LOCAL INTEGRABILITY AND LINEARIZABILITY OF A $(1:-1:-1)$ RESONANT QUADRATIC SYSTEM

MAŠA DUKARIĆ[1], REGILENE OLIVEIRA[2] AND VALERY G. ROMANOVSKI[1,3]

Abstract. In this paper we study the local integrability and linearizability of quadratic three dimensional systems of the form

$$\begin{aligned}\dot{x} &= \quad x + a_{12}xy + a_{13}xz + a_{23}yz = P_1(x,y,z)\\ \dot{y} &= -y + b_{12}xy + b_{13}xz + b_{23}yz = Q_1(x,y,z)\\ \dot{z} &= -z + c_{12}xy + c_{13}xz + c_{23}yz = R_1(x,y,z).\end{aligned}$$

First, we obtain necessary and sufficient conditions for the *complete* integrability and linearizability of this system. Then, we discuss the problem of existence of *one* first integral of the form $\psi^{(1)}(x,y,z) = xy + O(|x,y,z|^3)$. Computation of resonant focus quantities and the decomposition of the variety of the ideal that they generate in the ring of polynomials of parameters a_{ij}, b_{ij}, c_{ij} of the system were used to obtain necessary conditions of integrability and linearizability. The theory of Darboux integrability and some other methods are used to show the sufficiency. In the investigation of the conditions for the existence of one first integral the decomposition of the variety mentioned above was performed using modular computations, its consequences are discussed.

1. Introduction

The problem of integrability is one of the most studied problems in the theory of ordinary differential equations, especially the integrability of two dimensional polynomial systems. Poincaré and Lyapunov showed that existence of a local analytic first integral in a neighborhood of a singular point with pure imaginary eigenvalues of the matrix of the linear approximation yields existence of a center, that is, all trajectories in a neighborhood of the point are ovals. One of further first important contributions is due to Dulac [11] who classified integrable quadratic systems with $(1:-1)$ resonant singular point. The integrability of quadratic $(1:-2)$ resonant singularities was studied in [12] and some important results on integrability of $(p:-q)$-resonant singularities were obtained in [5, 6, 15, 26].

The integrability of three dimensional polynomial systems is not studied in such extension, the most researched three dimensional systems are the Lotka-Volterra systems. Some conditions for integrability and linearizability of such systems have been obtained in [2]. In [16, 18] the authors provided some results about Darbouxian integrability of

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higher dimensional systems. The integrability of three dimensional quadratic systems in a neighborhood of a $(0 : -1 : 1)$ resonant singular point was investigated in [13, 14]. Recently Aziz [1] considered a particular not Lotka-Volterra family of quadratic systems with a $(1 : -1 : 1)$ resonant singularity.

In this paper we look for sufficient and necessary conditions of integrability and linearizability of quadratic systems, which are written in the form

$$
\begin{aligned}
\dot{x} &= \quad x + a_{12}xy + a_{13}xz + a_{23}yz = P_1(x,y,z) \\
\dot{y} &= \ -y + b_{12}xy + b_{13}xz + b_{23}yz = Q_1(x,y,z) \\
\dot{z} &= \ -z + c_{12}xy + c_{13}xz + c_{23}yz = R_1(x,y,z),
\end{aligned}
\tag{1}
$$

where a_{ij}, b_{ij}, c_{ij} are the parameters of the systems. We say that these systems have a $(1 : -1 : -1)$ resonant critical point at the origin. Clearly, these systems do not belong to the Lotka-Volterra family.

In order to obtain the necessary conditions for the integrability and linearizability the computation of the resonant focus quantities and the decomposition of the variety of the ideal generated by the quantities were used. The theory of Darboux integrability and some other methods are used to prove the sufficiency of obtained conditions.

In this paper we also present conditions for the existence of one first integral of the form $\psi(x,y,z) = xy + O(|x,y,z|^3)$ for quadratic systems (1) with $a_{23} = 0$. In this case the decomposition of the variety mentioned above was performed using modular computations, so the obtained conditions of integrability represent the complete list of the integrability conditions only with very high probability and there remains an open problem to verify that all necessary and sufficient conditions are found.

The paper is organized as follows. Basic definitions and statements necessary to present our results are in Section 2. In Section 3 we prove the main result about the complete integrability and linearizability in this paper, Theorem 3.1. In Section 4 the problem of existence of one analytic integral in system (1) is discussed.

2. Basic definitions and results

The study of integrability of ordinary differential equations is closely connected to the theory of normal form. We recall here some well-known important results on the normal forms.

Consider n-dimensional autonomous system of the form

$$\dot{x} = Ax + f(x), \tag{2}$$

where A is $n \times n$ matrix, $x = (x_1, \ldots, x_n)^\tau$, $f(x) = (f_1(x), \ldots, f_n(x))^\tau$, and f_i are series starting with at least quadratic terms. For simplicity we assume that the matrix A is diagonal. We will also assume that it has at least one nonzero eigenvalue.

Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be the n–tuple of eigenvalues of A. Set $\mathbb{Z}_+ = \mathbb{N} \cup 0$. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$ denote

$$(\lambda, \alpha) = \sum_{i=1}^{n} \alpha_i \lambda_i$$

and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Let

$$\mathfrak{R} = \{\alpha \in \mathbb{Z}_+^n | \quad (\lambda, \alpha) = 0, \ |\alpha| > 0\},$$

and denote by r_λ the rank of $\mathbb{Z}$-module spanned by the elements of $\mathfrak{R}$.

A substitution

$$x = \Phi(y) := y + \varphi(y), \tag{3}$$

transforms (2) to its Poincaré–Dulac normal form, that is, to a system of the form

$$\dot{y} = Ay + g(y), \tag{4}$$

where $g(y) = (g_1(y), \dots, g_n(y))^\tau$ contains only *resonant terms*, that is, each monomial in g_k, $k = 1, \dots, n$, is of the form $g^{(\alpha)} y^\alpha e_k$ with

$$(\lambda, \alpha) - \lambda_k = 0,$$

where e_k is the n–dimensional unit vector with its nth component equal to 1 and the others all equal to zero. The transformation (3) is called a *normalization.* The normalization containing only nonresonant terms is unique. We call this normalization a *distinguished normalization* and term the corresponding Poincaré–Dulac normal form a *distinguished normal form.* We also recall that in a series

$$\psi(x) = \sum_{\alpha : |\alpha| > 0} \psi_\alpha x^\alpha$$

the term $\psi_\alpha x^\alpha$ is a *resonant term* if $(\lambda, \alpha) = 0$.

If there is a transformation (3) which brings (2) to a linear system, then we say that system (2) is *linearizable.*

Normalization (3) does not necessarily converge, so generally speaking φ and g are formal power series. The first important result on convergence is due to Poincaré. *Poincaré domain* in $\mathbb{C}^n$ is the set of all points $(z_1, \dots, z_n)$ such that the convex hull of the set $\{z_1, \dots, z_n\} \subset \mathbb{C}$ does not contain the origin. If the vector $(\lambda_1, \dots, \lambda_n)$ of eigenvalues of A in (2) lies in the Poincaré domain then there exists a convergent normalizing transformation.

Latter on it was proved by Siegel [20], that if there exist positive constants $C > 0$ and $\nu > 0$ such that for all $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| > 1$ and for all $k \in \{1, \dots, n\}$ the inequality

$$\left| \sum_{i=1}^{n} \alpha_i \lambda_i - \lambda_k \right| \geq C |\alpha|^{-\nu} \tag{5}$$

holds, then there exists a convergent transformation of (2) to normal form.

An essential further step in the investigation of the convergence of normalizing transformation is due to V. A. Pliss [65], who proved that if for system (2):

(i) the nonzero elements among the $\sum_{j=1}^{n}\alpha_j\lambda_j - \lambda_k$ satisfy condition (5)
(ii) some formal normal form of (2) is linear,

then there exists a convergent transformation to normal form.

The further fundamental result on convergence of the normalizing transformation is due to Bryuno [3, 4], who gave the two conditions that together are sufficient for existence of a convergent normalizing transformation:

(i) *Condition* ω: for $w_\ell = \min(\alpha, \lambda)$ over all $\alpha \in \mathbb{N}_0^n$ for which $(\alpha, \lambda) \neq 0$ and $|\alpha| \leq 2^l$, $\sum 2^{-\ell} \ln w_\ell < \infty$;
(ii) *Condition A* (simplified version): some normal form has the form

$$\dot{\mathbf{y}} = (1 + g(\mathbf{y}))A\mathbf{y}, \tag{6}$$

that is, $\dot{y}_j = \lambda_j y_j (1 + g(\mathbf{y}))$ for some scalar function $g(\mathbf{y})$.

It is said that (2) satisfies the Pliss-Bryuno condition if it can be transformed to (6) by a normalizing transformation.

Definition 2.1. *System* (2) *is (locally) analytically (or formally) integrable if it has* $n-1$ *functionally independent analytic (or formal) first integrals in a neighborhood of the origin.*

The following theorem was proven in [17, 24, 25].

Theorem 2.2. *System* (2) *has* $n-1$ *functionally independent analytic first integrals in a neighborhood of the origin if and only if the rank of* $\mathfrak{R}$ *is* $r_\lambda = n-1$ *and the distinguished normal form of* (2) *satisfies the Pliss-Bryuno condition.*

It follows from the theorem and its proof that in order to find independent first integrals we can choose $n-1$ linearly independent vectors from $\mathfrak{R}$, let say $\alpha_1, \ldots, \alpha_{n-1} \in \mathfrak{R}$. Then $x^{\alpha_1}, \ldots, x^{\alpha_{n-1}}$ are functionally independent integrals of the system of the linear approximation and we look for $n-1$ functions

$$\psi^{(s)}(x) = x^{\alpha_s} + \sum_{\alpha:|\alpha|>\alpha_s} \psi_\alpha^{(s)} x^\alpha \tag{7}$$

satisfying

$$\mathcal{X}(\psi^{(s)}(x)) = \sum_{\alpha\in\mathfrak{R}} p_\alpha^{(s)} x^\alpha.$$

Indeed, if we look for a series (7) such that $\mathcal{X}(\psi^{(s)}(x)) \equiv 0$, then it is easy to see that coefficients $\psi_\alpha^{(s)}$ of (7) are determined recursively from equations of the form

$$(\lambda, \alpha)\psi_\alpha^{(s)} = f_\alpha^{(s)}, \tag{8}$$

where $f_\alpha^{(s)}$ is a polynomial in parameters of system (2). If $(\lambda, \alpha) \neq 0$, that is, the coefficient $\psi_\alpha^{(s)}$ of (7) is non-resonant, $\psi_\alpha^{(s)}$ is uniquely determined from (8). If it is resonant, then $\psi_\alpha^{(s)}$ can be chosen arbitrary, however the equality in (8) holds only for those values of parameters of system (2), where the polynomial $f_\alpha^{(s)}$ vanishes. Thus, for a resonant α the polynomial $f_\alpha^{(s)}$ represents an obstacle for existence of integral (7) of system (2). Let us write for resonant α's $p_\alpha^{(s)}$ instead of $f_\alpha^{(s)}$. Then using (8) one can compute polynomials $p_\alpha^{(s)}$ ($\alpha \in \mathfrak{R}$) and system (2) has a first integral (7) only for those values of parameters for which it holds that

$$p_\alpha^{(s)} = 0 \text{ for all } \alpha \in \mathfrak{R}.$$

By the analogy with the two-dimensional case we call polynomials $p_\alpha^{(s)}$ the focus quantities of system (2). As we have seen above they are not uniquely defined, however the following theorem tells us that in the case when system (2) has $n-1$ independent analytic first integrals, for any choice of polynomials $p_\alpha^{(s)}$ despite of ideals they define can be different the variety[1] of the ideals is the same.

Theorem 2.3 ([22])**.** *For system* (2) *the following statements hold.*
(a) *There exist series $\psi(x)$ with its resonant monomials arbitrary such that*

$$\mathcal{X}(\psi(x)) = \sum_{\alpha \in \mathfrak{R}} p_\alpha x^\alpha, \tag{9}$$

where p_α are polynomials in the coefficients of (2)*.*
(b) *If the vector field* (2) *has $n-1$ functionally independent analytic or formal first integrals, then for any ψ satisfying* (9)*, we have*

$$p_\alpha = 0, \qquad \text{for all} \quad \alpha \in \mathfrak{R}. \tag{10}$$

(c) *Assume that the rank of $\mathfrak{R}$ is k, i.e. $r_\lambda = k$, and there are k functionally independent $\psi^{(1)}, \dots, \psi^{(k)}$, such that for the corresponding coefficients in* (9) *hold $p_\alpha^{(i)} = 0$, for all $\alpha \in \mathfrak{R}$, $i = 1, \dots, k$. Then the vector field $\mathcal{X}$ has exactly k functionally independent analytic or formal first integrals.*

Denote by $\mathcal{B}$ the ideal generated by the polynomials $p_\alpha^{(s)}$ ($s = 1, \dots, n-1$), for some choice of $n-1$ functionally independent functions $\psi^{(1)}, \dots, \psi^{(n-1)}$ satisfying (9), i.e.

$$\mathcal{B} = \langle p_\alpha^{(i)} | \ \alpha \in \mathfrak{R}, \quad i = 1, \dots, n-1 \rangle. \tag{11}$$

By the equivalence of (b) and (c) with $k = n-1$ the variety of $\mathcal{B}$, $\mathbf{V}(\mathcal{B})$, is the set of *all points* in the space of parameters of system (2), such that the corresponding systems have

[1]Recall that by the definition the variety of an ideal I generated by $f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n)$ of the polynomial ring $\mathbb{F}[x_1, \dots, x_n]$ is the set of all points in $\mathbb{F}^n$ where all polynomials of I vanish. The variety of I is denoted by $\mathbf{V}(I)$.

$n-1$ functionally independent integrals. We call $\mathbf{V}(\mathcal{B})$ the *integrability variety* of system (2). It follows from the theorem that when computing we can set $\psi_\alpha^{(i)} = 0$ $(\alpha \in \mathfrak{R})$.

In actual calculations we can find only a finite number of polynomials $p_\alpha^{(s)}$, so we compute until the chain of the radicals $\sqrt{\mathcal{B}_1} \subseteq \sqrt{\mathcal{B}_2} \subseteq \sqrt{\mathcal{B}_3} \subseteq \dots$, where $\mathcal{B}_k = \langle p_\alpha^{(1)}, \dots, p_\alpha^{(n-1)} \mid \alpha \in \mathfrak{R}, |\alpha| \leq k,\ k \in \mathbb{N}\rangle$, stabilizes (that is, until we find an m such that $\sqrt{\mathcal{B}_m} = \sqrt{\mathcal{B}_{m+1}}$).

Then,

(1) we find the irreducible decomposition of $\mathbf{V}(\mathcal{B}_m)$ (that is we "solve" the polynomial system $p_\alpha^{(s)} = 0$, $\alpha \in \mathfrak{R}$, $s = 1, \dots, n-1$),
(2) using different methods we try to show that $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{B}_m)$, that is, all systems corresponding to points from $\mathbf{V}(\mathcal{B}_m)$ have $n-1$ functionally independent analytic or formal first integrals.

One of the most powerful methods to find first integrals of polynomial systems is the Darboux method [8, 18] which we briefly recall here. For the system of differential equations

$$\dot{x}_1 = P_1(x), \dots, \dot{x}_n = P_n(x), \tag{12}$$

where $x = (x_1, \dots, x_n) \in \mathbb{C}^n$ we suppose that $P_i = P_i(x) \in \mathbb{C}[x]$, and P_i and P_j have no common factor if $i \neq j$. Let $\mathcal{X}$ denote the vector field on $\mathbb{C}^n$ associated to (12),

$$\mathcal{X} = \sum_{i=1}^{n} P_i(x) \frac{\partial}{\partial x_i}.$$

The *degree* of $\mathcal{X}$ is the number $d = \max\{\deg P_1, \dots, \deg P_n\}$.

An analytic function $f(x)$ is called a *Darboux factor* of system (12) if there exists a polynomial $K(x) \in \mathbb{C}[x]$ of degree at most $d-1$ such that $\mathcal{X}f = Kf$. The polynomial K is termed a *cofactor* of f. A *Darboux first integral* of system (12) is a first integral of the form

$$f_1^{\alpha_1} \cdots f_s^{\alpha_p}, \tag{13}$$

where f_i are Darboux factors.

If sufficiently many Darboux factors can be found then they can be used to construct a Darboux first integral. Indeed, if a polynomial vector field $\mathcal{X}$ of degree d in $\mathbb{C}^n$ admits p Darboux factors such that $\sum_{i=1}^{p} a_i K_i = 0$, then the function

$$H = f_1^{a_1} \cdots f_p^{a_p} \tag{14}$$

is a first integral of $\mathcal{X}$.

Similarly, one can look for Darboux linearization of system (2). In such case we look not for a transformation (3), but for its inverse. If it can be found in the form

$$y_k = f_1(x)^{a_1} \cdots f_p(x)^{a_p},$$

then it is said that system (2) is Darboux linearizable.

Darboux functions are usually polynomials or exponential functions of the form $\exp(g/h)$ (exponential Darboux factors), where g and h are polynomials. They can be also hypergeometric functions [6]. Below we give examples of Darboux functions which are polynomials of polynomial and exponential Darboux factors, see (21),(26),(27). To our knowledge such Darboux function did not appear in the literature before.

3. Local integrability and linearizability of system (1)

In this section we apply the methods described in the previous section to find the sets in the space of parameters of (1) corresponding to locally integrable and linearizable systems.

Theorem 3.1. *Consider the quadratic three dimensional systems* (1)*. It is locally integrable if and only if one of the following conditions is satisfied:*

(1) $c_{13} = c_{12} = b_{13} = b_{12} = 0$;
(2) $c_{23} = c_{12} = b_{23} = b_{13} = b_{12} + c_{13} = a_{13} = a_{12} = 0$;
(3) $c_{12} = b_{13} = a_{23} = a_{12}b_{23} + a_{13}c_{23} + b_{23}c_{23} = 0$,
$a_{13}b_{12} - a_{13}c_{13} - b_{23}c_{13} = a_{12}b_{12} - a_{12}c_{13} + b_{12}c_{23} = 0$;
(4) $c_{23} = c_{13} = c_{12} = b_{12} = a_{13} - b_{23} = a_{12} = 0$;
(5) $c_{23} = b_{23} = a_{23} = a_{13} = a_{12} = 0$;
(6) $c_{13} = b_{23} = b_{13} = b_{12} = a_{13} = a_{12} - c_{23} = 0$.

Moreover, systems (1) *are linearizable if and only if either one of conditions* (1)*,* (2)*,* (4)*,* (5)*,* (6) *is satisfied or one of two following conditions holds:*

(7) $c_{12} = b_{13} = b_{12} = b_{23} = a_{23} = a_{13} = a_{12} = 0$
(8) $b_{13} = c_{13} = c_{12} = c_{23} = a_{23} = a_{13} = a_{12} = 0$

Proof. We split the proof of this theorem in two steps: the computation of the conditions of integrability and linearizability and the proof of sufficiency of these conditions.

Computation of the conditions of integrability. Since xy and xz are two functionally independent integrals of the system of the linear approximation of (1) we look for two independent first integrals of the forms

$$\psi^{(1)}(x,y,z) = xy + h.o.t \text{ and } \psi^{(2)}(x,y,z) = xz + h.o.t. \tag{15}$$

For each case we computed the first 11 focus quantities $f_1 = p^{(1)}_{(2,1,1)}, f_2 = p^{(1)}_{(3,1,2)}, \ldots$ and $g_1 = p^{(2)}_{(2,1,1)}, g_2 = p^{(2)}_{(3,1,2)}, \ldots$ (resonant coefficients in (15) were set to zero as justified by

Theorem 2.3). Few first focus quantities are as follows:

$$\begin{aligned}
f_1 =&(a_{13}+b_{23})c_{12};\\
f_2 =&-2a_{13}b_{12}-b_{12}b_{23}+b_{12}(a_{13}+b_{23})+(a_{13}+b_{23})c_{13}-b_{13}c_{23};\\
f_3 =&-2a_{13}b_{13}+b_{13}(a_{13}+b_{23});\\
f_4 =&(-9a_{12}^2(b_{12}^2+b_{13}c_{12}))/2+(b_{12}(9a_{12}^2b_{12}+6a_{12}a_{13}c_{12}-a_{23}b_{12}c_{12}+\\
&+9a_{12}b_{23}c_{12}+a_{23}c_{12}c_{13}+3a_{13}c_{12}c_{23}+3b_{23}c_{12}c_{23}))/2+\\
&+(c_{12}(12a_{12}a_{13}b_{12}+a_{23}b_{12}^2+9a_{12}^2b_{13}+18a_{12}b_{12}b_{23}+6a_{13}^2c_{12}+\\
&+4a_{23}b_{13}c_{12}+12a_{13}b_{23}c_{12}+6b_{23}^2c_{12}+6a_{12}a_{13}c_{13}-2a_{23}b_{12}c_{13}+\\
&+9a_{12}b_{23}c_{13}+a_{23}c_{13}^2+6a_{13}b_{12}c_{23}++6b_{12}b_{23}c_{23}+3a_{13}c_{13}c_{23}+\\
&+3b_{23}c_{13}c_{23}))/6;\\
g_1 =&-2a_{12}c_{12}+c_{12}(a_{12}+c_{23});\\
g_2 =&-(b_{23}c_{12})-2a_{12}c_{13}-c_{13}c_{23}+b_{12}(a_{12}+c_{23})+c_{13}(a_{12}+c_{23});\\
g_3 =&b_{13}(a_{12}+c_{23});\\
g_4 =&(-9a_{12}^2(b_{12}c_{12}+c_{12}c_{13}))/2-(b_{12}c_{12}(-3a_{12}^2+4a_{23}c_{12}-6a_{12}c_{23}-\\
&+3c_{23}^2))/2+(c_{12}(2a_{12}^2b_{12}+4a_{12}a_{13}c_{12}+a_{23}b_{12}c_{12}+2a_{12}b_{23}c_{12}+\\
&+a_{12}^2c_{13}-a_{23}c_{12}c_{13}+4a_{12}b_{12}c_{23}+6a_{13}c_{12}c_{23}+2b_{23}c_{12}c_{23}+\\
&+2a_{12}c_{13}c_{23}+2b_{12}c_{23}^2+c_{13}c_{23}^2))/2.
\end{aligned}$$

The expressions for other focus quantities are too long so we do not present them here. Using the routine `minAssGTZ` [9] of computer algebra system SINGULAR [10] we computed irreducible decomposition of the variety of the ideal $\mathcal{B}_{11}=\langle f_1,f_2,f_3,\dots,f_{11},g_1,g_2,g_3,\dots,g_{11}\rangle$ and obtained that it consists of sets (1)–(6) listed in the statement of the theorem.

Computation of the conditions of linearizability. The necessary conditions for linearizability were computed in a similar way as the conditions for integrability. The first few linearizability quantities were computed, more specifically, first twenty-three ones. Then, with `minAssGTZ` we computed the irreducible decomposition of the variety of the ideal generated by these quantities and obtained seven conditions for linearizability mentioned in the statement of Theorem 3.1.

Proof of sufficiency of the conditions. In order to prove the sufficiency of the conditions given in Theorem 3.1 we mainly used the Darboux method.

Case 1. In this case system (1) takes the form

$$\begin{aligned}
\dot{x} &= x+a_{12}xy+a_{13}xz+a_{23}yz\\
\dot{y} &= y(-1+b_{23}z)\\
\dot{z} &= z(-1+c_{23}y).
\end{aligned}\tag{16}$$

We use a similar arguments as in the proof of Case 14 of Theorem 2 in [1]. The second and third equations of (16) define a linearizable node, so there exists a change of coordinates $Y = y(1 + O(y, z)), Z = z(1 + O(y, z))$ such that $\dot{Y} = -Y$ and $\dot{Z} = -Z$. In order to prove the linearizability for this case it is sufficient to find $X = \alpha(Y, Z) + \beta(Y, Z)x$ such that $\dot{X} = X$. This is equivalent to the following equations

$$\dot{\alpha} + \beta a_{23}yz = \alpha, \tag{17}$$

$$\dot{\beta} + \beta(a_{12}y + a_{13}z) = 0. \tag{18}$$

First solving equation (18), where $\beta(Y, Z) = \sum c_{i,j}Y^iZ^j$ and then using obtained $\beta(Y, Z)$ to solve the equation (17), obtaining $\alpha(Y, Z)$, we have $X = \alpha(Y, Z) + \beta(Y, Z)x$ such that $\dot{X} = X$. Thus the linearizability of this case and, hence, the integrability is proven.

Case 2. Under this condition the corresponding system has the form

$$\begin{aligned} \dot{x} &= x + a_{23}yz \\ \dot{y} &= -y(1 - b_{12}x) \\ \dot{z} &= -z(1 + b_{12}x). \end{aligned} \tag{19}$$

It has three invariant surfaces, $l_1 = y$, $l_2 = z$, $l_3 = x + \frac{a_{23}}{3}yz$ and one exponential factor $l_4 = e^{2x+a_{23}yz}$.

So the system can be linearized by the transformation $X = l_3$,$Y = yl_4^{\frac{-b_{12}}{2}}$, $Z = zl_4^{\frac{b_{12}}{2}}$. Consequently, this system admits two independent first integral as desired.

Case 3. In this case by the change of variables $x \mapsto y, y \mapsto x, z \mapsto z$ and a time rescaling we obtain a family of systems studied in [2]. The integrability of such systems follows from Theorem 4 in [2].

Case 4. The fourth condition yields the system

$$\begin{aligned} \dot{x} &= x + a_{13}xz + a_{23}yz \\ \dot{y} &= -y + b_{13}xz + a_{13}yz \\ \dot{z} &= -z. \end{aligned} \tag{20}$$

Darboux factors and exponential factors obtained in this case are

$$\begin{aligned} l_1 =& z, \qquad l_2 = y - b_{13}xz + \sqrt{a_{23}}\sqrt{b_{13}}yz, \\ l_3 =& y - b_{13}xz - \sqrt{a_{23}}\sqrt{b_{13}}yz, \\ l_4 =& e^{\sqrt{a_{23}}\sqrt{b_{13}}z}, \qquad l_5 = e^{a_{13}z}. \end{aligned}$$

Case 4.1. In case $a_{23}b_{13} \neq 0$, by the change of coordinates

$$\begin{aligned} X =& \frac{1}{2\sqrt{a_{23}}\sqrt{b_{13}^3}} l_1^{-2} l_4^{-1} l_5 l_6, \\ Y =& l_3 l_4^{\frac{a_{13}+\sqrt{a_{23}}\sqrt{b_{13}}}{\sqrt{a_{23}}\sqrt{b_{13}}}}, \\ Z =& z, \end{aligned}$$

where $l_1, .., l_5$ are Darboux factors defined above, and

$$l_6 = l_2 + l_4^2 l_3 = (y - b_{13}xz + \sqrt{a_{23}}\sqrt{b_{13}}yz) + e^{2\sqrt{a_{23}}\sqrt{b_{13}}z}(y - b_{13}xz - \sqrt{a_{23}}\sqrt{b_{13}}yz), \tag{21}$$

we obtain the linear system.

Case 4.2. If $a_{23} = 0$ the system has the form

$$\begin{aligned} \dot{x} &= x(1 + a_{13}z) \\ \dot{y} &= -y + b_{13}xz + a_{13}yz \\ \dot{z} &= -z. \end{aligned} \tag{22}$$

The change of coordinates

$$\begin{aligned} X =& l_3 l_5 \\ Y =& l_2 l_5 \\ Z =& z, \end{aligned}$$

linearizes system (22).

Case 4.3. The system under the condition $b_{13} = 0$ is

$$\begin{aligned} \dot{x} &= x + a_{13}xz + a_{23}yz \\ \dot{y} &= y(-1 + a_{13}z) \\ \dot{z} &= -z. \end{aligned} \tag{23}$$

The change of coordinates that linearizes the system (23) is

$$\begin{aligned} X =& l_5 l_7 \\ Y =& y l_5 \\ Z =& z, \end{aligned}$$

where $l_7 = x + \frac{a_{23}}{3}yz$.

Case 4.4. The system (20) where $a_{23} = 0$ and $b_{13} = 0$ is

$$\begin{aligned} \dot{x} &= x(1 + a_{13}z) \\ \dot{y} &= y(-1 + a_{13}z) \\ \dot{z} &= -z. \end{aligned} \tag{24}$$

It is linearized by the substitution

$$\begin{aligned} X =& xe^{a_{13}z} \\ Y =& ye^{a_{13}z} \\ Z =& z. \end{aligned}$$

Remark: The Darboux factor l_6 was obtained using the first integral $\psi = \psi_1^2 - \psi_2^2$, where
$\psi_1 = l_1^{-1}(-l_2)l_4^{-1}l_5$ and $\psi_2 = l_1^{-1}(-l_2)^{\frac{1}{2}}(-l_3)^{\frac{1}{2}}l_5$ are first integrals of the system (20). From the factorization of the first integral $\psi = l_1^{-2}l_2l_4^{-2}l_5^2l_6$ we obtained l_6.

Case 5. Now the corresponding system is

$$\begin{array}{rcl} \dot{x} &=& x \\ \dot{y} &=& -y + b_{12}xy + b_{13}xz \\ \dot{z} &=& -z + c_{12}xy + c_{13}xz. \end{array} \tag{25}$$

The Darboux factors are $l_1 = x$, $\quad l_{2,3} = y - \frac{m_\pm}{2c_{12}}z$, $\quad l_4 = e^{b_{12}x}$, $\quad l_5 = e^{c_{13}x}$ and $l_6 = e^{\sqrt{4b_{13}c_{12}+(b_{12}-c_{13})^2}x}$, where $m_\pm = (b_{12} - c_{13} \pm \sqrt{b_{12}^2 + 4b_{13}c_{12} - 2b_{12}c_{13} + c_{13}^2})$ and $c_{12} \neq 0$.

Case 5.1 The change of variables in case $c_{12} \neq 0$ and $\sqrt{4b_{13}c_{12} + (b_{12} - c_{13})^2} \neq 0$ is

$$\begin{aligned} X =& x \\ Y =& k(l_4l_5l_6)^{-\frac{1}{2}}l_7 \\ Z =& 2c_{12}k(l_4l_5l_6)^{-\frac{1}{2}}l_8, \end{aligned}$$

where $k = (2\sqrt{4b_{13}c_{12} + (b_{12} - c_{13})^2})^{-1}$,

$$l_7 = m_+l_3 - m_-l_2l_6, \qquad l_8 = l_3 - l_2l_6. \tag{26}$$

Case 5.2 In case $c_{12} = 0$ and $\sqrt{4b_{13}c_{12} + (b_{12} - c_{13})^2} \neq 0$, by the linear transformation

$$\begin{aligned} X =& x \\ Y =& l_{10} \\ Z =& zl_5^{-1}, \end{aligned}$$

where $l_9 = y + \frac{b_{13}}{b_{12}-c_{13}}z$ and

$$l_{10} = -\frac{b_{13}}{b_{12} - c_{13}}zl_5^{-1} + l_4^{-1}l_9, \tag{27}$$

the system is linearized.

Case 5.3 The linear transformation of the system, where $c_{12} \neq 0$ and $\sqrt{4b_{13}c_{12} + (b_{12} - c_{13})^2} = 0$, which linearizes the system is

$$X = x$$
$$Y = (l_4 l_5)^{-\frac{1}{2}} l_{11}$$
$$Z = -c_{12}(l_4 l_5)^{-\frac{1}{2}} l_{12},$$

where $l_{11} = xy - \frac{2+b_{12}x - c_{13}x}{2c_{12}} z$ and $l_{12} = y - \frac{b_{12}}{2}xy + \frac{c_{13}}{2}xy + \frac{b_{12}^2}{4c_{12}}xz - \frac{b_{12}c_{13}}{2c_{12}}xz + \frac{c_{13}^2}{4c_{12}}xz.$

Case 5.4 The change of variables in case $c_{12} = 0$ and $b_{12} = c_{13}$ is

$$X = x$$
$$Y = (y - b_{13}xz) l_5^{-1}$$
$$Z = z l_5^{-1}.$$

Remark: As in Case 4, Darboux factors l_7, l_8, l_{10} were obtained using first integrals of system (25).

Case 6. Under these conditions system (1) is written as

$$\begin{aligned} \dot{x} &= x + a_{12}xy + a_{23}yz \\ \dot{y} &= -y \\ \dot{z} &= -z + c_{12}xy + a_{12}yz. \end{aligned} \tag{28}$$

By the linear transformation $X = x, Y = z, Z = z$ and $a_{12} = a_{13}, c_{12} = b_{13}$ (28) is transformed to a system of the form (20).

Case 7. The system under condition (7) is

$$\begin{aligned} \dot{x} &= x \\ \dot{y} &= -y \\ \dot{z} &= -z(1 - c_{13}x - c_{23}y). \end{aligned} \tag{29}$$

By the change of coordinates $X = x$, $Y = y$, $Z = ze^{-c_{13}x + c_{23}y}$ we obtain $\dot{X} = X$, $\dot{Y} = -Y$, $\dot{Z} = -Z$.

Case 8. The system of this case is transformed to (29) by the transformation $X = \frac{c_{13}}{b_{12}}x$, $Y = z$, $Z = \frac{c_{23}}{b_{23}}y$. □

4. One first integral of system (1) with $a_{23} = 0$

In the previous section we have considered the problem of complete integrability of a polynomial system, that is, the problem of existence of maximal number of independent local first integrals. If we can find such integrals then using them we can determine trajectories of the system. However completely integrable systems are rare phenomena. So it is of the fundamental importance to know if a differential system admits even one first integral. If such an integral exists it gives a conservation law for the system and enables to reduce the dimension of the system.

In this section we study the problem of existence of a first integral of the form

$$\psi^{(1)}(x,y,z) = xy + O(|x,y,z|^3) \tag{30}$$

for system (1).

It appears, with the computational point of view this problem is more complicate that the problem of complete integrability (of simultaneous existence of two first integrals) due to two reasons. First, in this case we have to deal with ideals involving polynomials of higher degrees. Second, and more important, in this case we do not have an analog of Theorem 2.3, so we cannot set resonant coefficients in the function $\psi^{(1)}$ to zero. The arising computations are so laborious that we were not able to complete them for system (1), and to be able to complete them we have restricted our consideration to the case of system (1) with $a_{23} = 0$. The result is given in the following statement.

Theorem 4.1. *The necessary conditions to system* (1) *with* $a_{23} = 0$ *to have a first integral of the form* $xy + O(|x,y,z|^3)$ *are the following*

(1) $c_{12} = b_{13} = a_{13}b_{12} - a_{13}c_{13} - b_{23}c_{13} = 0$;
(2) $c_{13} = c_{12} = a_{13} - b_{23} = b_{12}b_{23} + b_{13}c_{23} = 0$;
(3) $c_{12} = a_{12}b_{12} - a_{12}c_{13} + b_{12}c_{23} = b_{12}b_{23} - 2b_{23}c_{13} + b_{13}c_{23} = 0$
$a_{12}b_{23}c_{13} - a_{12}b_{13}c_{23} + 2b_{23}c_{13}c_{23} - b_{13}c_{23}^2 = a_{13} - b_{23} = 0$;
(4) $c_{12} = a_{13} - b_{23} = b_{12}b_{23} - 2b_{23}c_{13} + b_{13}c_{23} = 3a_{12}b_{12} - 5a_{12}c_{13} + 4b_{12}c_{23} - 6c_{13}c_{23} = 0$,
$a_{12}c_{13}^2 - \frac{143}{30}b_{12}^2c_{23} + \frac{3}{163}b_{12}c_{13}c_{23} - \frac{162}{31}c_{13}^2c_{23} = 0$,
$a_{12}b_{23}c_{13} - 3a_{12}b_{13}c_{23} + 2b_{23}c_{13}c_{23} - 4b_{13}c_{23}^2 = 0$,
$a_{12}b_{13}c_{13} + \frac{137}{107}b_{23}c_{13}^2 + \frac{143}{90}b_{12}b_{13}c_{23} - \frac{73}{39}b_{13}c_{13}c_{23} = 0$,
$a_{12}^2c_{13} + \frac{175}{39}a_{12}c_{13}c_{23} - \frac{93}{136}b_{12}c_{23}^2 - \frac{124}{73}c_{13}c_{23}^2 = a_{12}^3 - \frac{29}{136}a_{12}^2c_{23} + \frac{115}{104}a_{12}c_{23}^2 - \frac{29}{176}c_{23}^3 = 0$,
$a_{12}^2b_{13} + \frac{149}{39}a_{12}b_{13}c_{23} - \frac{7}{125}b_{23}c_{13}c_{23} + \frac{105}{58}b_{13}c_{23}^2 = b_{12}^3 + \frac{164}{61}b_{12}^2c_{13} + \frac{91}{59}b_{12}c_{13}^2 + \frac{107}{7}c_{13}^3 = 0$,
$b_{23}c_{13}^3 - \frac{78}{115}b_{12}^2b_{13}c_{23} - \frac{74}{95}b_{12}b_{13}c_{13}c_{23} - \frac{134}{69}b_{13}c_{13}^2c_{23} = 0$,
$b_{23}^2c_{13}^2 + \frac{71}{130}a_{12}b_{13}^2c_{23} + \frac{77}{90}b_{13}b_{23}c_{13}c_{23} - \frac{103}{79}b_{13}^2c_{23}^2 = 0$;
(5) $c_{12} = 152b_{12} + 13c_{13} = a_{13} - b_{23} = 39a_{12} + 7c_{23} = 71b_{23}c_{13} - 135b_{13}c_{23} = c_{13}b_{23} = 0$;
(6) $c_{12} = b_{12} - c_{13} = a_{13} - b_{23} = a_{12} + c_{23} = b_{23}c_{13} - b_{13}c_{23} = 0$;
(7) $c_{23} = b_{23} = a_{13} = 0$;
(8) $b_{23} = b_{13} = a_{13} = 0$;
(9) $c_{13} = b_{13} = b_{12} = a_{13} + b_{23} = 0$;
(10) $b_{13} = b_{12} = a_{13} + b_{23} = a_{12} = 0$;
(11) $b_{13} = b_{12} = a_{13} + b_{23} = b_{23}c_{12} + a_{12}c_{13} = 0$;
(12) $b_{13} = b_{12} = a_{13} + b_{23} = b_{23}c_{12} + c_{13}c_{23} = 0$.

Proof. Computing necessary conditions for existence of first integral (30) was done in the similar way as computing necessary conditions for complete integrability. The first three focus quantities f_1, f_2, f_3 are the same as presented in the proof of Theorem 3.1 (they do not contain resonant coefficients $\psi^{(1)}_\alpha$) and f_4 is

$$\begin{aligned} f_4 &= (b_{12}(9a_{12}^2 b_{12} + 6a_{12}a_{13}c_{12} - a_{23}b_{12}c_{12} + 9a_{12}b_{23}c_{12} + \\ &+ a_{23}c_{12}c_{13} + 3a_{13}c_{12}c_{23} + 3b_{23}c_{12}c_{23} + 12a_{12}\psi^{(1)}_{\alpha_1}))/2 + \\ &+ (3a_{12}(-3a_{12}b_{12}^2 - 3a_{12}b_{13}c_{12} - 4b_{12}\psi^{(1)}_{\alpha_1} - 2c_{12}\psi^{(1)}_{\alpha_2}))/2 + \\ &+ (c_{12}(12a_{12}a_{13}b_{12} + a_{23}b_{12}^2 + 9a_{12}^2 b_{13} + 18a_{12}b_{12}b_{23} + \\ &+ 6a_{13}^2 c_{12} + 4a_{23}b_{13}c_{12} + 12a_{13}b_{23}c_{12} + 6b_{23}^2 c_{12} + \\ &+ 6a_{12}a_{13}c_{13} - 2a_{23}b_{12}c_{13} + 9a_{12}b_{23}c_{13} + a_{23}c_{13}^2 + \\ &+ 6a_{13}b_{12}c_{23} + 6b_{12}b_{23}c_{23} + 3a_{13}c_{13}c_{23} + 3b_{23}c_{13}c_{23} + 12a_{13}\psi^{(1)}_{\alpha_1} + \\ &+ 12b_{23}\psi^{(1)}_{\alpha_1} + 12a_{12}\psi^{(1)}_{\alpha_2} + 6c_{23}\psi^{(1)}_{\alpha_2}))/6, \end{aligned}$$

where $\psi^{(1)}_{\alpha_1} = \psi^{(1)}_{(2,1,1)}, \psi^{(1)}_{\alpha_2} = \psi^{(1)}_{(3,1,2)}$ represent coefficients of (7). We also computed the next polynomials $f_5, \dots, f_{13}$. The expressions of these polynomials are too long, so we do not present them here, however the interested reader can easily compute them with any available computer algebra system.

To obtain necessary conditions for existence of the first integral (30) we first should eliminate from the system $f_1 = ... = f_{13} = 0$ variables $\psi^{(1)}_\alpha$ corresponding to the resonant coefficients and then find the irreducible decomposition of the variety of the obtained ideal. The most laborious is the elimination procedure which is based on the Elimination Theorem [7, 21]. In Singular it is implemented as the routine `eliminate`. We were not able to complete the computations over the field of rational numbers, but we have succeed to complete them in the field of characteristic 32003. So, in this field with `eliminate` we have performed elimination of variables $\psi^{(1)}_\alpha$ for resonant α. Geometrically it means that we found the projection P of the variety of the ideal $F = \langle f_1, \dots, f_{13}\rangle$ on the space of parameters (a, b, c) of the system, and then with `minAssGTZ` we carried out the decomposition of $\mathbf{V}(P)$. After lifting the obtained ideals to the field of rational numbers using the rational reconstruction algorithm of [23] we obtained the conditions (1)–(4), (6)–(12) of Theorem 4.1, but instead of (5) we got the condition

$$(5') \quad c_{12} = 152b_{12} + 13c_{13} = a_{13} - b_{23} = 39a_{12} + 7c_{23} = 71b_{23}c_{13} - 135b_{13}c_{23} = 0.$$

Then with the radical membership test [7, 21] we have checked that each of conditions (1)–(4), (6)–(12) yields vanishing all polynomials $f_1, \dots, f_{13}$ for some values of $\psi^{(1)}_\alpha$. However for the condition (5′) given above this is true only if $c_{13} = 0$ or $b_{23} = 0$. For this reason we added the polynomials $c_{13}b_{23}$ into condition (5). Thus, the conditions (1)–(12) of the theorem are necessary conditions for integrability of system (1) with $a_{23} = 0$. □

Remark 1. Since modular computations were employed we cannot guarantee that Theorem 4.1 gives the complete list of the necessary conditions. To verify this we need to check that the Groebner basis obtained after the lifting to the field of characteristic 0 is a true Groebner basis of the original ideal. Recently an efficient algorithm to perform this task was proposed in [19], however it is not yet implemented in widely available computer algebra systems.

Remark 2. To show that the obtained conditions are also sufficient for the existence of first integral (30) we have to construct such integral in each case or prove its existence. We were able to do this only in cases (4), (8) and (10). However for the other conditions we have checked that under each of them not only first thirteen, but first 25 focus quantities vanish. That is, most probably, all conditions of the theorem are also sufficient for existence of a first integral of the form (30).

Remark 3. As we have seen above, basing on Theorem 2.3 when looking for two first integrals in system (1) we can chose resonant coefficients $\psi_\alpha^{(1)}$ and $\psi_\alpha^{(2)}$ ($\alpha \in \mathfrak{R}$) in series $\psi^{(1)}$ and $\psi^{(2)}$ equal to zero. However it appears it is not the case when we look for just one integral. For the system studied in Theorem 4.1 if resonant coefficients $\psi_\alpha^{(1)}$ are chosen equal to zero then after computing the minimal decomposition we obtain the following conditions:

(1) $c_{13} = c_{12} = b_{13} = b_{12} = 0$;
(2) $c_{12} = b_{13} = a_{12}b_{23} + a_{13}c_{23} + b_{23}c_{23} = a_{13}b_{12} - a_{13}c_{13} - b_{23}c_{13} = a_{12}b_{12} - a_{12}c_{13} + b_{12}c_{23} = 0$;
(3) $c_{23} = c_{13} = c_{12} = b_{12} = a_{13} - b_{23} = a_{12} = 0$;
(4) $c_{23} = b_{23} = a_{13} = a_{12} = 0$;
(5) $b_{23} = b_{13} = a_{13} = 0$;
(6) $b_{13} = b_{12} = a_{13} + b_{23} = a_{12} = 0$;
(7) $c_{13} = b_{13} = b_{12} = a_{13} + b_{23} = 0$.

Denote by I_i ideal generated by polynomials given by conditions (i) of Theorem 4.1 and J_i ideal generated by polynomials of conditions (i) given above. First, with `intersect` of Singular, we computed intersection[2] of ideals I_i, $\cap_{i=1}^{12} I_i = M_1$ and intersection of ideals J_i, $\cap_{i=1}^{7} J_i = M_2$. We compare ideals M_1 and M_2 using the Singular routine `reduce` which reduces an ideal F with respect to another ideal H represented by a standard basis. For such ideals F and H `reduce(F,H)` returns 0 if and only if $F \subset H$. As the result of the reduction of the ideal M_1 with respect to a standard basis of M_2 we obtain 0, which means that that $M_1 \subset M_2$. But the opposite reduction shows that $M_2 \not\subset M_1$. Thus the variety of M_2 is a proper subvariety of M_1. It means that setting all resonant coefficients zero we most probably have lost some cases of the existence of the first integral (30) for system (1).

[2]The intersection of two polynomial ideals F and H is an ideal $F \cap H$ formed by polynomials contained in both ideals, F and H. The variety of the intersection of ideals F and H, $F \cap H$, is equal to the union of variety of ideal F and variety of ideal H.

Theorem 4.2. *If one of conditions* (4), (8) *or* (10) *of Theorem 4.1 holds then the system* (1) *with* $a_{23} = 0$ *admits an analytic first integral of the form (30).*

Proof. Case 4. System (1) with $a_{23} = 0$ under these conditions is

$$\begin{aligned}\dot{x} =& x(1 + \frac{1}{\gamma}(4147 + 11^{2/3}(13(-\alpha + 612\beta))^{1/3} - (11)^{2/3}(13(\alpha + 612\beta))^{1/3})y + b_{23}z)\\ \dot{y} =& -y(1 + b_{23}z)\\ \dot{z} =& -z(1 + c_{23}y)\end{aligned}$$

where $\alpha = 5485580347$, $\beta = \sqrt{2125208259580367}$ and $\gamma = 58344$. The system is Lotka-Volterra system, so, similarly as in [2], we can show that this is a linearizable system.

In this case the equations in y and z are independent of x and give a linearizable node. Hence there exists a change of coordinates $Y = y(1 + O(y, z))$ and $Z = z(1 + O(y, z))$ such that $\dot{Y} = -Y$ and $\dot{Z} = -Z$. The equation in x has the form $\dot{x} = x(1 + b_{23}z(Y, Z) + ky(Y, Z))$, where $k = 4147 + 11^{2/3}13^{1/3}(-\alpha + 612\beta)^{1/3} - 11^{2/3}13^{1/3}(\alpha + 612\beta)^{1/3}$.

Moreover if there exists a function $m(Y, Z)$ such that $\dot{m} = b_{23}z(Y, Z) + ky(Y, Z)$ then the transformation $X = xe^{-m}$ yields $\dot{X} = X$. Writing $b_{23}z(Y, Z) + ky(Y, Z) = \sum_{i+j>0} f_{ij}Y^iZ^j$ we obtain $m = \sum_{i+j>0} \frac{f_{ij}}{i+j} Y^iZ^j$ which is convergent, so the system is linearizable and consequently admits a first integral, as desired.

Case 8. Under this condition the corresponding system is

$$\begin{aligned}\dot{x} &= x(1 + a_{12}y)\\ \dot{y} &= y(-1 + b_{12}x)\\ \dot{z} &= -z + c_{12}xy + c_{13}xz + c_{23}yz.\end{aligned} \tag{31}$$

It is also easy to see that system (31) has the first integral $\psi^{(1)} = xye^{-b_{12}x+a_{12}y}$.

Case 10. The corresponding system (1) is

$$\begin{aligned}\dot{x} &= x(1 + a_{13}z),\\ \dot{y} &= -y(1 + a_{13}z),\\ \dot{z} &= -z + c_{12}xy + c_{13}xz + c_{23}yz.\end{aligned}$$

Clearly $\psi^{(1)} = xy$ is a first integral of the system.

□

5. Conclusion

We have proved that system (1) is locally integrable if one of conditions of Theorem 3.1 is satisfied. The necessary and sufficient conditions for the linearizability are obtained as well. The problem of existence of a first integral of form $\psi^{(1)}(x, y, z) = xy + O(|x, y, z|^3)$ for system (1) with $a_{23} = 0$ was discussed. Necessary conditions were obtained and for some of these conditions the sufficiency was proof as well. From the proof of Theorem 4.1 and the remarks after the theorem we can conjecture that, unlike in the case of complete integrability, setting the resonant coefficient to zero when looking for integral (30) can lead to the lost of some conditions of existence of the integral. However to confirm this conjecture one has to show that for all cases of Theorem 4.1 there is a first integral (30).

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[1] Center for Applied Mathematics and Theoretical Physics, University of Maribor, Krekova 2, SI-2000 Maribor, Slovenia

E-mail address: `masa.dukaric@gmail.com,valery.romanovsky@uni-mb.si`

[2] ICMC - Universidade de São Paulo, Caixa Postal 668, 13566-590, São Carlos, SP, Brazil

E-mail address: `regilene@icmc.usp.br`

[3] Faculty of Natural Science and Mathematics, University of Maribor, Koroška c. 160, SI-2000 Maribor, Slovenia