Count and symmetry of global and local minimizers of the Cahn-Hilliard energy over cylindrical domains.

Arnaldo Simal do Nascimento, João Biesdorf*

Departamento de Matemática

Universidade Federal de São Carlos, S. Carlos, São Paulo, Brasil

e-mail: arnaldon@dm.ufscar.br, jbiesdorf@dm.ufscar.br

Janete Crema Simal

Departamento de Matemática - ICMC

Universidade São Paulo, S. Carlos, S. Paulo, Brasil

e-mail: janete@icmc.usp.br

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Abstract

We address the problem of minimization of the Cahn-Hilliard energy functional under a mass constraint over two and three-dimensional cylindrical domains. Although existence is presented for a general case the focus is mainly on rectangles, parallelepipeds and circular cylinders. According to the symmetry of the domain the exact number of global and local minimizers are given as well as their geometric profile and interface location; all are one-dimensional increasing/decreasing and odd functions for domains with lateral symmetry in all axes and also for circular cylinders. The selection of global minimizers by the energy functional is made via the smallest interface area criterion.

The approach utilizes $\Gamma-$convergence techniques to prove existence of an one-parameter family of local minimizers of the energy functional for any cylindrical domain. The exact number of global and local minimizers as well as their geometric profiles are accomplished via suitable applications of the unique continuation principle while exploring the domain geometry in each case and also the preservation of global minimizers through the process of $\Gamma-$convergence.

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1 Introduction

¿From the physical standpoint minimizers of energy functionals, either global or local, are the most interesting critical points as they are the most likely observable configurations. In this work we are concerned with global and local minimizers of the so-called Cahn-Hilliard energy

$$E_\varepsilon(u) = \int_\Omega [\varepsilon|\nabla u|^2 + \varepsilon^{-1}(1-u^2)^2]dx \tag{1.1}$$

under the constraint $\int_\Omega u = 0$ over domains $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ of cylindrical type, i.e.,

$$\Omega = (-l, l) \times D$$

where $D \subset \mathbb{R}^n$ $(n = 1, 2)$ is a Lipschitz bounded domain.

Our goal is to find, for $\varepsilon > 0$ small, the exact number of solutions to

$$\inf_{u \in \mathcal{M}} \quad E_\varepsilon(u) \tag{1.2}$$

where

$$\mathcal{M} \stackrel{\text{def}}{=} \{u \in H^1(\Omega) : \int_\Omega u = 0\}$$

as well as local (which are not global) minimizers of E_ε in $\mathcal{M}$ according to the symmetry and dimension of the domain. Moreover for rectangles, squares, parallelepipeds, cubes and circular cylinders we present a complete picture of its symmetry properties, geometric profile and location of the interfaces $\{u_\varepsilon = 0\}$ thus concluding that global and local minimizers are one-dimensional increasing/decreasing odd functions, a type of Sturm-Liouville result. As expected the selection of global minimizers by the energy functional is made via the smallest interface area criterion.

If the double-well potential $F_1(u) = (1-u^2)^2$ in (1.1) is replaced with $F_2(u) = (u-\alpha)^2(u-\beta)^2$ where $0 < \alpha < \beta < \infty$ and the mass constraint replaced with $\int_\Omega u = \frac{\mathcal{L}^n(\Omega)}{2}(\alpha+\beta)$ (here $\mathcal{L}^n$ stands for the n-dimensional Lebesgue measure) then the functional (1.1) represents the free energy of a binary alloy and u the relative concentration of one of the two phases where the total mass $m = \frac{\mathcal{L}^n(\Omega)}{2}(\alpha + \beta)$ is preserved.

The reader interested in the physical background of this model is referred to [22], [3] or [5], for instance, and the references therein. We take for simplicity $\alpha = -1$ and $\beta = 1$ since all the results presented here, concerning existence of local and global minimizers (Sections 3 and 4) as well as their inherited symmetry, translate verbatim to the meaningful physical case $0 < \alpha < \beta < \infty$.

Let us mention some related works in order to set our own into perspective.

The authors in [5] showed existence of global and local minimizers of (1.2) over the flat torus $\mathbb{T}^n$ $(n = 2, 3)$, in which case it is called the periodic Cahn-Hilliard problem.

In [4], for smooth domains $\Omega \subset \mathbb{R}^2$ and $\int_\Omega u = m \approx 0$ or $m \approx \mathcal{L}^2(\Omega)$, the authors showed existence of local minimizers of E_ε (for $\varepsilon > 0$ small) whose transition layers are close to circular arcs and intersect the boundary orthogonally. In particular, under these hypotheses, they conclude that there are at least two such local minimizers and each of these arcs encloses a point on the boundary where the curvature of $\partial\Omega$ attains its local maximum.

The problem of minimizing $\Lambda_\varepsilon = \varepsilon E_\varepsilon$ over the unit square $\Omega = (0\,,1) \times (0\,,1)$ has been considered in [10] via a bifurcation approach where either ε or the total mass $\int_\Omega u = m \in \mathbb{R}$ is the bifurcation parameter. Bifurcation of critical points from a particular class of eigenfunctions is considered and in particular for m fixed the author shows that the pointwise limit, as $\varepsilon \to 0$, of some conditionally critical points are global minimizers of the limiting problem Λ_0.

Monotonicity, regularity and other symmetry properties of local minimizers of E_ε under the constraint $\int_\Omega u = m$ over cylindrical domains has been studied in [14] but existence is not considered.

In [2] the author gives some accounts on symmetry and monotonicity of local minimizers of some variational problems subjected to a integral constraint in cylinders and annuli with periodic phase separation and also zero Neumann boundary condition. Again existence is not the issue.

It follows from [21] that any local minimizer of E_ε in $\mathcal{M}$ over any smooth strictly convex domain must have a connected phase boundary. Herein we draw the same conclusion for just convex domain of cylindrical type. Let us recall some terminology which is going to be used:

1. A solution u to (1.2) is called a global minimizer of E_ε in $\mathcal{M}$.
2. A function $u \in \mathcal{M}$ is called a local minimizer of E_ε in $\mathcal{M}$ if there is a neighborhood $\mathcal{V}(u)$ of u in $H^1(\Omega)$ such that $E_\varepsilon(u) \le E_\varepsilon(v)$, $\forall\, v \in \mathcal{M} \cap \mathcal{V}$. If the inequality is strict then u is called a strict local minimizer.
3. Any $u \in \mathcal{M}$ satisfying $E'_\varepsilon(u_\varepsilon)\varphi = 0$ and $E''_\varepsilon(u)\varphi \ge 0$, $\forall\, \varphi \in \mathcal{M}$ is called a conditionally stable critical point of E_ε.

First we state our main result when the domain is a rectangle. As usual we denote the characteristic function of a set A by χ_A.

Theorem 1.1 *Let $\Omega = R = (-l, l) \times (-r, r)$ with $l \ge r > 0$. Then, for ε small, (1.2) has a family of solutions $\{u_\epsilon\}_{\varepsilon > 0}$ in $C^2(R) \cap C^0(\overline{R})$ (global minimizers of E_ε in $\mathcal{M}$) satisfying*

1. *u_ϵ depends just on the first variable,*
2. *u_ϵ is odd and increasing in $(-l, l)$ and*
3. *$\|u_\varepsilon - u_0\|_{L^1(-l,l)} \overset{\varepsilon \to 0}{\longrightarrow} 0$, where $u_0 = \chi_{(-l,0)} - \chi_{(0,-l)}$.*

(1.1.i) *If $l > r$, i.e., R is a rectangle, problem (1.2) has, for ε small, only two solutions: u_ε (given above) and $-u_\varepsilon$. There are only two other local (but not global) minimizers v_ε and $-v_\varepsilon$, where v_ε depends only on the second variable and is increasing and odd in $(-r, r)$. Moreover $\|v_\varepsilon - v_0\|_{L^1(-r,r)} \stackrel{\varepsilon\to 0}{\longrightarrow} 0$, where $v_0 = \chi_{(-r,0)} - \chi_{(0,r)}$.*

(1.1.ii) *If R is a square, then problem (1.2) has, for ε small, only four solutions $u_\varepsilon(x_1)$, $-u_\varepsilon(x_1)$, $v_\varepsilon(x_2)$, $-v_\varepsilon(x_2)$ where v_ε is obtained from u_ε by a rotation of $\pi/2$ with respect to x_1-axis. There is no other local minimizer.*

A similar result holds also for three-dimensional cylindrical domains. We label the points $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ as $x = (x_1, x')$ where $x' = (x_2, x_3)$.

Theorem 1.2 *Let $\Omega = (-l, l) \times D$ where $D \subset \mathbb{R}^2$ is any Lipschitz bounded domain. Then, for ε small, E_ε has a family of local minimizers $\{u_\epsilon\}_{\varepsilon>0}$ in $C^2(\Omega) \cap C^0(\overline{\Omega})$ such that*

1. *u_ϵ depends just on x_1,*
2. *u_ϵ is odd and increasing in $(-l, l)$ and*
3. *$\|u_\varepsilon - u_0\|_{L^1(-l,l)} \stackrel{\varepsilon\to 0}{\longrightarrow} 0$, where $u_0 = \chi_{(-l,0)} - \chi_{(0,l)}$.*

In particular if

(1.2.i) *$D = B_R = \{x \in \mathbb{R}^2 : |x| < R\}$ then (1.2) has only two solutions (global minimizers of E_ε in $\mathcal{M}$): $u_\epsilon(x_1)$ and $-u_\epsilon(x_1)$, as above. Moreover E_ε has no other local minimizer in $\mathcal{M}$.*

(1.2.ii) *$\Omega = (-l, l) \times (-r, r) \times (-q, q)$ with $l > r > q$ then (1.2) has only two solutions, $u_\epsilon(x_1)$ and $-u_\epsilon(x_1)$, and there are only four other local (but not global) minimizers $v_\varepsilon(x_2)$, $-v_\varepsilon(x_2)$, $w_\varepsilon(x_3)$ and $-w_\varepsilon(x_3)$; v_ε is increasing and odd in $(-r, r)$ whereas w_ε is increasing and odd in $(-q, q)$. Moreover $\|v_\varepsilon - v_0\|_{L^1(-r,r)} \stackrel{\varepsilon\to 0}{\longrightarrow} 0$ (v_0 as in Theorem 1.1) and $\|w_\varepsilon - w_0\|_{L^1(-r,r)} \stackrel{\varepsilon\to 0}{\longrightarrow} 0$ where $w_0 = \chi_{(-q,0)} - \chi_{(0,q)}$.*

(1.2.iii) *Ω is a cube ($l = r = q$) then problem (1.2) has only six solutions $u_\varepsilon(x_1)$, $-u_\varepsilon(x_1)$, $v_\varepsilon(x_2)$, $-v_\varepsilon(x_2)$, $w_\varepsilon(x_3)$ and $-w_\varepsilon(x_3)$ and there is no other local minimizer.*

The results above regarding existence are sharp and show that, for ε small, the geometry of global and local minimizers of the energy functional restricted to $\mathcal{M}$ is quite simple as they present no oscillation, a type of Sturm-Liouville property.

Regarding existence the main tool utilized in our approach is a theorem [11] based on the theory of $\Gamma-$convergence. It is well known that the family of functional E_ε $\Gamma-$converges, as $\varepsilon \to 0$, to the perimeter or area functional

$$E_0(v) = \begin{cases} \frac{8}{3}\mathrm{Per}_\Omega\{u = 1\}, & v \in BV(\Omega, \{\pm 1\}) \text{ and } \int_\Omega v = 0 \\ \infty, & \text{otherwise.} \end{cases}$$

Existence of local minimizers is obtained by proving that E_0 has an isolated $L^1(\Omega)$-local minimizer, say u_0, and then the aforementioned theorem assures that close (in $L^1(\Omega)$ norm) to u_0 there is a local minimizer of the original problem. The many cases of uniqueness of local and global minimizers are obtained using a property of Γ-convergence (preservation of global minimizers) and a uniqueness result of a one-dimensional monotone solutions of the corresponding elliptic Neumann boundary value problem.

In particular we prove that when Ω is a cylindrical domain the admissible function

$$u_0(x) \stackrel{\text{def}}{=} \chi_{\{x_1>0\}}(x) - \chi_{\{x_1<0\}}(x), \quad x \in \Omega$$

is indeed a local minimizer of E_0. This gives a positive answer to a question raised in [11], p. 80, for the special case of a rectangle.

Note that for competitors with fixed boundary a more general result has recently been established in [17], Theorem 1.2. The authors consider a smooth Riemannian manifold of dimension $n \leq 7$ and let S be a smooth oriented constant-mean-curvature hypersurface with boundary with positive second variation for fixed volume and boundary. Then the authors prove that S is uniquely homologically area minimizing for fixed volume among oriented hypersurfaces in a small L^1 neighborhood.

Finally it is worthwhile to comment on the procedure used to conclude that global and local minimizers depend only on one variable and is increasing (or decreasing) and odd in this variable. This procedure is not new and has been used in different scenarios and in a systematic way in [12] and [13] (although previously used in [14]) to study radial symmetry or symmetry with respect to a hyperplane [23] of global minimizers of a class of functionals under a class of constraints which, by the way, do not comprise ours.

Basically the technique consists of taking a global minimizer U and then defining a function V as the reflection, with respect to a convenient hyperplane, of U restricted to one of the two sides of the partitioned domain. For the class of functionals considered, U and V have the same energy and thus V is also a global minimizer and then the Unique Continuation Principle is used to obtain (in case of an infinite cylinder) that U is symmetric with respect to this hyperplane. If applied to our case such a procedure would produce even functions whereas we are looking for odd ones. Essentially this difficulty is avoided by taking a change of dependent variable $w = 1 - u^2$ and then using the information gathered while obtaining the minimizers via $\Gamma-$convergence. Another difficulty in our scenario is that, as opposed to the above cases, we cannot reflect the local minimizer around the plane of symmetry of the domain. This will become clear in Section 5.

2 Background and notation

In this section we give some notation and basic results of geometric measure theory which are going to be used. For more details on this matter the reader is referred to [6], [1] and [24], for instance.

We denote the $n-$dimensional Lebesgue measure by $\mathcal{L}^n$ and the $n-$dimensional Hausdorff measure by $\mathcal{H}^n$.

The density of a set E at a point x is defined by

$$D(E,x) = \lim_{r\to 0} \frac{\mathcal{L}^n(E\cap B_r(x))}{\mathcal{L}^n(B_r(x))},$$

with $B_r(x)$ denoting the n-dimensional ball centered at x with radius r. Let f be any real-valued function in $\mathbb{R}^n$ and let $x \in \mathbb{R}^n$. The approximate upper limit and lower limit of f at x are defined by

$$f_+(x) = \inf\{t : D(\{f > t\}, x) = 0\} \text{ and } f_-(x) = \sup\{t : D(\{f < t\}, x) = 0\},$$

respectively. If $f_+(x) = f_-(x) < \infty$ then the function f is said to be approximately continuous at x.

If μ is a Borel measure on a domain Ω with values in $[0, +\infty)$ its total variation is denoted by $|\mu|$ and the integral of a $|\mu|$-integrable function f will be denoted by $\int_R f d|\mu|$.

For $\Omega \subset \mathbb{R}^N$ a bounded set, the space $BV(\Omega)$ of functions of bounded variation in Ω is defined as the set of all functions $v \in L^1(\Omega)$ whose distributional gradient Dv is a Radon measure with bounded total variation in Ω, i.e.,

$$|Dv|(\Omega) = \sup \{\int_R v(x) \operatorname{div} \sigma(x)\, d\mathcal{L}^n \ < \infty,\ \sigma \in C_0^1(\Omega, \mathbb{R}^n),\ |\sigma| \le 1\}.$$

If $v \in W^{1,1}_{loc}$ then the total variation measure satisfies $|Dv| = \mathcal{L}^n \ \lfloor\ |\nabla v|$, i.e., $|Dv| = |\nabla v|\, d\mathcal{L}^n$, where ∇ denotes the usual gradient.

A set $E \subset \Omega$ is said to have finite perimeter in Ω if $D\chi_E$ (here χ_E stands for the characteristic function of E) is a vector valued Radon measure in Ω with finite total variation. The perimeter of E in Ω is defined by

$$\mathrm{Per}_\Omega(E) = |D\chi_E|(\Omega).$$

If E has finite perimeter in Ω and $\mathcal{L}^n(E\cap\Omega) < \infty$ then $\chi_E \in \mathrm{BV}(\Omega)$. Denoting by ν_j^E $(j = 1,\ldots,n)$ the derivative of the measure $D_j\chi_E$ with respect to $|D\chi_E|$ we have for every $x \in \Omega$

$$\nu_j^E(x) = \lim_{r\to 0} \frac{D_j\chi_E(B_r(x))}{|D\chi_E|(B_r(x))}\ (j = 1,\ldots,n).$$

as long as the limit exists.

The reduced boundary of E, denoted by $\partial^* E$, is the set of all points $x \in E$ such that the vector $\nu^E(x) = (\nu_1^E(x),\ldots,\nu_N^E(x))$ exists and $|\nu^E(x)| = 1$. If E has finite perimeter then $\partial^* E$ is a rectifiable set,

$$D\chi_E = \nu^E\, \mathcal{H}^{N-1} \lfloor \partial^* E$$

and consequently

$$|D\chi_E| = \mathcal{H}^{N-1} \lfloor \partial^* E \tag{2.1}$$

and

$$|D_j\chi_E| = |\nu_j^E|\ \mathcal{H}^{N-1}\lfloor\partial^* E\ (j = 1,\ldots,N). \tag{2.2}$$

If $E \subset \mathbb{R}^n$ then we say $x \in \partial_M E$, the measure theoretic boundary of E, if

$$\limsup_{r\to 0} \frac{\mathcal{L}^n(B(x,r)\cap E)}{r^n} > 0 \text{ and } \limsup_{r\to 0} \frac{\mathcal{L}^n(B(x,r)-E)}{r^n} > 0.$$

It holds that $\partial^* E \subset \partial_M E$ and $\mathcal{H}^{n-1}(\partial_M E \backslash \partial^* E) = 0$. For future references we recall the well-known isoperimetric inequality (see [9], for instance): if $E \subset \mathbb{R}^n$ ($n \geq 2$) is a set of finite perimeter then

$$\mathcal{H}^n(E)^{\frac{n-1}{n}} \leq c_1 \mathcal{H}^{n-1}(\partial^* E) \tag{2.3}$$

where the constant c_1 depends only on n.

We denote by $BV(\Omega;\{\pm 1\})$ the class of all $u \in BV(\Omega)$ which take values 1 and -1 only. If $u \in BV(\Omega;\{\pm 1\})$ then we call $\mu_1,\ldots,\mu_n$ the Radon (signed) measures defined by the partial derivatives $\frac{\partial u}{\partial x_j}$ ($j = 1\ldots,n$) and $\mu = Du$ (in the sense of distribution). Its total variation measures $|\mu_j|$ and $|\mu|$ satisfy for any measurable Borel set B

$$|\mu|(B) \geq (\Sigma_{j=1}^n (|\mu_j|(B))^2)^{1/2}. \tag{2.4}$$

Our approach relies strongly on the concept of $\Gamma-$ convergence and therefore we rather give the version which is going to be used.

Definition 2.1 The extended-real family of functionals $\{\Lambda_\varepsilon\}_{\varepsilon>0}$ defined in $L^1(\Omega)$, Γ-converges, as $\varepsilon \to 0$, to a real-extended functional Λ_0 in a point $u \in L^1(\Omega)$ if

1. $\forall\ \{u_\varepsilon\} \in L^1(\Omega) : \|u_\varepsilon - u\|_{L^1(\Omega)} \overset{\varepsilon\to 0}{\longrightarrow} 0 \Longrightarrow \liminf_{\varepsilon\to 0} \Lambda_\varepsilon(u_\varepsilon) \geq \Lambda_0(u)$.

2. $\exists\ \{w_\varepsilon\} : \|w_\varepsilon - u\|_{L^1(\Omega)} \overset{\varepsilon\to 0}{\longrightarrow} 0$ and $\Lambda_0(u) \geq \limsup_{\varepsilon\to 0} \Lambda_\varepsilon(w_\varepsilon)$.

The following result relates isolated local minimizers of the Γ-limit to local minimizers of the original problem.

Theorem 2.1 *[11] Suppose that a family of extended-real functionals $\{\Lambda_\varepsilon\}_{\varepsilon>0}$ defined in $L^1(\Omega)$ Γ-converges, as $\varepsilon \to 0$, to a extended functional Λ_0 and that the following hypotheses are satisfied*

(2.1.1) $\forall\ \{v_\varepsilon\}_{\varepsilon>0} : \Lambda_\varepsilon(v_\varepsilon) \leq$ *constant* $< \infty$ *is compact in* $L^1(\Omega)$ *and*

(2.1.2) *There exists an isolated L^1-local minimizer v_0 of Λ_0.*

Then there exists $\varepsilon_0 > 0$ and a family $\{v_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$ such that

(2.1.i) *v_ε is an L_1-local minimizer of Λ_ε and*

(2.1.ii) $\|v_\varepsilon - v_0\|_{L^1(\Omega)} \overset{\varepsilon\to 0}{\longrightarrow} 0$.

3 Existence of local minimizer via $\Gamma-$convergence

We want to work in $L^1(\Omega)$ in order to explore the compact imbedding $BV(\Omega) \hookrightarrow L^1(\Omega)$ and hence we define $E_\varepsilon : L^1(\Omega) \longmapsto \mathbb{R} \cup \{\infty\}$ by

$$E_\varepsilon(u) \stackrel{\text{def}}{=} \begin{cases} \int_\Omega [\varepsilon|\nabla u|^2 + \varepsilon^{-1}(1-u^2)^2]\, dx, & \text{if } u \in \mathcal{M} \\ \infty, & \text{otherwise.} \end{cases} \tag{3.1}$$

Lemma 3.1 *The family of functionals $\{E_\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ $\Gamma-$converges in $L^1(\Omega)$, as $\varepsilon \to 0$, to*

$$E_0(v) \stackrel{\text{def}}{=} \begin{cases} \frac{8}{3} Per_\Omega\{v=1\}, & v \in BV(\Omega, \{\pm 1\}) \text{ and } \int_\Omega v = 0 \\ \infty, & \text{otherwise.} \end{cases} \tag{3.2}$$

A proof can be found in [20], for instance; in this case the $\Gamma-$limit has long been known and it is just the perimeter or area functional given by (3.2).

Let us define the sets

$$\Omega_l^1 \stackrel{\text{def}}{=} \{x \in \Omega : x_1 < 0\} \text{ and } \Omega_r^1 \stackrel{\text{def}}{=} \{x \in \Omega : x_1 > 0\}$$

and the function

$$u_0(x) \stackrel{\text{def}}{=} \chi_{\Omega_r^1}(x) - \chi_{\Omega_l^1}(x), \quad x \in \Omega \tag{3.3}$$

Note that $\int_\Omega u_0 = 0$.

When

$$\Omega = R \stackrel{\text{def}}{=} (-l, l) \times (-r, r) \quad (l \geq r)$$

then, in addition to u_0, we define the function v_0 as

$$v_0(x) \stackrel{\text{def}}{=} \chi_{\Omega_r^2}(x) - \chi_{\Omega_l^2}(x), \quad x \in \Omega \tag{3.4}$$

where

$$\Omega_l^2 \stackrel{\text{def}}{=} \{x \in \Omega : x_2 < 0\} \text{ and } \Omega_r^2 \stackrel{\text{def}}{=} \{x \in \Omega : x_2 > 0\}$$

When

$$\Omega = Q \stackrel{\text{def}}{=} (-l, l) \times (-r, r) \times (-q, q)\, (l \geq r \geq q)$$

then, in addition to u_0 and v_0, we define the function w_0 as

$$w_0(x) \stackrel{\text{def}}{=} \chi_{\Omega_r^3}(x) - \chi_{\Omega_l^3}(x), \quad x \in \Omega \tag{3.5}$$

where

$$\Omega_l^3 \stackrel{\text{def}}{=} \{x \in \Omega : x_3 < 0\} \text{ and } \Omega_r^3 \stackrel{\text{def}}{=} \{x \in \Omega : x_3 > 0\}.$$

Theorem 3.1 *Let $\Omega = (-l, l) \times D$ where $D \subset \mathbb{R}^2$ is any Lipschitz bounded domain. There exists $\varepsilon_0 > 0$ and a family $\{u_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$ in $C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfying*

(3.1.i) *u_ε is a local minimizer of E_ε in $\mathcal{M}$ and*

(3.1.ii) $\|u_\varepsilon - u_0\|_{L^1(\Omega)} \stackrel{\varepsilon\to 0}{\longrightarrow} 0$, *with* u_0 *given by (3.3).*

When $\Omega = R$ *(rectangle) then in addition to* $\{u_\varepsilon\}$ *we obtain another family* $\{v_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$ *of local minimizers satisfying* $\|v_\varepsilon - v_0\|_{L^1(\Omega)} \stackrel{\varepsilon\to 0}{\longrightarrow} 0$, *with* v_0 *given by (3.4).*

When $\Omega = Q$ *(parallelepiped) then, in addition to* $\{u_\varepsilon\}$ *and* $\{v_\varepsilon\}$, *we obtain another family* $\{w_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$ *of local minimizers satisfying* $\|w_\varepsilon - w_0\|_{L^1(\Omega)} \stackrel{\varepsilon\to 0}{\longrightarrow} 0$, *with* w_0 *given by (3.5).*

This theorem follows straight from Theorem 2.1 once the hypotheses (2.1.1) and (2.1.2) are verified. The latter hypothesis, which is by far the most difficult to prove will be the subject of the next section. Therefore the proof of this theorem is postponed until we verify (2.1.2) for the cases.

4 Proof Theorem 3.1

Most of this section is devoted to proving that u_0, given by (3.3), is an isolated local minimizer of E_0 in $\mathcal{M}$. But before doing that an important remark is worth being made. For $n = 2$ this result could have been obtained by generalizing [11], Proposition 2.4, where the result pursued herein is established for $n = 2$ in the unconstrained case for non-convex domains which have a narrow region. However the proof provided in [11] is not suited for generalization to dimension $n \geq 3$. We will explain where and why it fails for $n \geq 3$.

In order to prove Proposition 2.4 ([11]) the authors split the admissible functions in four classes. In one of these classes the admissible functions $\mathrm{BV}(R; \pm 1)$ in the rectangle $R = (-l, l) \times (-r, r)$, for some $a \in (l/2, l)$ are either

1. not constant along the interval $\{(x, a),\ -l < x < l\}$ or
2. not constant along the interval $\{(x, -a),\ -l < x < l\}$.

In either case the admissible function v changes sign at least once in at least one of these intervals enabling the authors to conclude (see [11], first inequality in p. 77)

$$\int_{-l}^{l} \left(\left| \frac{\partial v}{\partial x_1} \right| (x_1, a) + \left| \frac{\partial v}{\partial x_1} \right| (x_1, -a) \right) \geq 2. \tag{4.1}$$

Note that in this case the measure of integration, in view of (2.2), can be thought of as the Hausdorff counting measure $\mathcal{H}^0$ of points of discontinuities of $v(\cdot, a)$.

When $n = 3$, under the same hypotheses, (4.1) depends on the $\mathrm{Per}_D\{v = 1\}$. In such case the lower positive bound given by (4.1) would no longer be available, i.e., when taking the limit of (4.1) over this class of functions the limit inferior could be zero and the procedure utilized in [11] no longer works. The proof rendered here for the three-dimensional domain $(-l, l) \times D$ is of geometric nature and the basic idea is to find $t \in (l/2, l)$ such that

$$\mathcal{H}^2((D \times \{t\}) \cap \{v = -u_0\}) \leq \mathrm{Per}_{D\times[t,\, l]}\{v = -u_0\}$$

holds for any admissible function v where u_0 is given by (3.3). This is accomplished by a series of technical results which to some extent is the price paid for working in the topology of L^1. This topology, by its turn, is chosen in order check hypothesis (2.1.1) in Theorem 2.1 (in order to prove Theorem 3.1) by exploring the compact imbedding $BV(\Omega) \hookrightarrow L^1(\Omega)$ (see [16] or [20], for instance).

We now state the main result of this section.

Lemma 4.1 *Let $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ be a cylindrical domain $\Omega = (-l, l) \times D$. Then the function u_0, given by (3.3), is a local isolated minimizer of E_0 given by (3.2), i.e., there is $\rho > 0$ such that for any $v \in BV(\Omega, \{\pm 1\})$, satisfying $0 < \|v - u_0\|_{L^1(\Omega)} < \rho$ and $\int_\Omega v = 0$, we have $E_0(u_0) < E_0(v)$.*

As preparation for the proof of this lemma we state and prove some technical results.

Lemma 4.2 *Let $D \subset \mathbb{R}^m$ $(m = 1, 2)$ a bounded Lipschitz domain with a finite number of vertices if $m = 2$ and an interval if $m = 1$. There exists $\theta > 0$ such that for any $A \subset D$ of finite perimeter satisfying $\mathcal{H}^m(A) < \frac{1}{2}\mathcal{H}^m(D)$ it holds that*

$$0 < \theta \leq \frac{\mathcal{H}^{m-1}(\partial^* A \backslash \partial^* D)}{\mathcal{H}^{m-1}(\partial^* A)}. \tag{4.2}$$

Proof.
We may suppose that A is connected and then the general case follows at once.

When $m = 1$, $\theta = 1/2$ since $\mathcal{H}^0(\partial^* A) = 2$ and $\mathcal{H}^0(\partial^* A \backslash \partial^* D) \geq 1$. Thus suppose $m = 2$ and let N_D denote the finite set of points where $\partial^* D$ is not C^1.

We abbreviate connected component as c.c., for short, and set

$$\text{diag}(N_D) = \{(P, P) : P \in N_D\}.$$

Then we define a function $\gamma : (\partial^* D \times \partial^* D) \backslash \text{diag}(N_D) \to \mathbb{R}^+$ by

$$\gamma(P, Q) = \begin{cases} \min \left\{ \mathcal{H}^1(C) : C \text{ is c.c. of } \partial^* D \backslash \{P, Q\} \right\} & \text{if } P \neq Q, \\ 0, & \text{se } P = Q, \end{cases}$$

Note that $\partial^* D \backslash \{P, Q\}$ consists of exactly two connected components when $P \neq Q$ given that now $D \subset \mathbb{R}^2$. Hence γ is well defined and continuous.

Next we define a function $\sigma : (\partial^* D \times \partial^* D) \backslash \text{diag}\{N_D\} \to \mathbb{R}^+$ by

$$\sigma(P, Q) \stackrel{\text{def}}{=} \begin{cases} \frac{|P-Q|}{\gamma(P, Q) + |P-Q|} & \text{if } P \neq Q, \\ 1/2, & \text{if } P = Q, \end{cases}$$

¿From the fact that D is a Lipschitz domain one can easily check that

$$\liminf_{(P,Q) \to \text{diag}(N_D)} \sigma(P, Q) > 0 \tag{4.3}$$

On the account that $\partial^* D$ is compact and σ is continuous there is a constant $m_\sigma > 0$ satisfying

$$\sigma \geq m_\sigma > 0 \ \text{ on } \ (\partial^* D \times \partial^* D)\backslash \mathrm{diag}\{N_D\}. \tag{4.4}$$

With c_1 as in (2.3) (the isoperimetric inequality) and m_σ as in (4.4) we define

$$\theta \stackrel{\text{def}}{=} \min \left\{ m_\sigma \, , \, \frac{m_\sigma \sqrt{2}\, \mathcal{H}^2(D)^{1/2}}{m_\sigma \sqrt{2}\, \mathcal{H}^2(D)^{1/2} + 2c_1 \mathrm{Per}(D)} \right\}.$$

Hence $0 < \theta < 1$ and we claim that θ satisfies (4.2). Indeed given $A \subset D$ satisfying our hypotheses then three cases may occur:

(i) $\mathcal{H}^1(\partial^* A \cap \partial^* D) = 0$ or

(ii) $0 < \mathcal{H}^1(\partial^* A \cap \partial^* D) < \mathrm{Per}(D)$.

(iii) $\mathcal{H}^1(\partial^* A \cap \partial^* D) = \mathrm{Per}(D)$.

¿From now on, in order to simplify the notation, we replace the $\mathcal{H}^1$ measure of a set in Ω with its perimeter in Ω.

If (i) holds then $\mathrm{Per}_D(A) = \mathrm{Per}(A)$ and as such $\mathrm{Per}_D(A)/\mathrm{Per}(A) = 1 > \theta$.

We will say that P and Q, with $P \neq Q$ and $P, Q \in \overline{\partial^* A \cap \partial^* D}$, are extremal points of $\partial^* A \cap \partial^* D$ if $\partial^* A \cap \partial^* D \backslash \{P, Q\}$ is contained in one of the two c.c. of $\partial^* D \backslash \{P, Q\}$.

If (ii) holds then two cases may occur.

(ii-a) $\exists$ P and Q, extremal points of $\partial^* A \cap \partial^* D$ so that $\partial^* A \cap \partial^* D$ is contained in the smaller c.c. (in the $\mathcal{H}^1$ metric) of $\partial^* D \backslash \{P, Q\}$.

In this case using that $|P - Q| \leq \mathrm{Per}_D(A)$ and if a, b and c are strictly positive real numbers with $a \leq b$, then $\frac{a}{c+a} \leq \frac{b}{b+c}$, we have

$$\begin{aligned} \frac{\mathrm{Per}_D(A)}{\mathrm{Per}(A)} &= \frac{\mathrm{Per}_D(A)}{\mathrm{Per}_D(A) + \mathcal{H}^1(\partial^* A \cap \partial^* D)} \geq \frac{\mathrm{Per}_D(A)}{\mathrm{Per}_D(A) + \gamma(x,y)} \\ &\geq \frac{|x-y|}{|x-y| + \gamma(x,y)} = \sigma(x,y) \geq m_\sigma \geq \theta. \end{aligned}$$

(ii-b) For any two points P and Q, extremals of $\partial^* A \cap \partial^* D$, the set $\partial^* A \cap \partial^* D$ is contained in the largest c.c. of $\partial^* D \backslash \{P, Q\}$.

In this case each c.c. of $D \backslash A$, say $\mathcal{C}$, satisfies (ii-a) with A replaced with $\mathcal{C}$ and therefore

$$\frac{\mathrm{Per}_D(D \backslash A)}{\mathrm{Per}(D \backslash A)} \geq m_\sigma. \tag{4.5}$$

Invoking the hypothesis $\mathcal{H}^2(A) < \frac{1}{2}\mathcal{H}^2(D)$ (which has not been used yet) we obtain

$$\mathcal{H}^2(D \backslash A) > \frac{1}{2}\mathcal{H}^2(D). \tag{4.6}$$

Now (2.3), (4.5) and (4.6) yield

$$\mathrm{Per}_D(A) = \mathrm{Per}_D(D\backslash A) \geq m_\sigma \mathrm{Per}(D\backslash A) \geq m_\sigma \frac{1}{c_1} \mathcal{H}^2(D\backslash A)^{1/2} >$$

$$m_\sigma \frac{1}{c_1} \frac{1}{\sqrt{2}} \mathcal{H}^2(D)^{1/2}.$$

This implies

$$\begin{aligned} \frac{\mathrm{Per}_D(A)}{\mathrm{Per}(A)} &\geq \frac{\mathrm{Per}_D(A)}{\mathrm{Per}_D(A)+\mathrm{Per}(D)} > \frac{m_\sigma \frac{1}{c_1}\frac{1}{\sqrt{2}}\mathcal{H}^2(D)^{1/2}}{m_\sigma \frac{1}{c_1}\frac{1}{\sqrt{2}}\mathcal{H}^2(D)^{1/2}+Per(D)} \\ &= \frac{\sqrt{2} m_\sigma \mathcal{H}^2(D)^{1/2}}{\sqrt{2} m_\sigma \mathcal{H}^2(D)^{1/2}+2c_1 \mathrm{Per}(D)} \geq \theta. \end{aligned}$$

In case (iii) each c.c. of $D\backslash A$ fits in (i) and as such the very same argument used in (ii-b) establishes the proof when $m = 2$. ■

Remark 4.1 As it stands the difficulty we have found to generalize our results to dimension $n > 3$ is to prove Lemma 4.2 for higher dimensions.

Lemma 4.3 *Let $(-l, l) \times D = \Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ as in Section 3, c_1 the constant from the isoperimetric inequality (2.3) and θ as in Lemma 3. Then there exists a real number $\xi \in (\frac{l}{2}, l)$ and a continuous function $\eta : [\frac{l}{2}, \xi] \to [0, \infty)$ satisfying*

$$\begin{cases} \eta(t) = \frac{\theta}{2c_1} \int_t^\xi \eta(s)^{\frac{n-2}{n-1}} ds \\ \eta(\xi) = 0 \\ \eta(\frac{l}{2}) \leq \frac{1}{2}\mathcal{H}^{n-1}(D) \\ \eta > 0 \ \ on \ [\frac{l}{2}, \xi). \end{cases} \tag{4.7}$$

Proof.
One can easily check that

$$\eta(t) = \left(\frac{\theta}{(n-1)2c_1}\right)^{n-1} (\xi - t)^{n-1} \ (n = 2, 3)$$

is a solution to (4.7) where the point $\xi \in (l/2, l)$ should be chosen, if necessary, close to $l/2$ in order to satisfies the condition $\eta(\frac{l}{2}) \leq \frac{1}{2}\mathcal{H}^{n-1}(D)$. ■

Lemma 4.4 *Let Ω, c_1 and θ be as in Lemma 4.3. There exists $\rho > 0$ such that for any $\mathcal{L}$-measurable function $\psi : [\frac{l}{2}, l] \to (0, \mathcal{H}^{n-1}(D)]$ $(n = 2, 3)$ satisfying $\int_{\frac{l}{2}}^l \psi(s) d\mathcal{H}^1 < \rho$ there exists $t_0 \in (\frac{l}{2}, l)\backslash\ I$, where $I = \{t \in [\frac{l}{2}, l] : \psi(t) \geq \frac{1}{2}\mathcal{H}^{n-1}(D)\}$, such that*

$$\psi(t_0) < \frac{\theta}{2c_1} \int_{[t_0,l]\backslash I} \psi(s)^{\frac{n-2}{n-1}} ds.$$

Remark 4.2 Before rendering the proof let us give a geometric insight of the meaning of the function ψ. Given any function $v \in BV(\Omega, \pm 1)$ in a $L^1(\Omega)$-neighborhood of u_0 (see statement of Lemma 4.1) consider the set $A = \{v = -u_0\}$, say. Then ψ will be taken as $\psi(t) = \mathcal{H}^{(n-1)}(A \cap (D \times \{t\}))$. In general ψ will not be a continuous function and, in addition, its image must be in $[0, \mathcal{H}^{n-1}(D)]$. However in the proof of the main result in this section we will see that its domain must be restricted to $\{t : \psi(t) < \frac{1}{2}\mathcal{H}^{n-1}(D)\}$. That is why the set I was introduced.

Proof.
Recalling that $\Omega = (-l, l) \times D$ and taking η and ξ as in Lemma 4.3 we define

$$\rho = \min\{\int_{l/2}^{\xi} \eta(t)dt, \frac{(l-\xi)}{2}\mathcal{H}^{n-1}(D)\}, \ n = 2, 3. \tag{4.8}$$

We anticipate that ρ will be the radius of the neighborhood of u_0 in $L^1(\Omega)$ when proving that u_0 is an isolated local minimizer of E_0 in $\mathcal{M}$ as in Lemma 4.1.

Given ψ and I as in our hypotheses we define

$$\mathcal{O} = \left\{ t \in \left[\frac{l}{2}, l\right] \setminus I : \exists s \in \left[\frac{l}{2}, t\right) \setminus I \text{ with } \mathcal{L}([0, s] \setminus I) = \mathcal{L}([0, t] \setminus I) \right\}$$

$\tilde{I} = I \cup \mathcal{O}$, $\quad \vartheta = \mathcal{L}([0, l] \setminus I)$ and $\zeta : [\frac{l}{2}, l] \setminus \tilde{I} \to [\frac{l}{2}, \vartheta]$ given by

$$\zeta(t) = \mathcal{L}([0, t] \setminus \tilde{I}).$$

Note that $\mathcal{L}(\mathcal{O}) = 0$ and that $\mathcal{O}$ was introduced just to make ζ injective. Hence ζ is a measurable bijective function and we can define a $\mathcal{L}$-measurable function $g : [\frac{l}{2}, \vartheta] \to (0, \frac{1}{2}\mathcal{H}^{n-1}(D))$ by

$$g = \psi \circ \zeta^{-1}$$

which, from the hypothesis of ψ, satisfies

$$\int_{l/2}^{\vartheta} g(s)ds < \int_{l/2}^{\xi} \eta(t)dt. \tag{4.9}$$

Claim: $\exists\, t_1 \in (\frac{l}{2}, \vartheta)$ such that $g(t_1) < \frac{\theta}{2c_1} \int_{t_1}^{\vartheta} g(s)^{\frac{n-2}{n-1}} ds$.

Let us first prove that $\vartheta > \xi$; it suffices to show that $\mathcal{L}(I) < l - \xi$. Arguing by contradiction we have

$$\begin{aligned}
\rho &\leq \frac{(l-\xi)}{2}\mathcal{H}^{n-1}(D) \text{ (from the definition of } \rho) \\
&\leq \frac{1}{2}\mathcal{H}^{n-1}(D)\mathcal{L}(I) \text{ (contradiction hypothesis)} \\
&< \int_I \psi(s)ds \text{ (from definition of } I) \\
&< \int_{l/2}^{l} \psi(s)ds \text{ (since } I \subset [\frac{l}{2}, l]\ ) \\
&\leq \rho \text{ (from definition of } \rho).
\end{aligned}$$

Therefore $\vartheta > \xi$ and this make it possible to define the set

$$\Delta \stackrel{\text{def}}{=} \{t \in [\frac{l}{2}, \xi) : g \text{ is approximately continuous at } t \text{ and } g(t) \leq \eta(t)\}.$$

Note that g is approximately continuous a.e. in $[\frac{l}{2}, \xi]$ (cf. [7], p.47). Also $\Delta \neq \emptyset$, for if $\overline{\xi} = l/2$ then by definition of Δ we would have $g > \eta$ a.e. in $[l/2, \xi)$ and therefore $\int_{l/2}^{\vartheta} g > \int_{l/2}^{\xi} g > \int_{l/2}^{\xi} \eta$, thus contradicting (4.9). Therefore $l/2 < \overline{\xi}$.

There are two cases to be analyzed.

i) $\sup \Delta = \xi$.

In this case $\inf\{g(t) : t \in [\frac{l}{2}, \xi] : \; g \text{ aprox. continuous at } t\} = 0$. On the other hand since $g^{\frac{n-2}{n-1}} > 0$ we have $\int_{\xi}^{\vartheta} g^{\frac{n-2}{n-1}} ds > 0$ and this implies the existence of $t_1 \in [\frac{l}{2}, \xi]$ such that g is aprox. continuous at t_1 and

$$g(t_1) \leq \frac{\theta}{2c_1} \int_{\xi}^{\vartheta} g(s)^{\frac{n-2}{n-1}} ds \leq \frac{\theta}{2c_1} \int_{t_1}^{\vartheta} g(s)^{\frac{n-2}{n-1}} ds.$$

ii) $\sup \Delta = \overline{\xi} \neq \xi$.

In this case, since $\overline{\xi} < \xi$, it follows from the definition of Δ that $\int_{\overline{\xi}}^{\xi} g(t)^{\frac{n-2}{n-1}} dt > \int_{\overline{\xi}}^{\xi} \eta(t)^{\frac{n-2}{n-1}} dt$. Also $\vartheta > \xi$, $g > 0$ and g is approximate continuous a.e. in $[l/2, \xi)$ where $l/2 < \overline{\xi} < \xi$.

Together all these facts allow us to take $t_1 \in (\frac{l}{2}, \overline{\xi}]$ such that g is approximate continuous at t_1, $g(t_1) \leq \eta(t_1)$ and

$$\int_{t_1}^{\overline{\xi}} \eta(t)^{\frac{n-2}{n-1}} dt < \int_{\xi}^{\vartheta} g(t)^{\frac{n-2}{n-1}} dt.$$

If g is continuous then we may choose $t_1 = \tilde{\xi}$. ¿From the above properties of t_1, Lemma 4.3 and the three aforementioned properties we have

$$\begin{aligned}
g(t_1) &\leq \eta(t_1) = \frac{\theta}{2c_1} \int_{t_1}^{\xi} \eta(t)^{\frac{n-2}{n-1}} dt \\
&= \frac{\theta}{2c_1} \int_{t_1}^{\overline{\xi}} \eta(t)^{\frac{n-2}{n-1}} dt + \frac{\theta}{2c_1} \int_{\overline{\xi}}^{\xi} \eta(t)^{\frac{n-2}{n-1}} (t) dt \\
&< \frac{\theta}{2c_1} \int_{\xi}^{\vartheta} g(t)^{\frac{n-2}{n-1}} dt + \frac{\theta}{2c_1} \int_{\overline{\xi}}^{\xi} g(t)^{\frac{n-2}{n-1}} dt \\
&< \frac{\theta}{2c_1} \int_{t_1}^{\vartheta} g(t)^{\frac{n-2}{n-1}} dt,
\end{aligned}$$

establishing in this way our claim.

Now in order to complete the proof of the lemma we take $t_0 = \zeta^{-1}(t_1)$. ■

The above sequence of elementary lemmas is utilized to prove the next result which in turn will play an essential role in the proof of Lemma 4.1.

Lemma 4.5 *Let $\Omega = (-l, l) \times D \subset \mathbb{R}^n$ $(n = 2, 3)$ as above, c_1 as in (2.3), θ as in Lemma 4.2 and ρ given by (4.8).*

(4.5.i) *Given $A^+ \subset \overline{D \times \left[\frac{l}{2}, l\right]}$, with $\mathcal{H}^n(A^+) < \rho$, there exists $t_0 \in \left(\frac{l}{2}, l\right)$ such that*

$$\mathcal{H}^{(n-1)}(A^+ \cap (D \times \{t_0\})) \le \frac{1}{2}\mathcal{H}^{(n-1)}(\partial^*(A^+ \cap (D \times [t_0, l]))\backslash\partial^*(D \times [t_0, l])).$$

(4.5.ii) *Given $A^- \subset \overline{D \times \left[-l, \frac{-l}{2}\right]}$, with $\mathcal{H}^n(A^-) < \rho$, there exists $t_0' \in \left(-l, -\frac{l}{2}\right)$ such that*

$$\mathcal{H}^{(n-1)}(A^- \cap (D \times \{t_0'\})) \le \frac{1}{2}\mathcal{H}^{(n-1)}(\partial^*(A^- \cap (D \times [-l, t_0']))\backslash\partial^*(D \times [-l, t_0'])).$$

Proof.
(4.5.i) If $\mathcal{H}^{(n-1)}(A^+ \cap (D \times \{t\})) = 0$ for some $t \in \left(\frac{l}{2}, l\right)$, then we may choose $t_0 = t$. Otherwise let $\psi : \left[\frac{l}{2}, l\right] \to (0, \mathcal{H}^{n-1}(D)]$ defined by

$$\psi(t) \stackrel{\text{def}}{=} \mathcal{H}^{(n-1)}(A^+ \cap (D \times \{t\})).$$

Then due to our hypotheses, ψ satisfies the hypotheses of Lemma 4.4 and there follows the existence of $t_0 \in \left(\frac{l}{2}, l\right) \backslash I$ (recall that $I = \{t \in [\frac{l}{2}, l] : \psi(t) \ge \frac{1}{2}\mathcal{H}^{n-1}(D)\}$) satisfying

$$\begin{aligned}
& \mathcal{H}^{(n-1)}(A^+ \cap (D \times \{t_0\})) \\
\le\ & \frac{\theta}{2c_1} \int_{[t_0, l]\backslash I} (\mathcal{H}^{(n-1)}(A^+ \cap (D \times \{s\})))^{\frac{n-2}{n-1}} d\mathcal{H}^1 \\
& \text{(using isoperimetric inequality (2.3))} \\
\le\ & \frac{\theta}{2} \int_{[t_0, l]\backslash I} \mathcal{H}^{(n-2)}(\partial^*(A^+ \cap (D \times \{s\}))) d\mathcal{H}^1 \\
\le\ & \frac{1}{2} \int_{[t_0, l]\backslash I} \text{Per}_{D \times \{s\}}(A^+ \cap (D \times \{s\}))) d\mathcal{H}^1 \text{ (from Lemma 4.2)} \\
\le\ & \frac{1}{2} \int_{t_0}^{l} \text{Per}_{D \times \{s\}}(A^+ \cap (D \times \{s\}))) d\mathcal{H}^1 \\
\le\ & \frac{1}{2}\mathcal{H}^{(n-1)}(\partial^*(A^+ \cap (D \times [t_0, l]))\backslash\partial^*(D \times [t_0, l])).
\end{aligned}$$

This establishes (4.5.i) and the proof of the other case is similar. ■

Finally, as a consequence of the above sequence of elementary results, we are in position to prove the main result of this section.

Proof of Lemma 4.1 The coarea formula yields

$$E_\varepsilon(u_0) = \frac{8}{3}\,\mathcal{H}^{n-1}(D). \tag{4.10}$$

Next we take ρ as in (4.8) and $v \in BV(\Omega, \{-1, 1\})$ satisfying $0 < \|v - u_0\|_{L^1(\Omega)} < \rho$ and $\int_\Omega v \; = 0$.

Note that by fixing $t \in (-l, l)$ the trace of $v(t, \cdot)$ is well defined in each cross-section $D_t = \{t\} \times D$. Set

1. $A_v^+ \overset{\text{def}}{=} \{x \in \left(\frac{l}{2}, l\right) \times D : v(x) = -1\}$

2. $A_v^- \overset{\text{def}}{=} \{x \in \left(-l, -\frac{l}{2}\right) \times D : v(x) = 1\}$.

Since $A_v^+ \subset \{v = -u_0\}$ and $A_v^- \subset \{v = -u_0\}$ we have

$$\begin{aligned} \rho \quad &> \|v - u_0\|_{\mathcal{L}^1(\Omega)} > \tfrac{1}{2}\|v - u_0\|_{\mathcal{L}^1(\Omega)} = \tfrac{1}{2}\textstyle\int_{\{v \neq u_0\}} |v - u_0| d\mathcal{L}^n \\ &= \tfrac{1}{2}\textstyle\int_{\{v = -u_0\}} 2 d\mathcal{L}^n = \mathcal{H}^n(\{v = -u_0\}) \\ &\geq \mathcal{H}^n(A_v^+). \end{aligned} \tag{4.11}$$

Thus $\mathcal{H}^n(A_v^+) < \rho$ and similarly $\mathcal{H}^n(A_v^-) < \rho$.

Therefore A_v^+ e A_v^- satisfy the hypotheses of (4.5.i) and in (4.5.ii), respectively, with A^+ and A^- replaced with A_v^+ and A_v^-, respectively. In this way by Lemma 4.5 there are $t_0 \in \left(\frac{l}{2}, l\right)$ and $t_0' \in \left(-l, \frac{-l}{2}\right)$ such that

$$\mathcal{H}^{n-1}(A_v^+ \cap (\{t_0\} \times D)) \leq \frac{1}{2}\mathcal{H}^{n-1}(\partial^*(A_v^+ \cap ([t_0, l] \times D))\backslash\partial^*([t_0, l] \times D))$$

and

$$\mathcal{H}^{n-1}(A_v^- \cap (\{t_0'\} \times D)) \leq \frac{1}{2}\mathcal{H}^{n-1}(\partial^*(A_v^- \cap ([-l, t_0'] \times D)\backslash\partial^*([-l, t_0'] \times D)).$$

By fixing t_0 e t_0' and, for the sake of simplification in notation, setting

$$\begin{aligned} D_v &= \{x' \in D : (t_0, x') \in A_v^+\} \cup \{x' \in D : (t_0', x') \in A_v^-\}, \\ A_v^{[t_0, l]} &= \partial^*(A_v^+ \cap ([t_0, l] \times D))\backslash\partial^*([t_0, l] \times D) \quad \text{and} \\ A_v^{[-l, t_0']} &= \partial^*(A_v^- \cap ([-l, t_0'] \times D)\backslash\partial^*([-l, t_0'] \times D) \end{aligned}$$

the two inequalities above yield

$$\mathcal{H}^{n-1}(D_v) \leq \frac{1}{2}\mathcal{H}^{n-1}(A_v^{[t_0,l]}) + \mathcal{H}^{n-1}(A_v^{[-l,t_0']}). \tag{4.12}$$

Our next goal is to use these inequalities to prove

$$\mathcal{H}^{n-1}(\partial^*\{v = 1\} \cap \Omega) > \mathcal{H}^{n-1}(D). \tag{4.13}$$

By definition of D_v

$$\begin{aligned} &\mathcal{H}^{n-1}(D_v) \\ \leq\ & \frac{1}{2}\left[\mathcal{H}^{n-1}(A_v^{[t_0,l]}) + \mathcal{H}^{n-1}(A_v^{[-l,t_0']})\right] \quad \text{(from 4.12)} \\ =\ & \frac{1}{2}\mathcal{H}^{n-1}(\partial^*\{v = -u_0\} \cap (((-l, t_0') \cup (t_0, l)) \times D)) \\ =\ & \frac{1}{2}\mathcal{H}^{n-1}(\partial^*\{v = 1\} \cap (((-l, t_0') \cup (t_0, l)) \times D)). \end{aligned} \tag{4.14}$$

On the other hand note that by definition of D_v,

$$v(t_0, x') = u_0(t_0, x') = 1 \text{ and } v(t_0', x') = u_0(t_0', x') = -1, \quad \text{a.e. in} D\setminus D_v. \quad (4.15)$$

Denoting the interior of a set A by $\text{int}(A)$, we obtain

$$\begin{aligned} &\mathcal{H}^{n-1}(\partial^*\{v=1\} \cap ((t_0', t_0) \times \text{int}(D\setminus D_v)) \\ &\quad \geq \mathcal{H}^{n-1}(D\setminus D_v) \end{aligned} \quad (4.16)$$

If (4.16) is a strict inequality then from (4.14) we would obtain

$$\begin{aligned} \mathcal{H}^{n-1}(\partial^*\{v=1\} \cap \Omega) &\geq \mathcal{H}^{n-1}(\partial^*\{v=\} \cap ((t_0', t_0) \times \text{int}(D\setminus D_v)) \\ &+ \mathcal{H}^{n-1}(\partial^*\{v=1\} \cap (((-l, t_0') \cup (t_0, l)) \times D)) \\ &> \mathcal{H}^{n-1}(D\setminus D_v) + \mathcal{H}^{n-1}(D_v) \\ &= \mathcal{H}^{n-1}(D), \end{aligned} \quad (4.17)$$

thus establishing (4.13).

If (4.16) is an equality then necessarily $\partial^*\{v=1\} \cap ((t_0', t_0) \times \text{int}(D\setminus D_v)$ and $D\setminus D_v$ are parallel cross-sections of Ω and therefore two cases have to be analyzed

(4.16.i) $\mathcal{H}^{n-1}(D_v) > 0$ and

(4.16.ii) $\mathcal{H}^{n-1}(D_v) = 0$

If the former one holds then (4.14) yields

$$\begin{aligned} &\mathcal{H}^{n-1}(D_v) \leq \\ &\frac{1}{2}\mathcal{H}^{n-1}(\partial^*\{v=1\} \cap (((-l, t_0') \cup (t_0, l)) \times D)) < \\ &\mathcal{H}^{n-1}(\partial^*\{v=1\} \cap (((-l, t_0') \cup (t_0, l)) \times D)) \end{aligned} \quad (4.18)$$

and hence from (4.16) and (4.18) we obtain

$$\begin{aligned} \mathcal{H}^{n-1}(\partial^*\{v=1\} \cap \Omega) &\geq \mathcal{H}^{n-1}(\partial^*\{v=1\} \cap ((t_0', t_0) \times \text{int}(D\setminus D_v)) \\ &+ \mathcal{H}^{n-1}(\partial^*\{v=1\} \cap (((-l, t_0') \cup (t_0, l)) \times D)) \\ &> \mathcal{H}^{n-1}(D\setminus D_v) + \mathcal{H}^{n-1}(D_v) \\ &= \mathcal{H}^{n-1}(D) \end{aligned} \quad (4.19)$$

thus proving (4.13).

Now if (4.16.ii) holds then taking into account that (4.16) is an equality, $\partial^*\{v = 1\} \cap ((t_0', t_0) \times \text{int}(D\setminus D_v)$ and $D\setminus D_v$ are parallel cross-sections of Ω, $\|v - u_0\|_{L^1(\Omega)} > 0$ and $\int_\Omega v\, dx = 0$, we necessarily have

$$\mathcal{H}^{n-1}(\partial^*\{v = -u_0\} \cap (((-l, t_0') \cup (t_0, l)) \times D)) > 0.$$

Hence

$$
\begin{aligned}
\mathcal{H}^{n-1}(\partial^*\{v=1\}\cap\Omega) &= \mathcal{H}^{n-1}(\partial^*\{v=1\}\cap(((-l,t_0')\cup(t_0,l))\times D)) \\
&+ \mathcal{H}^{n-1}(\partial^*\{v=1\}\cap((t_0',t_0)\times D)) \\
&= \mathcal{H}^{n-1}(\partial^*\{v=-u_0\}\cap(((-l,t_0')\cup(t_0,l))\times D)) \\
&+ \mathcal{H}^{n-1}(\partial^*\{v=1\}\cap((t_0',t_0)\times D)) \\
&> \mathcal{H}^{n-1}(\partial^*\{v=1\}\cap((t_0',t_0)\times D)) \\
&= \mathcal{H}^{n-1}(D), \qquad (4.20)
\end{aligned}
$$

thus establishing (4.13) in this case too.

Altogether (4.20), (4.19) and (4.17) imply (4.13). Finally from (4.13) we obtain

$$
\begin{aligned}
E_0(v) &= \tfrac{8}{3}\int_{\Omega\cap\partial^*\{v=1\}} d\mathcal{H}^{n-1} = \tfrac{8}{3}\mathcal{H}^{n-1}(\Omega\cap\partial^*\{v=1\}) > \\
&\qquad \tfrac{8}{3}\mathcal{H}^{n-1}(D) = E_0(u_0),
\end{aligned}
$$

and this completes the proof of Lemma 4.1. ■

Proof of Theorem 3.1

Note that (3.1.i) and (3.1.ii) follow from Theorem 2.1, (2.1.i) and (2.1.ii) respectively, once we verify hypotheses (2.1.1) and (2.1.2).

But the proof of (2.1.1) can be found in [16] and [20], for instance, and (2.1.2) has been verified in Lemma 4.1. Regularity of the solutions follows from classic elliptic regularity theory and can be found in [8], for instance.

The proof of existence of a family of local minimizers v_ε satisfying $\|v_\varepsilon - v_0\|_{L^1(\Omega)} \stackrel{\varepsilon\to 0}{\longrightarrow} 0$ (v_0 as in (3.4)), in case of a rectangle, and also of w_ε satisfying $\|w_\varepsilon - w_0\|_{L^1(\Omega)} \stackrel{\varepsilon\to 0}{\longrightarrow} 0$ (w_0 as in (3.5)), in case of a parallelepiped, is the same proof given for u_ε since each domain has the same symmetry property in each direction.

5 Symmetry inherited by local minimizers

The objective in this section is to give a complete geometric picture of the global and local minimizers of E_ε, found in Theorem 3.1, when Ω is any of the cylindrical domains considered in Theorem 1.2 as well as the exact number of these minimizers in each case.

In particular it follows that among all critical points of E_ε only the ones whose zero-level set are connected (and intersects the boundary) can minimize the energy functional. In other words, the more a function oscillates the more energy it carries; a kind of Sturm-Liouville result.

Since from now on we will deal with the Euler-Lagrange equation for E_ε, for simplicity in notation, unless otherwise stated, we re-scale the family of functionals E_ε as

$$E_\varepsilon(u) = \int_\Omega \left[\frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{4\varepsilon}(1-u^2)^2\right] dx.$$

In the next lemma we exploit the symmetry of the domains to obtain symmetry properties of the local minimizers via an application of the Unique Continuation Principle under very weak hypotheses.

In cylindrical coordinates (x_1, ρ, θ), for any fixed $\bar{\theta} \in [0\,, 2\pi)$, we set

$$\Omega_r(\bar{\theta}) \stackrel{\text{def}}{=} \{(x_1, \rho, \theta) : -l < x_1 < l,\, 0 < \rho < R,\, \bar{\theta} < \theta < \bar{\theta} + \pi\}$$

and denote by $\Omega_l(\bar{\theta})$ the interior of the set $\Omega \setminus \Omega_r(\bar{\theta})$. Also, as usual, $B_R(0, 0) = \{(\rho, \theta),\, 0 < \rho \leq R,\, 0 \leq \theta < 2\pi\}$.

Lemma 5.1 *Consider the family $\{u_\varepsilon\}$ of local minimizers of E_ε in $\mathcal{M}$, found in Theorem 3.1, satisfying $\|u_\varepsilon - u_0\|_{L^1(\Omega)} \stackrel{\varepsilon \to 0}{\longrightarrow} 0$ (u_0 as in (3.3)), for the domain $\Omega = (-l, l) \times D$ where $D \subset \mathbb{R}^n$ $(n = 2, 3)$ is any bounded Lipschitz domain. Then*

(5.1.i) *u_ε is increasing in the first variable and $u_\varepsilon(x_1, x') = -u_\varepsilon(-x_1, x')$, $\forall\, x' \in D$.*

In particular, $u_\varepsilon < 0$ on Ω_l and $u_\varepsilon > 0$ on Ω_r.

Proof.
(5.1.*i*) A regularity result presented in [14] for cylindrical domains assures that $u_\varepsilon \in W^{2,3}(\Omega) \cap C^{3+\delta}(\Omega)$, $0 < \delta < 1$. Any critical point of E_ε in $\mathcal{M}$ satisfies the Euler-Lagrange equation

$$\varepsilon \triangle u_\varepsilon + \varepsilon^{-1}(u_\varepsilon - u_\varepsilon^3) = \lambda$$

where the constant λ is the Lagrange multiplier corresponding to the constraint.

Since $u_\varepsilon \longrightarrow u_0$ in $L^1(\Omega)$ it holds that $\partial_1 u_\varepsilon \not\equiv 0$, for ε small enough. Since $\partial_1 u_\varepsilon \not\equiv 0$ we may evoke [2], Theorem 3.3, to conclude

$$\partial_1 u_\varepsilon > 0 \quad \text{on} \quad \Omega \tag{5.1}$$

This fact is going to be essential to our analysis. From $\int_\Omega u_\varepsilon = 0$, u_ε changes sign in Ω and we set

$$Z(u_\varepsilon) \stackrel{\text{def}}{=} \{x \in \Omega : u_\varepsilon(x) = 0\}.$$

Next we make the change of dependent variables $w = 1 - u_\varepsilon^2$ (the subscript ε has been dropped in the notation for w since it is kept fixed throughout the proof) which transforms the above equation into

$$\varepsilon \triangle w + \frac{\varepsilon}{2} \frac{|\nabla w|^2}{1 - w} + 2\varepsilon^{-1} w(w - 1) = \lambda_w \sqrt{1 - w}. \tag{5.2}$$

Hence $0 < w \leq 1$ and the Lagrange multiplier $\lambda_w = \int_\Omega w\sqrt{1 - w}dx$. Note that

$$w(x) = 1 \Leftrightarrow x \in Z(u_\varepsilon)$$

and also

$$\frac{|\nabla w|^2}{1 - w} = 4|\nabla u_\varepsilon|^2. \tag{5.3}$$

One sees from (5.3) and the regularity of u_ε that, on the set of $\mathcal{H}^2$-measure zero $\{u_\varepsilon = 0\}$, the second term in (5.2) is finite.

Claim 1: $w(x) = 1 \Longleftrightarrow x = (0, x'),\ x' \in D$.

Indeed suppose by contradiction that $\exists\, \bar{x} = (\bar{x}_1, \bar{x}')$ such that $w(\bar{x}) = 1$ and $\bar{x}_1 \neq 0$. We may suppose $\bar{x}_1 > 0$. Since $\partial_1 u_\varepsilon > 0$ (from (5.1)), by the Implicit Function Theorem, there exists a neighborhood of $\bar{x}' \in D$,

$$\mathcal{V}_\delta \stackrel{\text{def}}{=} \{x' : \|x' - \bar{x}'\| < \delta\}$$

and a function $g \in C^2(\mathcal{V}_\delta)$ satisfying

$$w = 1 \ \text{ on } \ Z_\delta(u_\varepsilon) \stackrel{\text{def}}{=} \{(g(x'), x'),\ x' \in \mathcal{V}_\delta\} \subset Z(u_\varepsilon).$$

From $\bar{x}_1 > 0$, g can be taken so that $g > 0$ and therefore $-l < 2g(x') - l$, $\forall x' \in D$.

Next we define

$$\overline{w}(x_1, x') \stackrel{\text{def}}{=} \begin{cases} w(x_1, x'), & \text{if } g(x') < x_1 < l, \quad x' \in \mathcal{V}_\delta \\ w(2g(x') - x_1, x'), & \text{if } 2g(x') - l < x_1 < g(x'),\ x' \in \mathcal{V}_\delta \end{cases}$$

and let

$$\mathcal{D}(\overline{w}) \stackrel{\text{def}}{=} \{(x_1, x') :\ 2g(x') - l < x_1 < l,\ \ x' \in \mathcal{V}_\delta\},$$

$$\mathcal{D}^-(\overline{w}) \stackrel{\text{def}}{=} \{(x_1, x') :\ 2g(x') - l < x_1 < g(x'),\ \ x' \in \mathcal{V}_\delta\},$$

$$\mathcal{D}^+(\overline{w}) \stackrel{\text{def}}{=} \{(x_1, x') :\ g(x') < x_1 < l,\ \ x' \in \mathcal{V}_\delta\}.$$

Note that

$$\overline{w}(x) = 1,\ x \in \mathcal{D}(\overline{w}) \Longleftrightarrow x = (g(x'), x'),\ \ x' \in \mathcal{V}_\delta.$$

Also $\partial_1 u_\varepsilon > 0$ on Ω implies that $u_\varepsilon(x_1, x') = 0$, for $x' \in \mathcal{V}_\delta$, if and only if $x_1 = g(x')$, i.e., $u_\varepsilon \neq 0$ on $\{(2g(x') - l, x'),\ x' \in \mathcal{V}_\delta\}$.

We deduce

$$\partial_1 w = -2u_\varepsilon \partial_1 u_\varepsilon \neq 0 \ \text{ on } \ \{(2g(x') - l, x'),\ x' \in \mathcal{V}_\delta\}. \tag{5.4}$$

Next we define the function

$$\theta \stackrel{\text{def}}{=} w - \overline{w}.$$

Since $\overline{w} \equiv w$ on $\mathcal{D}^+(\overline{w})$ one uses the constraint to verify that $\lambda_w = \lambda_{\overline{w}}$. Bearing in mind that $\partial_1 w = \partial_1 \overline{w} = 0$ on $Z_\delta(u_\varepsilon)$ (this is crucial here) one verifies that $\theta \in W^{2,2}(\mathcal{D}(\overline{w}))$ and, in addition, is a weak solution to

$$\varepsilon \triangle \theta + \frac{\varepsilon}{2} V(x) \cdot \nabla \theta + \varepsilon^{-1}[U(x) + W(x) + \lambda_w \Lambda(x)]\, \theta = 0 \quad \text{in } \ \mathcal{D}(\overline{w}) \tag{5.5}$$

where

$$V(\cdot) = \int_0^1 \frac{(t\nabla w(\cdot) + (1-t)\nabla \overline{w}(\cdot))}{1 - (tw(\cdot) + (1-t)\overline{w}(\cdot))}\, dt,$$

$$U(\cdot) = \frac{1}{2}\int_0^1 \frac{|(t\nabla w(\cdot) + (1-t)\nabla \overline{w}(\cdot))|^2}{[1-(tw(\cdot)+(1-t)\overline{w}(\cdot))]^2}\,dt$$

$$W(\cdot) = \frac{1}{2}\int_0^1 \frac{1}{[1-(tw(\cdot)+(1-t)\overline{w}(\cdot))]^{1/2}}\,dt$$

and

$$\Lambda(\cdot) = -\frac{1}{2}\int_0^1 \frac{1}{\sqrt{tw(\cdot)+(1-t)\overline{w}(\cdot)}dt}\,,$$

with U, W, Λ and each component of V being in $L^\infty_{loc}(\mathcal{D}(\overline{w}))$ (recall that $u_\varepsilon \in C^{3+\delta}(\Omega)$). Therefore, since $\theta \in W^{2,2}_{loc}$, the Unique Continuation Principle applied to (5.5) yields $\theta \equiv 0$ on the account that $\theta = 0$ on $\mathcal{D}^+(\overline{w})$. Therefore $w = \overline{w}$ on $\mathcal{D}(\overline{w})$ and given that $\partial_1 w(l, x') = 0,\ x' \in \mathcal{V}_\delta$ (recall that $\partial_1 u_\varepsilon(l, x') = 0$, $u_\varepsilon(l, x') \neq 0$ for $x' \in \mathcal{V}_\delta$ and w is C^1 in $\partial\Omega$ up to the edges and corners of Ω) we conclude

$$\partial_1 w(2g(x') - l, x') = 0,\ x' \in \mathcal{V}_\delta. \tag{5.6}$$

But (5.6) contradicts (5.4) and therefore Claim 1 is proved.

Claim 2: It holds that $w(x_1, x') = w(-x_1, x')$, $\forall\, x \in \overline{\Omega}$.

Recalling that $\Omega_l = \{x \in \Omega : x_1 < 0\}$ and $\Omega_r = \{x \in \Omega : x_1 > 0\}$ we now define $\overline{w}$ as the reflection with respect to $\{x_1 = 0\}$ of w restricted to Ω_r, i.e.,

$$\overline{w}(x_1, x_2) \stackrel{\text{def}}{=} \begin{cases} w(x_1, x_2) & \text{on } \Omega_r \\ \\ w(-x_1, x_2) & \text{on } \Omega_l. \end{cases}$$

Now, thanks to Claim 1 (which is crucial here), one verifies that $\overline{w} \in W^{2,2}(\Omega)$. Going through the same procedure as above we now define $\theta \stackrel{\text{def}}{=} w - \overline{w}$ in Ω. One easily verifies, using the symmetry of the domain, that θ is a weak solution to (5.5). Again since $\theta \equiv 0$ on Ω_r and $\theta \in W^{2,2}(\Omega)$ the Unique Continuation Principle implies $w = \overline{w}$ on Ω.

This establishes our claim on the evenness of w with respect to the first variable. The next step is to note that being $w = 1 - u_\varepsilon^2$ then $u_\varepsilon^2(x_1, x_2) = u_\varepsilon^2(-x_1, x_2)$, which amounts to saying that

$$u_\varepsilon(x_1, x_2) = \pm u_\varepsilon(-x_1, x_2),\ \forall\,(x_1, x_2) \in \overline{\Omega}. \tag{5.7}$$

Finally in view of (5.1) and (5.7) we conclude (5.1.i). ■

Lemma 5.2 *Consider the family $\{u_\varepsilon\}$ of local minimizers of E_ε in $\mathcal{M}$, as in Lemma 5.1, when*

(5.2.i) $\Omega = (-l, l) \times (-r, r)\,(l \geq r)$ *or*

(5.2.ii) $\Omega = (-l, l) \times (-r, r) \times (-q, q)\,(l \geq r \geq q)$ *or*

(5.2.iii) $\Omega = (-l, l) \times B_R(0, 0)$.

Then $u_\varepsilon(x_1, x') = u_\varepsilon(x_1)$, $\forall\, x \in \Omega$, *i.e.,* u_ε *depends just on the first variable and is odd and increasing in* $(-l, l)$.

Proof.
The proofs of the first two cases are analogous therefore just rectangular domains are considered. So suppose by contradiction that $\partial_2 u_\varepsilon \not\equiv 0$. Then, by symmetry, the very same argument used in the proof of Lemma 5.1 ([2], Theorem 3.3) yields $u_\varepsilon < 0$ on $\Omega \cap \{x_2 < 0\}$ and $u_\varepsilon > 0$ on $\Omega \cap \{x_1 > 0\}$. But this contradicts the fact, proved in Lemma 5.1, that $u_\varepsilon < 0$ on $\Omega_l = \Omega \cap \{x_1 < 0\}$ and $u_\varepsilon > 0$ on $\Omega_r = \Omega \cap \{x_1 > 0\}$. Hence we must have $\partial_2 u_\varepsilon \equiv 0$.

As for the case $\Omega = (-l, l) \times B_R(0, 0)$ it suffices to note that the property $u_\varepsilon|_{\Omega_r(\bar\theta)} = \pm u_\varepsilon|_{\Omega_l(\bar\theta)}$ (from Lemma 5.1) for any $0 < \bar\theta < 2\pi$ implies that $u_\varepsilon(x_1, \rho, \theta)$ cannot depend on the angular variable. In other words, $u_\varepsilon(x_1, x_2, x_3)$ depends on $x' = (x_2, x_3)$ only through its absolute value $|x'|$ and then, using that u_ε is a conditionally stable critical point of E_ε, we can resort to [14], p. 309, to conclude that u_ε depends only on its first variable x_1.

The oddness and monotonicity have been proved in Lemma 5.1. ■

In the next two lemmas the question of number as well as characterization of local and global minimizers is addressed. Since the argument for all the three types of domains considered in Lemma 5.2 is analogous we focus on the simplest case, the rectangular domain, in the hope that the main ideas can be easier enhanced in a simpler setting. In particular, the one-dimension character of minimizers will be used to conclude a uniqueness result.

Lemma 5.3 *Let* $\Omega = (-l, l) \times (-r, r)\,(l \geq r)$. *Then, for* ε *small,* E_ε *has four local minimizers in* $\mathcal{M}$ *which depend on just one variable:* $u_\varepsilon(x_1)$ *and* $v_\varepsilon(x_2)$ *(as in Theorem 3.1) and also* $-u_\varepsilon(x_1)$ *and* $-v_\varepsilon(x_2)$.

Proof.
Existence of $-u_\varepsilon$ and of $-v_\varepsilon$ follows as in Theorem 3.1 by replacing u_0 with $-u_0$ and v_0 with $-v_0$, respectively.

In order to obtain the dependence of just one variable in each case we follow the proofs of Lemmas 5.1 and 5.2. ■

Lemma 5.4 *Let* $\Omega = R = (-l, l) \times (-r, r)$ *with* $l > r$. *Then*

(5.4.i) E_ε *has only two global minimizers in* $\mathcal{M}$*:* u_ε *and* $-u_\varepsilon$ *(as in Lemma 5.3)*

(5.4.ii) E_ε *has only two other local minimizers* v_ε *and* $-v_\varepsilon$ *(as in Lemma 5.3) which are increasing and decreasing, respectively, and odd in* $(-r, r)$.

Proof.
(5.4.i) Let U_ε be a solution to (1.2). Existence of such a global minimizer is obtained as usual via the direct method of Calculus of Variations. Since $U_\varepsilon \in \mathcal{M}$, it is not a constant function in R.

Claim 1: U_ε depends only on one variable.

Suppose by contradiction that $\partial_1 U_\varepsilon \not\equiv 0$ and also $\partial_2 U_\varepsilon \not\equiv 0$. Since U_ε is also a local minimizer of E_ε in $\mathcal{M}$ we conclude, as in (5.1), that $\partial_1 U_\varepsilon > 0$ on Ω and $\partial_2 U_\varepsilon > 0$ on Ω.

Then we follow the proofs of Lemma 5.1 and Lemma 5.2 (eventually (5.1) may be reversed) to conclude from $\partial_1 U_\varepsilon \not\equiv 0$ that U_ε depends only on the first variable and then, from $\partial_2 U_\varepsilon \not\equiv 0$ that U_ε depends only on the second variable: a contradiction.

Claim 2: Either $U_\varepsilon(x_1) = u_\varepsilon(x_1)$ or $U_\varepsilon(x_1) = -u_\varepsilon(x_1)$.

Since, by Claim 1, U_ε depends only on one variable let us suppose by contradiction that it depends on the second one, i.e., $U_\varepsilon(x_1, x_2) = U_\varepsilon(x_2)$. Hence, from symmetry, we can follow the proof of Lemma 5.1 (note that $\partial_2 U_\varepsilon \not\equiv 0$ allows us to use [2], Theorem 3.3, so that (5.1) holds) to conclude that U_ε is monotone and odd in x_2. For definiteness we may suppose that $U_\varepsilon(x_2)$ is increasing and compare it with $v_\varepsilon(x_2)$ (given in Lemma 5.3), as follows.

Actually since U_ε, v_ε are odd functions in $(-r\,, r)$ and $f(u) = u - u^3$, the corresponding Lagrange multipliers satisfy

$$\lambda_{U_\varepsilon} = \int_{-r}^{r} f(U_\varepsilon)\, dx_2 = 0 \quad \text{and} \quad \lambda_{v_\varepsilon} = \int_{-r}^{r} f(v_\varepsilon)\, dx_2 = 0.$$

Therefore we have

$$\begin{cases} \varepsilon U_\varepsilon'' + \varepsilon^{-1} f(U_\varepsilon) = 0, & \text{in}\,(-r\,, r) \\ U_\varepsilon'(-r) = U_\varepsilon'(r) = 0 \end{cases} \tag{5.8}$$

and

$$\begin{cases} \varepsilon v_\varepsilon'' + \varepsilon^{-1} f(v_\varepsilon) = 0, & \text{in}\,(-r\,, r) \\ v_\varepsilon'(-r) = v_\varepsilon'(r) = 0 \end{cases} \tag{5.9}$$

Thus U_ε and v_ε are both monotone functions and satisfy, for ε small, the equation $\varepsilon^2 u'' + u - u^3 = 0$, subjected to Neumann boundary condition $u'(-r) = u'(r) = 0$. Hence by uniqueness of such solutions (see [19] or [18], p. 3130, for instance) we must have $U_\varepsilon \equiv v_\varepsilon$.

Therefore v_ε would be a global minimizer of E_ε in $\mathcal{M}$. Recall from Lemma 5.3 that v_ε is obtained through the same procedure used to obtain u_ε and as such

$$\|v_\varepsilon - v_0\|_{L^1(R)} \overset{\varepsilon \to 0}{\longrightarrow} 0, \tag{5.10}$$

where v_0 is given by (3.4).

It is also a basic fact in the theory of $\Gamma-$convergence that since v_ε is a global minimizer of E_ε then, due to (5.10), v_0 is also a global minimizer of E_0. But this is impossible due to the fact that u_0, given by (3.3), is a admissible function for E_0 and, since $l > r$, satisfies

$$E_0(v_0) = \frac{8}{3} l > \frac{8}{3} r = E_0(u_0), \tag{5.11}$$

against the fact that v_0 is a global minimizer of E_0.

Hence U_ε depends only on the first variable x_1 and therefore comparing it with $u_\varepsilon(x_1)$ (both satisfy equations (5.8) and (5.9), respectively, and are monotone) the same uniqueness argument used above yields $U_\varepsilon \equiv u_\varepsilon$, for ε small enough.

We conclude the proof of Claim 2 after remarking that since $E_\varepsilon(u_\varepsilon) = E_\varepsilon(-u_\varepsilon)$ it follows that $-u_\varepsilon$ is also a solution to (1.2).

(5.4.ii) Suppose now that V_ε is local minimizer of E_ε in $\mathcal{M}$ but not a solution to (1.2), i.e., not a global minimizer of E_ε in $\mathcal{M}$. As in Claim 1 we conclude that V_ε depends on just one variable and is monotone. Suppose that it depends on the first one and, for definiteness, suppose that V_ε is increasing. Hence comparing it with u_ε we would conclude as in Claim 2 (by uniqueness) that $V_\varepsilon \equiv u_\varepsilon$ and therefore, since u_ε is a global minimizer, so would be V_ε: a contradiction.

Therefore V_ε depends just on the second variable. Comparing it with v_ε we go through the same procedure used in Claim 2 to finally conclude that $V_\varepsilon(x_2) \equiv v_\varepsilon(x_2)$.

To complete the proof it suffices to show that necessarily $-v_\varepsilon$ is a also a local minimizer. By hypothesis, $\exists\, \delta > 0$ and

$$\mathcal{V}_\delta(v_\varepsilon) \stackrel{\text{def}}{=} \{v \in \mathcal{M} : \|v_\varepsilon - v\|_{H^1(\Omega)} < \delta\}$$

so that

$$E_\varepsilon(v_\varepsilon) \le E_\varepsilon(v), \, \forall\, v \in \mathcal{V}_\delta(v_\varepsilon). \tag{5.12}$$

Claim 3: $E_\varepsilon(-v_\varepsilon) \le E_\varepsilon(w), \, \forall\, w \in B_\delta(-v_\varepsilon)$.

Indeed, for any $w \in B_\delta(-v_\varepsilon)$ it holds $\delta \ge \|w-(-v_\varepsilon)\|_{H^1(\Omega)} = \|(-w)-v_\varepsilon\|_{H^1(\Omega)}$, thus implying that $-w \in B_\delta(v_\varepsilon)$. Since E_ε is quadratic functional and $-w \in B_\delta(v_\varepsilon)$, we use (5.12) to conclude

$$E_\varepsilon(-v_\varepsilon) = E_\varepsilon(v_\varepsilon) \le E_\varepsilon(-w) = E_\varepsilon(w).$$

This completes the proof. ■

We are now in condition to complete the proofs of our main results.

Proof of Theorem 1.1

In view of the above lemmas it remains only the case in which the domain is a square to be analyzed. Let u_ε and $-u_\varepsilon$ be the global minimizers found in Lemma 5.4. Since in this case $l = r$ we see that $v_\varepsilon(x_1, x_2) = u_\varepsilon(x_2, x_1)$, i.e., v_ε is obtained from u_ε by a rotation of $\pi/2$ with respect to x_1-axis.

Since $E_\varepsilon(u_\varepsilon) = E_\varepsilon(-u_\varepsilon) = E_\varepsilon(v_\varepsilon) = E_\varepsilon(-v_\varepsilon)$ it follows that all four functions u_ε, $-u_\varepsilon$, v_ε and $-v_\varepsilon$ are solutions to (1.2) and the only ones (either local or global) by Lemma 5.4. ■

Proof of Theorem 1.2

When $\Omega = (-l, l) \times D$,with $D \subset \mathbb{R}^2$ any Lipschitz bounded domain, the proof of existence of a family of local minimizers $\{u_\varepsilon\}$ has been established in Theorem 3.1. ¿From Lemma 5.1 it is increasing and odd on the first variable.

(1.2.i) When $D = B_R(0,0)$ the existence of the two local minimizers $u_\varepsilon(x)$ and $-u_\varepsilon(x)$ follows from Theorem 3.1. From Lemma 5.2 we have that $u_\varepsilon(x_1, x') = u_\varepsilon(x_1)$ is increasing and odd in $(-l, l)$.

It is our goal now to prove that these are the only two solutions to (1.2) in the circular cylinder.

Claim 1: Any local minimizer U_ε of E_ε in $\mathcal{M}$ satisfies either $U_\varepsilon = u_\varepsilon$ or $U_\varepsilon = -u_\varepsilon$.

Unfortunately the argument used in the previous cases for a rectangle and a parallelepiped does not work for a circular cylinder. So for $\varphi \in \mathcal{M}$, $\varphi \not\equiv 0$ and $F(\xi) = (1/4)(1-\xi^2)^2$ we define

$$I(U_\varepsilon; \varphi) \stackrel{\text{def}}{=} \int_\Omega (|\nabla \varphi|^2 + \varepsilon^{-2} F''(U_\varepsilon)\varphi^2)\, dx \tag{5.13}$$

Therefore, from our hypothesis,

$$I(U_\varepsilon; \varphi) \geq 0, \ \forall\, \varphi \in \mathcal{M}. \tag{5.14}$$

Also, letting for simplicity in notation $U_\varepsilon = U$, we have, in cylindrical coordinates,

$$\begin{cases} U_{\rho\rho} + U_\rho/\rho + U_{\theta\theta}/\rho^2 + U_{x_1x_1} + \varepsilon^{-2} f(U_\varepsilon) + \lambda_U = 0, & \text{in } (-l,l)\times D \\ U_\rho(x_1, R, \theta) = 0, & \text{for } -l < x_1 < l,\ 0 < \theta \leq 2\pi \\ U_{x_1}(\pm l, \rho, \theta) = 0 & \text{for } 0 < \rho < R,\ 0 < \theta \leq 2\pi \end{cases} \tag{5.15}$$

where λ_U is a Lagrange multiplier. By defining

$$V \stackrel{\text{def}}{=} U_\theta$$

one easily verifies that $V \in \mathcal{M}$, $V \in C^2(\Omega)$, and

$$\begin{cases} V_{\rho\rho} + V_\rho/\rho + V_{\theta\theta}/\rho^2 + V_{x_1x_1} + \varepsilon^{-2} f'(U_\varepsilon)V = 0, & \text{in } (-l,l)\times D \\ V_\rho(x_1, R, \theta) = 0, & \text{for } -l < x_1 < l,\ 0 < \theta \leq 2\pi \\ V_{x_1}(\pm l, \rho, \theta) = 0 & \text{for } 0 < \rho < R,\ 0 < \theta \leq 2\pi \end{cases} \tag{5.16}$$

Hence from (5.14) and (5.16) we deduce that $I(U;V) = 0$ and, given that $V \in \mathcal{M}$,

$$I(U;V) = 0 = \min\{I(U;\varphi) : \varphi \in \mathcal{M}, \varphi \not\equiv 0\}.$$

It then follows from a variant of Krein-Rutman's Theorem (see, for instance, [2], p. 4, for a proof) that V does not change sign in Ω. Therefore, due to the periodicity of U in the angular variable θ, we must have $U(x_1,\rho,\theta) = U(x_1,\rho)$.

We now evoke a result in [14], p. 308, to finally conclude that being a local minimizer in a domain with rotational symmetry we actually have that U depends just on the first variable. Recovering the sub-script ε, (5.15) becomes

$$\begin{cases} \varepsilon U''_\varepsilon + \varepsilon^{-1} f(U_\varepsilon) + \lambda_{U_\varepsilon} = 0, & \text{in } (-l\,,l) \\ U'_\varepsilon(-l) = U'_\varepsilon(l) = 0 \end{cases} \tag{5.17}$$

Lemma 5.1 (for one-dimensional domains) yields $U_\varepsilon(x_1) = -U_\varepsilon(-x_1)$ and therefore

$$\lambda_{U_\varepsilon} = \int_{-l}^{l} f(U_\varepsilon(x_1))\, dx_1 = 0.$$

This allows us to utilize the very same argument used in Claim 2 of Lemma 5.4, for the equations (5.8) and (5.9), to finally conclude that either $U_\varepsilon = u_\varepsilon$ or $U_\varepsilon = -u_\varepsilon$.

Therefore there are only two local minimizers, u_ε and $-u_\varepsilon$, and on the account that $E_\varepsilon(u_\varepsilon) = E_\varepsilon(-u_\varepsilon)$, they must be global, i.e., the only solutions to (1.2).

(1.2.ii) When $\Omega = (-l,l) \times (-r,r) \times (-q,q)$ with $l > r > q$, the proof that (1.2) has only two solutions, u_ε and $-u_\varepsilon$, depending only on the first variable with u_ε increasing and odd, follows from the same argument used in Lemma 5.4 since dimension of the domain is not an issue here. We briefly describe the steps.

Existence of the other three local minimizers, $-u_\varepsilon(x_1)$, $-v_\varepsilon(x_2)$ and $-w_\varepsilon(x_3)$, follows as in Theorem 3.1 by replacing u_0, v_0 and w_0 with $-u_0$, $-v_0$ and $-w_0$, respectively.

To assure that any solution to (1.2), say U_ε, depends only on one variable we argue as in Claim 1 of Lemma 5.4 except that now we must consider the possibilities

1. $\partial_1 U_\varepsilon \not\equiv 0$, $\partial_2 U_\varepsilon \not\equiv 0$ and $\partial_3 U_\varepsilon \not\equiv 0$,

   and

2. $\partial_1 U_\varepsilon \not\equiv 0$ and $\partial_2 U_\varepsilon \not\equiv 0$ or
3. $\partial_1 U_\varepsilon \not\equiv 0$ and $\partial_3 U_\varepsilon \not\equiv 0$ or
4. $\partial_2 U_\varepsilon \not\equiv 0$ and $\partial_3 U_\varepsilon \not\equiv 0$.

Next we prove that either $U_\varepsilon(x_1) = u_\varepsilon(x_1)$ or $U_\varepsilon(x_1) = -u_\varepsilon(x_1)$. Since U_ε depends only on one variable, suppose first that it depends on the second one. Since U_ε is a global minimizer, as in the rectangular domain we conclude that U_ε is monotone and, for instance, increasing $(-r, r)$.

Following the same argument utilized in Lemma 5.4, Claim 2, we obtain, by uniqueness, that $U_\varepsilon \equiv v_\varepsilon$. Hence v_ε would be a global minimizer of E_ε over $\mathcal{M}$ and, by the $\Gamma-$convergence property, v_0 would be a global minimizer of E_0. Hence since u_0 is a admissible function and $l > r > q$ it would hold that

$$E_0(u_0) = \frac{8}{3} rq < \frac{8}{3} ql = E_0(v_0). \tag{5.18}$$

But then (5.18) would contradict the fact that v_0 is a global minimizer. Therefore U_ε cannot depend on the second variable. Likewise we conclude that U_ε cannot depend on the third variable. Therefore, since U_ε is not a constant function, U_ε must depend only on its first variable.

Had we supposed that U_ε depended on its third variable then the procedure would have been the same except that now U_ε would have been compared to $w_\varepsilon(x_3)$, if U_ε is supposed increasing, and to $-w_\varepsilon(x_3)$ if it were decreasing.

Having established that U_ε depends only on its first variable we again utilize an argument as in Lemma 5.4, Claim 2, to obtain, by uniqueness, that $U_\varepsilon \equiv u_\varepsilon$. Therefore any solution to (1.2) must be either u_ε or $-u_\varepsilon$.

Arguing as in Lemma 5.4 (5.4.ii), using the Γ-convergence property, we conclude that any local (which is not global) minimizer must be one of the four functions: $v_\varepsilon(x_2)$, $-v_\varepsilon(x_2)$, $w_\varepsilon(x_3)$ or $-w_\varepsilon(x_3)$.

(1.2.iii) When Ω is a cube ($l = r = q$) the existence of the six local minimizers $u_\varepsilon(x_1)$, $-u_\varepsilon(x_1)$, $v_\varepsilon(x_2)$, $-v_\varepsilon(x_2)$, $w_\varepsilon(x_3)$ and $-w_\varepsilon(x_3)$, has been established in (2.ii).

We argue as in Lemma 5.4 (5.4.i) to conclude that any solution to (1.2) depends only on one variable and then as in Claim 2 to prove that it must equal $u_\varepsilon(x_1)$, for instance. Here it is irrelevant which of the six functions we choose first.

We remark that from the fact that the energy functional is quadratic and, due to the symmetry of the domain, it holds

$$E_\varepsilon(u_\varepsilon) = E_\varepsilon(-u_\varepsilon) = E_\varepsilon(v_\varepsilon) = E_\varepsilon(-v_\varepsilon) = E_\varepsilon(w_\varepsilon) = E_\varepsilon(-w_\varepsilon)$$

thus showing that they all are solutions to (1.2).

Suppose now that V_ε is any local minimizer of E_ε in $\mathcal{M}$. Note that in the proofs of Claim 1 and Claim 2 in Lemma 5.4 we did not used the fact that the minimizers were global so that both conclusion still hold for local minimizers. Therefore, as in Claim 1, we conclude that V_ε depends on just one variable and is monotone. Suppose that it depends on the first one and, for definiteness, suppose that V_ε is increasing. Hence comparing it with u_ε we would conclude by uniqueness as in Lemma 5.4, Claim 2, that $V_\varepsilon \equiv u_\varepsilon$ thus concluding that there is no other local minimizer than the six global ones found above. ■

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