mathbb{R}^2, t \geqslant \tau \in \mathbb{R}\} \subset L(E_0)$ *associated in* $E_0 = L^2(\Omega) \times H_0^{-1}(\Omega)$ *with*

$$\frac{d}{dt}\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] + \left[\begin{smallmatrix} 0 & -I \\ -\Delta & \eta(t)(-\Delta)^{\frac{1}{2}}+\nu I \end{smallmatrix}\right]\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] = 0,\ t > 0, \qquad \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]_{t=\tau} = \left[\begin{smallmatrix} u_\tau \\ v_\tau \end{smallmatrix}\right], \tag{4.9}$$

and $\{U(t,\tau) : (t,\tau) \in \mathbb{R}^2, t \geqslant \tau \in \mathbb{R}\}$ *enjoys the smoothing properties* (1.2), (1.3).

Remark 4.3. *Besides* (4.8) *we also have that* $A(\cdot) \in C^\mu_{loc}(\mathbb{R}, L(E_2, E_1))$ *with* $E_2 = H^2(\Omega) \cap H^1_0(\Omega) \times H^1_0(\Omega)$, $E_1 = H^1_0(\Omega) \times L^2(\Omega)$ *as whenever* $t, s \in [-T, T]$ (4.2) *yields*

$$\sup_{\|\left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right]\|_{E_2}=1} \|[A(t) - A(s)]\left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right]\|_{E_1} = \sup_{\|\left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right]\|_{E_2}=1} \left\|\left[\begin{smallmatrix} 0 \\ [\eta(t)-\eta(s)](-\Delta)^{\frac{1}{2}}\psi \end{smallmatrix}\right]\right\|_{E_1}$$

$$= |\eta(t) - \eta(s)| \sup_{\|\left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right]\|_{E_2}=1} \|(-\Delta)^{\frac{1}{2}}\psi\|_{Z^0} \leqslant c|t - s|^\mu.$$

Assuming $N \geqslant 3$ we now define a number

$$\rho_c := \frac{N+2}{N-2},$$

which in this example plays a role of a critical exponent for initial data in $H^1_0(\Omega) \times L^2(\Omega)$.

Remark 4.4. *To keep the notation short let us adapt throughout the rest of the paper the Landau symbols* $O(\varphi)$, $o(\varphi)$. *Namely we will write that* $h(t, x, s) = O(\varphi(s))$ *if, given a bounded time interval* $I \subset \mathbb{R}$, $|h(t, x, s)| \leqslant c|\varphi(s)|$ *for some constant* $c > 0$, *which does not depend on* $s \in \mathbb{R}$, $x \in \Omega$ *and* $t \in I$. *We will also write that* $h(t, x, s) = o(\varphi(s))$ *if, given a bounded time interval* $I \subset \mathbb{R}$, $\lim_{|s|\to\infty} \frac{|h(t,x,s)|}{\varphi(s)} = 0$ *uniformly with respect to* $x \in \Omega$ *and* $t \in I$.

Proposition 4.5. *Suppose that* $N \geqslant 3$, $f \in C(\mathbb{R}^2, \mathbb{R})$ *has partial derivative* $f'_u \in C(\mathbb{R}\times\mathbb{R}, \mathbb{R})$ *and* $E_\alpha = Z^{\frac{\alpha}{2}} \times Z^{\frac{\alpha-1}{2}}$ *for* $\alpha \in [0, 1 + \mu)$, *where* μ *is as in* (4.2).
i) If $f'_s(t, s) = O(c_\eta + \eta|s|^{\rho-1})$ *for some* $\eta > 0$ *and* $\rho \in (1, \rho_c)$, *then the map* $F(t, \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right])$ *in* (4.4) *is of the class* $\mathcal{L}(\varepsilon, \rho, \gamma(\varepsilon), \eta, C_\eta)$ *relative to* $\{E_\alpha, \alpha \in [0, 1 + \mu)\}$ *and is subcritical.*
ii) If $f'_s(t, s) = O(c_\eta + \eta|s|^{\rho_c-1})$ *for some* $\eta > 0$ *and i) does not apply, then the map* $F(t, \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right])$ *in* (4.4) *is of the class* $\mathcal{L}(\varepsilon, \rho, \gamma(\varepsilon), \eta, C_\eta)$ *relative to* $\{E_\alpha, \alpha \in [0, 1 + \mu)\}$ *and is critical.*
iii) If $f'_s(t, s) = o(|s|^{\rho_c-1})$ *and i) does not apply, then* $F(t, \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right])$ *in* (4.4) *is of the class* $\mathcal{L}(\varepsilon, \rho, \gamma(\varepsilon), \eta, C_\eta)$ *relative to* $\{E_\alpha, \alpha \in [0, 1 + \mu)\}$ *and is almost critical.*

Proof: Parts i)-ii) follow similarly as [11, Lemma 3 and Corollary 2]. Also iii) can be proved analogously as in [13, Lemma 3.1 and Corollary 3.1]. We thus omit the details. □

Corollary 4.6. *Suppose that Assumption 4.1 holds and let* $E_\alpha = Z^{\frac{\alpha}{2}} \times Z^{\frac{\alpha-1}{2}}$ *for* $\alpha \in [0, 1+\mu)$, *where* μ *is as in* (4.2). *Suppose also that the assumptions of Proposition 4.5 are satisfied; in particular that* $f'_s(t, s) = O(c_\eta + \eta|s|^{\rho_c-1})$ *for some* $\eta > 0$.

Then Theorem 1.7 applies and, given any $\tau \in \mathbb{R}$, $\left[\begin{smallmatrix} u_\tau \\ v_\tau \end{smallmatrix}\right] \in H^1_0(\Omega) \times L^2(\Omega)$, *the abstract counterpart* (4.3)-(4.4) *of* (4.1) *has the unique* $E_{1+\varepsilon}$*-solution* $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] = \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](\cdot, \tau, \left[\begin{smallmatrix} u_\tau \\ v_\tau \end{smallmatrix}\right])$ *defined on the maximal interval of existence* $[\tau, T_{u_\tau,v_\tau})$.

With additional assumption on f there will be functional $\mathcal{L}$ decreasing along $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t, \tau, \left[\begin{smallmatrix} u_\tau \\ v_\tau \end{smallmatrix}\right])$

$$\mathcal{L}\left(\left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right]\right) = \frac{1}{2}\|w_2\|^2_{L^2(\Omega)} + \frac{1}{2}\|(-\Delta)^{\frac{1}{2}} w_1\|^2_{L^2(\Omega)} - \int_\Omega \int_0^{w_1} f(s)dsdx, \quad \left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right] \in E_1. \tag{4.10}$$

Lemma 4.7. *Suppose that f does not depend on t; that is $f = f(u)$.*

Then $\mathcal{L}$ in (4.10) takes bounded subsets of E_1 into bounded subsets of $\mathbb{R}$ and, given $E_{1+\varepsilon}$-solution $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$ of (4.3) on the interval I_τ, $\mathcal{L}\left(\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]\right)$ is nonincreasing for $t \in I_\tau$.

If

$$\limsup_{|s|\to\infty} \frac{f(s)}{s} < \lambda_1, \tag{4.11}$$

where λ_1 is the first positive eigenvalue of the negative Dirichlet Laplacian in $L^2(\Omega)$, then $\mathcal{L}$ is also bounded from below and for some constants $d_1, d_2 > 0$,

$$\left\|\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t, \tau, \left[\begin{smallmatrix} u_\tau \\ v_\tau \end{smallmatrix}\right])\right\|_{E_1} \leqslant d_1 \mathcal{L}(\left[\begin{smallmatrix} u_\tau \\ v_\tau \end{smallmatrix}\right]) + d_2. \tag{4.12}$$

Proof: Multiplying the first equation in (4.1) by $v = u_t$, we have

$$\frac{d}{dt}\left(\mathcal{L}\left(\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]\right)\right) = -\eta(t)\|(-\Delta)^{\frac{1}{4}} v\|^2_{L^2(\Omega)} - \nu\|v\|^2_{L^2(\Omega)} \leqslant 0, \tag{4.13}$$

which yields that $\mathcal{L}(\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]) \leqslant \mathcal{L}(\left[\begin{smallmatrix} u_\tau \\ v_\tau \end{smallmatrix}\right])$ as long as the solution exists. On the other hand, using (4.11), we obtain for any $\delta > 0$ small enough that $-\int_\Omega \int_0^{w_1} f(s)dsdx$ is bounded from below by $-\frac{\lambda_1-\delta}{2}\|w_1\|^2_{L^2(\Omega)} - N_\delta|\Omega|$ for some $N_\delta > 0$. Consequently

$$\mathcal{L}(\left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right]) \geqslant \frac{\delta}{2\lambda_1}\|(-\Delta)^{\frac{1}{2}} w_1\|^2_{L^2(\Omega)} + \frac{1}{2}\|w_2\|^2_{L^2(\Omega)} - N_\delta|\Omega|,\ \left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right] \in E_1,$$

and the result follows easily. This proves (4.12) for smooth solutions, e.g. for solutions with smooth initial data which can be obtained within [26, Theorem 7] due to Remark 4.3. With $(1.14)_{\theta=0}$ (see Remark 1.9 iii)) it then extends to $E_{1+\varepsilon}$-solutions and the proof is complete. □

Theorem 1.10 now leads to the following conclusion.

Corollary 4.8. *Suppose that Assumption 4.1 holds and assume that $f \in C^1(\mathbb{R}, \mathbb{R})$ does not depend on time variable, (4.11) is satisfied and $f'_s(s) = o(|s|^{\rho_c - 1})$. Then, given $\tau \in \mathbb{R}$ and $\left[\begin{smallmatrix} u_\tau \\ v_\tau \end{smallmatrix}\right] \in E_1 = H^1_0(\Omega) \times L^2(\Omega)$, there exists the unique global $E_{1+\varepsilon}$-solution of (4.1).*

Suppose finally that we have $f'_s(s) = O(1 + |s|^{\rho_c - 1})$ but not $f'_s(s) = o(|s|^{\rho_c - 1})$. Note that (1.19) is rather uneasy to verify as an $E_{1+\varepsilon}$-estimate can hardly be derived. Nonetheless, since we know (4.12) and, in addition,

$$(-\Delta)^{-\frac{1}{2}}\dot{v} + \eta(t)v + \nu(-\Delta)^{-\frac{1}{2}} v + (-\Delta)^{\frac{1}{2}} u = (-\Delta)^{-\frac{1}{2}} f(u),$$

we infer that $u \in W^{1,1}((0, T_{u_\tau,v_\tau}), L^2(\Omega))$, $\dot{u} \in W^{1,1}((0, T_{u_\tau,v_\tau}), H^{-1}(\Omega))$ whenever $T_{u_\tau,v_\tau} < \infty$ and the map $[0, T_{u_\tau,v_\tau}) \ni t \longrightarrow \left[\begin{smallmatrix} u(t,u_\tau,v_\tau) \\ v(t,u_\tau,v_\tau) \end{smallmatrix}\right] \in E_0 = L^2(\Omega) \times H^{-1}(\Omega)$ is uniformly continuous (see [10, Theorem I.2.2]). Thus (1.26)-(1.27) hold and Theorem 1.12 applies.

Corollary 4.9. *Suppose that Assumption 4.1 holds, $f \in C^1(\mathbb{R}, \mathbb{R})$ does not depend on time variable, $f'_s(s) = O(1 + |s|^{\rho_c - 1})$ and (4.11) is satisfied.*

Whenever $\tau \in \mathbb{R}$, $\left[\begin{smallmatrix} u_\tau \\ v_\tau \end{smallmatrix}\right] \in E_1$ are such that T_{u_τ,v_τ} is finite, there exist $a \in (T_{u_\tau,v_\tau}, \infty]$ and an extension $\mathcal{U} : [\tau, a) \to E_1$ of maximally defined $E_{1+\varepsilon}$-solution of (4.1) such that $\mathcal{U}$ is a piecewise-$E_{1+\varepsilon}$-solution on $[\tau, a)$ and $a = \infty$ or a is an accumulation time of singular times.

4.2. **Nonautonomous parabolic problems.** Given

$$\mathcal{A}(t) = (-1)^m \sum_{|\sigma|\leqslant 2m} a_\sigma(t,x)D^\sigma, \ t\in\mathbb{R}, \ x\in\Omega, \tag{4.14}$$

$$B_j = \sum_{|\sigma|\leqslant m_j} b^j_\sigma(x)D^\sigma, \quad \text{where} \ \ j=1,\dots,m, \ m_j\in\{0,1,\dots,2m-1\}, \ x\in\partial\Omega, \tag{4.15}$$

and adapting the notion of a regular parabolic initial boundary value problem we say that $\{(\mathcal{A}(t),\{B_j\},\Omega,\partial\Omega), \ t\in\mathbb{R}\}$ is of the class $\mathcal{RPIBVP}$ of regular parabolic initial boundary value problems of order $2m$ if $(\mathcal{A}(t),\{B_j\},\Omega,\partial\Omega,\alpha)$ is a strongly α-regular elliptic boundary value problem of class C^0 and order $2m$ for every $t\in\mathbb{R}$ as in [7, p. 655] and, in addition,

(4.16)
there exists $\mu\in(0,1]$ such that for each bounded time interval $I\subset\mathbb{R}$ and for any $|\sigma|\leqslant 2m$ map $I\ni t\to a_\sigma(t,\cdot)\in C(\overline{\Omega},\mathbb{R})$ is of the class $C^\mu(I,C(\overline{\Omega},\mathbb{R}))$; in addition, whenever $|\sigma|=2m$, modulus of continuity of the maps $\overline{\Omega}\ni x\to a_\sigma(t,x)\in\mathbb{R}$ can be chosen uniformly for $t\in I$.

We will next consider spaces $H^s_p(\Omega)$ as in [28]. For $p=2$ they are Hilbert spaces and will be denoted by $H^s(\Omega)$. Following [28] we also define

$$H^s_{p,\{B_j\}}(\Omega) = \{\phi\in H^s_p(\Omega): \ \forall_{i\in\left\{j:\ m_j<s-\frac{1}{p}\right\}} \ B_i\phi_{|\partial\Omega}=0\}.$$

Assuming that $\{(\mathcal{A}(t),\{B_j\},\Omega,\partial\Omega), \ t\in\mathbb{R}\}$ is of the class $\mathcal{RPIBVP}$ we have the estimate

$$\|\varphi\|_{H^{2m}_p(\Omega)} \leqslant c^*(\|\mathcal{A}(t)\varphi\|_{L^p(\Omega)} + \|\varphi\|_{L^p(\Omega)}), \ \varphi\in H^{2m}_{p,\{B_j\}}(\Omega), \ t\in I, \tag{4.17}$$

where $I\subset\mathbb{R}$ is arbitrarily chosen bounded time interval. We emphasize that $c^*>0$ actually depends on Ω, m, N, p, α, moduli of continuity of the top order coefficients of $\mathcal{A}(t)$ with $t\in I$, coefficients of boundary operators B_j and certain constants related to the notion of α-regular elliptic boundary value problem which are specified in [7, Theorems 12.1] (see also [1, 2]). Thus the constant c^* in (4.17) is independent of t in a bounded time interval $I\subset\mathbb{R}$.

We remark that, due to (4.14), (4.16) and properties of $H^{2m}_p(\Omega)$-norm we also have

$$\|\mathcal{A}(t)\varphi\|_{L^p(\Omega)} \leqslant c_*\|\varphi\|_{H^{2m}_p(\Omega)}, \ \varphi\in H^{2m}_p(\Omega), \ t\in I, \tag{4.18}$$

where c_* depends on Ω, m and $L^\infty(I,C(\overline{\Omega},\mathbb{R}))$-norms of coefficients of $A(t)$.

With the above set-up we consider the $2m$-th order problem

$$\begin{cases} u_t + (-1)^m\sum_{|\xi|,|\zeta|\leqslant m} D^\zeta(a_{\xi,\zeta}(t,x)D^\xi u) = f(t,x,u), \ t>0, \ x\in\Omega\subset\mathbb{R}^N, \\ B_0u=\ldots=B_{m-1}u=0, \ t>0, \ x\in\partial\Omega, \quad u(0,x)=u_0(x), \ x\in\Omega. \end{cases} \tag{4.19}$$

Letting

$$A(t)u = (-1)^m\sum_{|\xi|,|\zeta|\leqslant m} D^\zeta(a_{\xi,\zeta}(t,x)D^\xi u)$$

we summarize conditions on (4.19).

Assumption 4.10. *$N>2m$, $\Omega\subset\mathbb{R}^N$ is a bounded C^{2m}-domain, the coefficients $a_{\xi,\zeta}(t,\cdot)\in C^m(\overline{\Omega},\mathbb{R})$ ($|\xi|\leqslant m,|\zeta|\leqslant m$) of $A(t)$ are such that the maps $I\ni t\to D^\beta a_{\xi,\zeta}(t,\cdot)\in C(\overline{\Omega},\mathbb{R})$ ($|\beta|\leqslant m$) belong to the class $C^\mu(I,C(\overline{\Omega},\mathbb{R}))$ and after rewriting $A(t)$ in the form* (4.14) *we have that $\{(\mathcal{A}(t),\{B_j\},\Omega,\partial\Omega), \ t\in\mathbb{R}\}$ is of the class $\mathcal{RPIBVP}$.*

Assumption 4.11. *$A(t)$, $t \in \mathbb{R}$, are selfadjoint operators in $L^2(\Omega)$ bounded from below uniformly for t in bounded time intervals $I \subset \mathbb{R}$; that is,*

$$\langle A(t)\phi, \phi\rangle_{L^2(\Omega)} \geqslant s_* \|\phi\|^2_{L^2(\Omega)}, \tag{4.20}$$

where $s_* > 0$ can depend on I but not on $t \in I$.

Proposition 4.12. *Suppose that Assumptions 4.10, 4.11 hold and*

$$\begin{array}{ll} E_\alpha = (H^{2m}_{\{B_j\}}(\Omega))',\ \alpha = 0, & E_\alpha = ([L^2(\Omega), H^{2m}_{\{B_j\}}(\Omega)]_{1-\alpha})',\ \alpha \in (0,1), \\ E_\alpha = L^2(\Omega),\ \alpha = 1, & E_\alpha = [L^2(\Omega), H^{2m}_{\{B_j\}}(\Omega)]_{\alpha-1},\ \alpha \in (1, 1+\mu). \end{array} \tag{4.21}$$

Then there exists a continuous process associated in $E_0 = (H^{2m}_{\{B_j\}}(\Omega))'$ with

$$\begin{cases} u_t + A(t)u = 0,\ t > 0,\ x \in \Omega \subset \mathbb{R}^N, \\ B_0 u = \ldots = B_{m-1}u = 0,\ t > 0,\ x \in \partial\Omega, \quad u(0,x) = u_0 \in L^2(\Omega), \end{cases} \tag{4.22}$$

and possessing smoothing properties (1.2), (1.3).

Proof: We will ensure that Theorem 3.13 applies with $X = L^2(\Omega)$ and $E_0 = (H^{2m}_{\{B_j\}}(\Omega))'$.

Note that, proceeding as in [17, Proposition 1.3.3], we get $\|(\lambda I - A(t))\phi\|_X \geqslant 2^{-\frac{1}{2}}|\lambda - s_*|$ whenever $\phi \in H^2_{\{B_j\}}(\Omega)$, $t \in I$, $Re(\lambda) \leqslant s_*$. From this we conclude that $\{A(t) : t \in \mathbb{R}\}$ is of the class $\mathcal{LUS}(D_X, X)$. On the other hand, since purely imaginary powers are unitary operators, we also have that $\{A(t) : t \in \mathbb{R}\}$ is of the class $\mathcal{BIP}(X)$.

We now fix a bounded time interval $I \subset \mathbb{R}$ and concentrate on points $t \in I$. Using (4.20), Schwartz's inequality and (4.17) we get with s_* as in (4.20)

$$\|\varphi\|_{H^{2m}(\Omega)} \leqslant c^*(1 + s_*^{-1})\|A(t)\varphi\|_{L^2(\Omega)},\ \varphi \in H^{2m}_{\{B_j\}}(\Omega),\ t \in I. \tag{4.23}$$

Next, to obtain (3.5), we apply (4.18) with $p = 2$, $\varphi = A^{-1}(s)\psi$, $\psi \in L^2(\Omega)$, and use (4.23) with $t = s$ and $\varphi = A^{-1}(s)\psi$ to conclude that

$$\|A(t)A^{-1}(s)\psi\|_{L^2(\Omega)} \leqslant c_*\|A^{-1}(s)\psi\|_{H^{2m}(\Omega)} \leqslant c_*c^*(1 + s_*^{-1})\|\psi\|_{L^2(\Omega)},\ \psi \in L^2(\Omega),\ t, s \in I.$$

In the proof of (3.8) we adapt the idea of [4, Remark 6.6 (c)]. Since from above we have $\sup_{t,s\in I}\|A(s)A^{-1}(t)\|_{L(L^2(\Omega))} \leqslant N$, using this and selfadjointness of the operators we get

$$|\langle \phi, A^{-1}(t)A(s)\psi\rangle_{L^2(\Omega)}| \leqslant N\|\phi\|_{L^2(\Omega)}\|\psi\|_{L^2(\Omega)},\ \phi \in L^2(\Omega),\ \psi \in H^{2m}_{\{B_j\}}(\Omega),\ t, s \in I.$$

This ensures that the set $\{A^{-1}(t)A(s)\psi : t, s \in I, \psi \in H^{2m}_{\{B_j\}}(\Omega), \|\psi\|_{L^2(\mathbb{R}^N)} \leqslant 1\}$ is bounded in $L^2(\Omega)$ and hence $\|\overline{A^{-1}(t)A(s)}\|_{L(L^2(\Omega))} \leqslant \bar{c}$, where $\bar{c} > 0$ does not depend on $t, s \in I$.

Letting $\alpha_0 = 0$, we define next spaces E_α, $\alpha \in [0, 1 + \mu_0] = [0, 2]$ as in (3.11), which are characterized here as in (4.21). To ensure that

$$A(\cdot) \in C^\mu_{loc}(\mathbb{R}, L(E_1, E_0)) \text{ with } E_1 = L^2(\Omega) \text{ and } E_0 = (H^{2m}_{\{B_j\}}(\Omega))' \tag{4.24}$$

we observe $\|(A(t)-A(s))\phi\|_{(H^{2m}_{\{B_j\}}(\Omega))'} = \sup_{\|\psi\|_{H^{2m}_{\{B_j\}}(\Omega)}=1} |\int_\Omega \phi(A(t)-A(s))\psi|$ for $t,s\in\mathbb{R}$, $\phi\in L^2(\mathbb{R}^N)$. Hence, for t,s in a bounded time interval $I\subset\mathbb{R}$, using (4.14), (4.16) we get

$$\sup_{\|\phi\|_{E_1}=1} \|[A(t)-A(s)]\phi\|_{E_0} = \sup_{\|\phi\|_{L^2(\Omega)}=1} \sup_{\|\psi\|_{H^{2m}_{\{B_j\}}(\Omega)}=1} \Big|\int_\Omega \phi(A(t)-A(s))\psi\Big|$$

$$\leqslant \sup_{\|\phi\|_{L^2(\Omega)}=1} \sup_{\|\psi\|_{H^{2m}_{\{B_j\}}(\Omega)}=1} \sum_{|\sigma|\leqslant 2m} \|a_\sigma(t,\cdot)-a_\sigma(s,\cdot)\|_{C(\overline{\Omega},\mathbb{R})}\|\phi\|_{L^2(\Omega)}\|D^\sigma\psi\|_{L^2(\Omega)} \leqslant c|t-s|^\mu.$$

We can now apply Theorem 3.13 to get the result. □

Remark 4.13. *Besides (4.24) we also have $A(\cdot)\in C^\mu_{loc}(\mathbb{R},L(E_2,E_1))$ with $E_2=H^{2m}_{\{B_j\}}(\Omega)$, $E_1=L^2(\Omega)$ as by (4.16), whenever $\phi\in E_2$ and s,t vary in a bounded time interval $I\subset\mathbb{R}^N$,*

$$\|(A(t)-A(s))\phi\|_{E_1} \leqslant \|a_\sigma(t,\cdot)-a_\sigma(s,\cdot)\|_{C(\overline{\Omega})} \sum_{|\sigma|\leqslant 2m} \|D^\sigma\phi\|_{E_1} \leqslant |t-s|^\mu\|\phi\|_{E_2}.$$

We now consider a nonlinear term, where we use the Landau symbols $O(\varphi)$, $o(\varphi)$ as in Remark 4.4. For (4.19) with initial data in $L^2(\Omega)$ a role of a critical exponent is played by

$$\rho_c := \frac{N+4m}{N}.$$

Proposition 4.14. *Assume $f, f'_u \in C(\mathbb{R}^{N+2},\mathbb{R})$, let E_α, $\alpha\in[0,1+\mu)$, be as in (4.21) and*

$$N > 4m. \tag{4.25}$$

i) If $f'_s(t,x,s)=O(c_\eta+\eta|s|^{\rho-1})$ for some $\eta>0$ and $\rho\in(1,\rho_c)$, then the map $F(t,u)$ in (4.19) is of the class $\mathcal{L}(\varepsilon,\rho,\gamma(\varepsilon),\eta,C_\eta)$ relative to $\{E_\alpha,\alpha\in[0,1+\mu)\}$ and is subcritical.
ii) If $f'_s(t,x,s)=O(c_\eta+\eta|s|^{\rho_c-1})$ for some $\eta>0$ and i) does not apply, then the map $F(t,u)$ in (4.19) is of the class $\mathcal{L}(\varepsilon,\rho,\gamma(\varepsilon),\eta,C_\eta)$ relative to $\{E_\alpha,\alpha\in[0,1+\mu)\}$ and is critical.
iii) If $f'_s(t,x,s)=o(|s|^{\rho_c-1})$ and i) does not apply, then $F(t,u)$ in (4.19) is of the class $\mathcal{L}(\varepsilon,\rho,\gamma(\varepsilon),\eta,C_\eta)$ relative to $\{E_\alpha,\alpha\in[0,1+\mu)\}$ and is almost critical.
Furthermore,
iv) parts i), ii) and iii) above hold with $\varepsilon>0$ as small as we wish. Actually, whenever t varies in a bounded time interval $I\subset\mathbb{R}$, there exists a certain $c>0$ such that

$$\|F(t,\phi)\|_{E_0} \leqslant c(1+\|\phi\|^{\rho_c}_{E_1}),\ \phi\in E_1. \tag{4.26}$$

Proof: Note that restricting time variable t to a bounded time interval I one needs to show that there are constants $c>0$, $C_\eta>0$ and $\varepsilon\in(0,\frac{1}{\rho})$, $\varepsilon<\mu$, $\rho\varepsilon\leqslant\gamma(\varepsilon)<1$ such that

$$\|F(t,v)-F(t,w)\|_{E_{\gamma(\varepsilon)}} \leqslant c\|v-w\|_{E_{1+\varepsilon}}\big(C_\eta+\eta\|v\|^{\rho-1}_{E_{1+\varepsilon}}+\eta\|w\|^{\rho-1}_{E_{1+\varepsilon}}\big),\ v,w\in E_{1+\varepsilon}. \tag{4.27}$$

We now describe admissible triples $(\rho,\varepsilon,\gamma(\varepsilon))$ for which (4.27) holds and prove that the map F is indeed critical for $\rho=\rho_c$ whilst it is subcritical for $\rho\in(1,\rho_c)$.

Observe that due to (4.21) we have

$$\begin{aligned} &E_{1+\varepsilon}\hookrightarrow L^s(\Omega),\ \varepsilon\in[0,\mu),\ 2m\varepsilon-\frac{N}{2}\geqslant-\frac{N}{s},\ s\geqslant 2,\\ &E_{\gamma(\varepsilon)}\hookleftarrow L^\sigma(\Omega),\ \gamma(\varepsilon)\in[0,1),\ \frac{2N}{N+4m(1-\gamma(\varepsilon))}\leqslant\sigma\leqslant 2,\ \sigma>1,\end{aligned} \tag{4.28}$$

where $\frac{2N}{N+4m(1-\gamma(\varepsilon))} > 1$ provided that $\gamma(\varepsilon) > \frac{4m-N}{4m} =: \tilde{\gamma} > 0$.

By (4.28) $\|F(t,v)-F(t,w)\|_{E_{\gamma(\varepsilon)}}$ is bounded by $\hat{c}\|F(t,v)-F(t,w)\|_{L^{\frac{2N}{N+4m(1-\gamma(\varepsilon))}}(\Omega)}$ and, whenever $f'_s(t,x,s) = O(c_\eta + \eta|s|^{\rho-1})$, we have

$$\|F(t,v)-F(t,w)\|_{E_{\gamma(\varepsilon)}} \leqslant \tilde{c}\||v-w|(c_\eta + \eta|v|^{\rho-1} + \eta|w|^{\rho-1})\|_{L^{\frac{2N}{N+4m(1-\gamma(\varepsilon))}}(\Omega)}.$$

Applying next Hölder's inequality with $q = \frac{N+4m(1-\gamma(\varepsilon))}{N-4m\varepsilon}$, $q' = \frac{N+4m(1-\gamma(\varepsilon))}{4m(1-\gamma(\varepsilon)+\varepsilon)}$, recalling the embedding $H^{2m\varepsilon}(\Omega) \hookrightarrow L^{\frac{2N}{N-4m\varepsilon}}(\Omega)$, and assuming that

$$H^{2m\varepsilon}(\Omega) \hookrightarrow L^{\frac{N(\rho-1)}{2m(1-\gamma(\varepsilon)+\varepsilon)}}(\Omega), \tag{4.29}$$

we obtain

$$\begin{aligned}\|F(t,v)-F(t,w)\|_{E_{\gamma(\varepsilon)}} &\leqslant \tilde{c}\|v-w\|_{L^{\frac{2N}{N-4m\varepsilon}}(\Omega)}\|c_\eta + \eta|v|^{\rho-1} + \eta|w|^{\rho-1}\|_{L^{\frac{N}{2m(1-\gamma(\varepsilon)+\varepsilon)}}(\Omega)} \\ &\leqslant c\|v-w\|_{E_{1+\varepsilon}}(C_\eta + \eta\|v\|^{\rho-1}_{E_{1+\varepsilon}} + \eta\|w\|^{\rho-1}_{E_{1+\varepsilon}}),\ v,w\in E_{1+\varepsilon},\end{aligned}$$

where (4.29) requires that

$$\overline{\gamma} := \frac{(4m\varepsilon - N)(\rho-1) + 4m(1+\varepsilon)}{4m} \geqslant \gamma(\varepsilon) \geqslant \frac{2m(1+\varepsilon) - N(\rho-1)}{2m} =: \underline{\gamma}. \tag{4.30}$$

We remark that $\overline{\gamma} > \tilde{\gamma}$ and that for $\rho \in (1, 1+\frac{4m}{N}]$ and $\varepsilon > 0$ we have $\overline{\gamma} > \underline{\gamma}$ and $\overline{\gamma} \geqslant \varepsilon\rho$. We also have $1 > \overline{\gamma}$ if $\varepsilon \in (0, \frac{N(\rho-1)}{4m\rho})$.

The above ensures that any triple $(\rho, \varepsilon, \gamma(\varepsilon))$, where $\rho \in (1, 1+\frac{4m}{N}]$, $\varepsilon \in (0, \min\{\mu, \frac{N(\rho-1)}{4m\rho}\})$ and $\gamma(\varepsilon) \in [\rho\varepsilon, \overline{\gamma}] \cap [\max\{0, \underline{\gamma}\}, \overline{\gamma}] \cap (\tilde{\gamma}, \overline{\gamma}] =: \mathcal{I}(\varepsilon)$ is admissible.

For any admissible triple $(\rho, \varepsilon, \gamma(\varepsilon))$ (4.30) implies $\rho \leqslant \frac{N+4m-4m\gamma(\varepsilon)}{N-4m\varepsilon}$, and since $\gamma(\varepsilon) \geqslant \rho\varepsilon$ we have $\rho \leqslant \frac{N+4m-4m\rho\varepsilon}{N-4m\varepsilon}$, which holds if and only if $\rho \leqslant \frac{N+4m}{N} = \rho_c$. Thus $\rho = \rho_c$ cannot be attained for any $\gamma(\varepsilon) > \rho_c\varepsilon$ and therefore $\rho = \rho_c$ necessitates $\gamma(\varepsilon) = \varepsilon\rho_c$. Note that $\overline{\gamma}_{|\rho=\rho_c} = \varepsilon\rho_c$; that is for $\rho = \rho_c$ we have $\mathcal{I}(\varepsilon) = \{\varepsilon\rho_c\}$. This completes the proof of i)-ii).

Note that having $|f'(t,x,s)| \leqslant O(c_\eta + \eta|s|^{\rho_c-1})$ for each $\eta > 0$ we obtain (4.27) for any $\eta > 0$, which leads to iii).

Describing admissible triples we have already ensured that $\varepsilon > 0$ can be chosen arbitrarily close to zero. Actually we also have

$$\|F(t,v)-F(t,0)\|_{E_0} \leqslant \hat{c}\|F(t,v)-F(t,0)\|_{L^{\frac{2N}{N+4m}}(\Omega)} \leqslant \tilde{c}\||v|(c_\eta + \eta|v|^{\rho-1})\|_{L^{\frac{2N}{N+4m}}(\Omega)}$$

which leads to (4.26) as $L^2(\Omega) \hookrightarrow L^{\frac{2N\rho}{N+4m}}(\Omega)$ for $\rho \in (1, \rho_c]$. □

Remark 4.15. *Note that not assuming (4.25) in Proposition 4.14 we may not have i)-iii) satisfied for $\varepsilon > 0$ arbitrarily small as stated in iv) (see* [18, §3.1] *for a similar proof).*

Corollary 4.16. *Suppose that Assumptions 4.10, 4.11 hold and spaces E_α, $\alpha \in [0, 1+\mu)$, are as in* (4.21)*. Suppose also that the asssumptions of Proposition 4.14 are satisfied; in particular that $f'_s(t,x,s) = O(c_\eta + \eta|s|^{\rho_c-1})$ for some $\eta > 0$. Then Theorem 1.7 applies and, hence, given any $\tau \in \mathbb{R}$, $u_\tau \in L^2(\Omega)$, the initial boundary value problem* (4.19) *has the unique $E_{1+\varepsilon}$-solution $u = u(\cdot, \tau, u_\tau)$ defined on the maximal interval of existence $[\tau, T_{u_\tau})$.*

We will now derive an $L^2(\Omega)$-estimate of the solutions.

Lemma 4.17. *Suppose that*

$$sf(t,x,s) \leqslant C(t,x)s^2 + D(t,x),\ t \in \mathbb{R},\ x \in \Omega, \tag{4.31}$$

for some $C \in L^\infty_{loc}(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$ and $D \in L^1_{loc}(\mathbb{R}, L^1(\Omega))$.

If $\tau \in \mathbb{R}$, $u_\tau \in L^2(\Omega)$, $T \in (\tau,\infty)$ and $E_{1+\varepsilon}$-solution u of (4.19) exists for $t \in [\tau,T)$ then

$$\|u(t,\tau,u_\tau)\|^2_{L^2(\mathbb{R}^N)} \leqslant g(\tau, \|u_\tau\|_{L^2(\Omega)}, T),\ t \in [\tau,T), \tag{4.32}$$

where $g : \mathbb{R}\times[0,\infty)\times(0,\infty) \to [0,\infty)$ is a certain continuous function.

Proof: We restrict here time variable to $[\tau,T)$, which allows us to choose the constant s_* such that (4.20) holds uniformly for $t \in [\tau,T)$. We also define $C^* := \sup_{(t,x)\in[\tau,T)\times\Omega} 2|C(t,x)|$.

From (4.19), (4.20) and (4.31) we obtain for any $\lambda \in (0,s_*)$ the estimate of the form

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} + (s_* - C^*)\|u(t)\|^2_{L^2(\Omega)} \leqslant \|D(t,\cdot)\|_{L^1(\Omega)},\ t \in [\tau,T).$$

Solving the above inequality we get

$$\|u(t)\|^2_{L^2(\Omega)} \leqslant \|u_\tau\|^2_{L^2(\Omega)} e^{-2t(s_*-C^*)} + 2\int_\tau^t \|D(s,\cdot)\|_{L^1(\Omega)} e^{-2(t-s)(s_*-C^*)} ds,\ t \in [\tau,T).$$

This proves (4.32) for smooth solutions, e.g. for solutions with smooth initial data which can be obtained within [26, Theorem 7] due to Remark 4.13. With $(1.14)_{\theta=0}$ (see Remark 1.9 iii)) it then extends to $E_{1+\varepsilon}$-solutions, which completes the proof. □

Theorem 1.10 now implies the following result.

Corollary 4.18. *Suppose that the assumptions of Corollary 4.16 and Lemma 4.17 hold.*

If $f'_s(t,x,s) = o(|s|^{\rho_c-1})$ then, given any $\tau \in \mathbb{R}$ and $u_\tau \in L^2(\Omega)$, the unique $E_{1+\varepsilon}$-solution of (4.19) exists globally in time.

Proof: With maximal time of existence $T_{u_\tau} < \infty$ we would have $\sup_{[\tau,T_{u_\tau})} \|u(t,\tau,u_\tau)\|_{L^2(\mathbb{R}^N)} < \infty$ (see Lemma 4.17) and Theorem 1.10 i) with $E_1 = L^2(\Omega)$ would lead to contradiction. □

In the critical case $\rho = \rho_c$ some better estimate of the solutions can be sometimes obtained if additional conditions are imposed on (4.19). For example, in the autonomous case, $H^1(\Omega)$-estimate can be found as in [29]. Also, if $m = 1$ and maximum principle applies then $L^\infty(\Omega)$-estimate may be known. However, without any such specific assumption, one can hardly find for (4.19) the estimate of the solutions in $E_{1+\varepsilon}$-norm needed to apply (1.19). On the other hand, Theorem 1.12 will yield the existence of a piecewise-$E_{1+\varepsilon}$-solution on some larger time interval than the maximal interval of existence of $E_{1+\varepsilon}$-solution.

Lemma 4.19. *Suppose that the assumptions of Corollary 4.16 and Lemma 4.17 are satisfied.*

If $\tau \in \mathbb{R}$, $u_\tau \in E_1 = L^2(\Omega)$ and $T_{u_\tau} < \infty$, the map $[\tau, T_{u_\tau}) \ni t \to u(t) \in E_0 = (H^{2m}_{\{B_j\}}(\Omega))'$, where u is $E_{1+\varepsilon}$-solution of (4.19), is uniformly continuous.

Proof: From (4.19) we infer that

$$\|u_t(t)\|_{(H^{2m}_{\{B_j\}}(\Omega))'} \leqslant \|A(t)u(t)\|_{(H^{2m}_{\{B_j\}}(\Omega))'} + \|f(t,\cdot,u)\|_{(H^{2m}_{\{B_j\}}(\Omega))'},\ t \in (\tau, T_{u_\tau}).$$

Since

$$\|A(t)u\|_{(H^{2m}_{\{B_j\}}(\Omega)')} = \sup_{\|\psi\|_{H^{2m}_{\{B_j\}}(\Omega)}=1} \left|\int_\Omega uA(t)\psi\right| \leqslant \|u(t)\|_{L^2(\Omega)} \max_{|\sigma|\leqslant 2m} \|a_\sigma(t,\cdot)\|_{C(\overline{\Omega},\mathbb{R})},$$

by (4.16), (4.32) we get $\|A(t)u\|_{L^\infty((\tau,T_{u_\tau}),(H^{2m}_{\{B_j\}}(\Omega))')} \leqslant cg(\tau,\|u_\tau\|_{L^2(\Omega)},T_{u_\tau})$. From (4.26) $\|f(t,\cdot,u)\|_{(H^{2m}_{\{B_j\}}(\Omega))'}$ is bounded by a multiple of $(1+\|u(t)\|^{\rho_c}_{L^2(\Omega)})$ and hence, by (4.32),

$$\|f(t,\cdot,u)\|_{L^\infty((\tau,T_{u_\tau}),(H^{2m}_{\{B_j\}}(\Omega))')} \leqslant c(1+[g(\tau,\|u_\tau\|_{L^2(\Omega)},T_{u_\tau})]^{\rho_c}).$$

Since the above estimates ensure that $u(\cdot,\tau,u_\tau)\in W^{1,1}((\tau,T),(H^{2m}_{\{B_j\}}(\Omega))')$, then (1.27) is satisfied (see [10, Theorem I.2.2]) and the proof is complete. □

Theorem 1.12 and Lemmas 4.17, 4.19 now lead to the following conclusion.

Corollary 4.20. *Suppose that the assumptions of Corollary 4.16 and Lemma 4.17 hold.*

Whenever $\tau\in\mathbb{R}$, $u_\tau\in L^2(\Omega)$ are such that $T_{u_\tau}<\infty$, there exist $a\in(T_{u_\tau},\infty]$ and an extension $\mathcal{U}$ of the maximally defined $E_{1+\varepsilon}$-solution of (4.19) such that $\mathcal{U}$ is a piecewise-$E_{1+\varepsilon}$-solution on $[\tau,a)$ and either $a=\infty$ or a is an accumulation time of singular times.

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