C^k-SOLVABILITY NEAR THE CHARACTERISTIC SET FOR A CLASS OF ELLIPTIC VECTOR FIELDS WITH DEGENERACIES

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Abstract. This paper deals with the solvability near the characteristic set $\Sigma = \{0\} \times S^1$ of operators of the form $L = \partial/\partial t + (x^n a(x) + ixb(x))\partial/\partial x$, $b(0) \neq 0$ and $n \geq 2$, defined on $\Omega_\epsilon = (-\epsilon, \epsilon) \times S^1$, $\epsilon > 0$, where a and b are real-valued smooth functions in $(-\epsilon, \epsilon)$. For fixed $k \geq 1$, it is shown that given f belonging to a subspace of finite codimension (depending on k) of $C^\infty(\Omega_\epsilon)$ there is $u \in C^k$ solution of the equation $Lu = f$ in a neighborhood of Σ.

1. Introduction

Let $\Omega_\epsilon = (-\epsilon, \epsilon) \times S^1$, $\epsilon > 0$, and let

$$L = \partial/\partial t + (a(x) + ib(x))\partial/\partial x, \quad b \not\equiv 0, \tag{1}$$

be a complex vector field defined on Ω_ϵ, where a and b are real-valued smooth functions in $(-\epsilon, \epsilon)$.

Assume that $\Sigma = \{0\} \times S^1$ is the characteristic set of the structure associated with L and that L is of infinity type along Σ. Hence, L is elliptic on $\Omega_\epsilon \setminus \Sigma$ and $a(0) = b(0) = 0$. In particular, $b(x) \neq 0$ if $x \neq 0$.

Under hypotheses above L satisfies the well-known *Nirenberg-Treves* condition $(\mathcal{P})$. Hence, the local solvability is well understood (see, for instance, [5], [13] and [14]).

In this paper we are concerned with solvability in a full neighborhood of Σ.

We are interested in solving the equation

$$Lu = f$$

near the characteristic set Σ, where $f \in C^\infty(\Omega_\epsilon)$, in the sense of Hörmander (see [11]).

We say that L is solvable at Σ if given f belonging to a subspace of finite codimension of $C^\infty(\Omega_\epsilon)$ there exists $u \in \mathcal{D}'(\Omega_\epsilon)$ solving the equation $Lu = f$ in a neighborhood of Σ.

The interplay between the order of vanishing of the functions a and b, at $x = 0$, has influence in the solvability of L at Σ (see [1], [2], [3], [4], [8], [9], and [10]). Indeed, in the case where the order of vanishing of the function b, at $x = 0$, is greater than 1 the solvability of L at Σ is well understood.

Hence we have the right to restrict ourselves to the case where b vanishes of order 1 at $x = 0$. Therefore, by choosing a smaller $\epsilon > 0$ if necessary, we can write

$$(a + ib)(x) = x^n a_0(x) + ixb_0(x),$$

where $n \geq 1$, a_0 and b_0 are real-valued smooth functions in $(-\epsilon, \epsilon)$, and $b_0(x) \neq 0$ for all $x \in (-\epsilon, \epsilon)$.

It follows from [12] that

$$\lambda = b_0(0) - ia_0(0) \tag{2}$$

is an invariant of L. Such invariant is known as *Meziani number*.

Assume that $\lambda \in \mathbb{C} \setminus \mathbb{Q}$. For each fixed $k \in \mathbb{Z}_+$, it follows from [9] (see also [7] and [12]) that for all $f \in C^\infty(\Omega_\epsilon)$, satisfying

$$\int_0^{2\pi} f(0,t)dt = 0, \tag{3}$$

the equation $Lu = f$ has a C^k solution in a neighborhood of Σ. Also, there is $f \in C^\infty(\Omega_\epsilon)$, satisfying (3), such that the equation $Lu = f$ does not have C^∞ solution in any neighborhood of Σ.

Note that (3) is a necessary condition for the existence of C^k solution of the equation $Lu = f$, in a neighborhood of Σ.

The remainder case to be studied is the case where $\lambda \in \mathbb{Q}$. Now, the problem is a bit different. Indeed, (3) is not a sufficient condition for existence of C^k solutions.

In this paper we deal with the solvability of L in the case where $\lambda \in \mathbb{Q}$.

By a change of coordinates if necessary, we can assume $b_0(0) > 0$. Let p and q be positive integers such that $b_0(0) = p/q$ and $\gcd(p, q) = 1$.

We will show that for fixed $k \in \mathbb{Z}_+$ there is $N = N(k) \in \mathbb{Z}_+$ such that for all $f \in C^\infty(\Omega_\epsilon)$ satisfying, in addition to (3), conditions involving the derivatives of f of order up to $j_0 q$, where $j_0 = \max\{j \in \mathbb{Z} : jq \leq N\}$, there is $u \in C^k$ solution of $Lu = f$ in a neighborhood of Σ. We will present two examples to clarify these additional conditions.

Note that our operator L restricted to $\Omega_\epsilon^+ = (0, \epsilon) \times S^1$ is elliptic. Hence, for all $f \in C^\infty(\Omega_\epsilon)$ there exists $u \in C^\infty(\Omega_\epsilon^+)$ solution of the equation $Lu = f$ in Ω_ϵ^+. A natural question appears: is it possible to extend u smoothly to Ω_ϵ? We will address to this question. Indeed, we will show that there is $f \in C^\infty(\Omega_\epsilon)$, satisfying the conditions mentioned above, such that there is no C^∞ function u defined in Ω_ϵ satisfying $Lu = f$ in $\Omega_\epsilon^+ = (0, \epsilon) \times S^1$.

2. Results

Let $\Omega_\epsilon = (-\epsilon, \epsilon) \times S^1$, $\epsilon > 0$, and let

$$L = \partial/\partial t + x(a_0(x) + ib_0(x))\partial/\partial x, \tag{4}$$

be a complex vector field defined on Ω_ϵ, where a_0 and b_0 are real-valued smooth functions in $(-\epsilon, \epsilon)$. Assume that $a_0(0) = 0$, $b_0(x) \neq 0$ for all

$x \in (-\epsilon, \epsilon)$ and $b_0(0) \in \mathbb{Q}$. Without loss of generality we may assume that $b_0(0) > 0$.

Proposition 2.1. *Let L be given by (4). Let p and q be positive integer numbers such that $b_0(0) = p/q$ and $\gcd(p, q) = 1$. For a fixed $N \in \mathbb{Z}_+$ define $j_0 = \max\{j \in \mathbb{Z} : jq \leq N\}$. Given $f \in C^\infty(\Omega_\epsilon)$ satisfying*

$$\int_0^{2\pi} f(0, t)dt = 0 \tag{5}$$

and conditions involving the derivatives of f of order up to $j_0 q$, there exists $v \in C^\infty(\Omega_\epsilon)$ such that $Lv - f = O(|x|^N)$.

Proof: Let N be a fixed positive integer. Given $f \in C^\infty(\Omega_\epsilon)$ we will seek $v \in C^\infty(\Omega_\epsilon)$ such that $Lv - f = O(|x|^N)$. By using formal Taylor expansions we write

$$f(x, t) \simeq \sum_{j\geq 0} f_j(t)x^j, \quad (a + ib)(x) \simeq \sum_{j\geq 0} c_j x^j \quad \text{and} \quad v(x, t) \simeq \sum_{j\geq 0} v_j(t)x^j.$$

Note that $c_0 = 0$ and $c_1 = i\dfrac{p}{q}$.

Hence, $Lv - f = O(|x|^N)$ leads to

$$v_0'(t) = f_0(t) \tag{6}$$

and,

$$v_j'(t) + i\frac{pj}{q}v_j(t) = f_j(t) - \sum_{l=0}^{j-1} lc_{j-l+1}v_l(t), \quad \text{if} \quad 1 \leq j \leq N. \tag{7}$$

Note that (5) is equivalent to

$$\int_0^{2\pi} f_0(s)ds = 0;$$

hence, (6) has a solution given by

$$v_0(t) = \int_0^t f_0(s)ds.$$

For $1 \leq j < q$, by a simple calculation, we have that (7) has a solution given by

$$v_j(t) = \int_0^t \left(f_j(s) - \sum_{l=0}^{j-1} lc_{j-l+1}v_l(s) \right) e^{i\frac{pj}{q}(s-t)}ds + K_j e^{-i\frac{pj}{q}t}, \tag{8}$$

where

$$K_j = (1 - e^{-\frac{p}{q}ji2\pi})^{-1} \int_0^{2\pi} \left(f_j(t) - \sum_{l=0}^{j-1} lc_{j-l+1}v_l(t) \right) e^{\frac{p}{q}ji(t-2\pi)}dt.$$

For $j = q$ we must to assume f satisfies the compatibility condition

$$\int_0^{2\pi} \left(f_q(t) - \sum_{\ell=0}^{q-1} \ell c_{q-\ell+1} v_\ell(t) \right) e^{ipt} dt = 0$$

in order to find a smooth 2π-periodic solution of (7), which is given by

$$v_q(t) = \int_0^t \left(f_q(s) - \sum_{\ell=0}^{q-1} \ell c_{q-\ell+1} v_\ell(s) \right) e^{ip(s-t)} ds.$$

Suppose that we have determined $v_0, \ldots, v_{j-1}$, for $2 \leq j \leq N$. We have that: either $j \notin q\mathbb{Z}_+$ or $j \in q\mathbb{Z}_+$.

If $j \notin q\mathbb{Z}_+$ then (7) has a solution v_j given by formula (8).

If $j = mq$, for some $m = 1, \ldots, j_0$, $j_0 = \max\{j \in \mathbb{Z} : jq \leq N\}$, then we must to assume that f satisfies the compatibility conditon

$$\int_0^{2\pi} \left(f_{mq}(t) - \sum_{\ell=0}^{mq-1} \ell c_{mq-\ell+1} v_\ell(t) \right) e^{ipmt} dt = 0 \tag{9}$$

in order to find a smooth 2π-periodic solution of (7) which is given by

$$v_{mq}(t) = \int_0^t \left(f_{mq}(s) - \sum_{\ell=0}^{mq-1} \ell c_{mq-\ell+1} v_\ell(s) \right) e^{imp(s-t)} ds.$$

Finally, the function $v \in C^\infty(\Omega_\epsilon)$ defined by $v(x,t) = \sum_{j=0}^N v_j(t) x^j$, where v_j are obtained above, is such that $Lv - f = O(|x|^N)$. ■

Next, we will give two examples for clarifying the compatibility conditions of Proposition 2.1.

Example 2.2. *Consider the complex vector field*

$$L = \partial/\partial t + \left(a(x) + i\frac{p}{q}x \right) \partial/\partial x,$$

defined on Ω_ϵ, where $p, q \in \mathbb{Z}_+$, $\gcd(p,q) = 1$, $a(x) \in C^\infty(-\epsilon, \epsilon)$ and, a is flat at $x = 0$.

Let $f \in C^\infty(\Omega_\epsilon)$. We will seek $v \in C^\infty(\Omega_\epsilon)$ such that $Lv - f = O(|x|^N)$, for fixed $N \in \mathbb{Z}_+$. By using formal Taylor expansions we write

$$f(x,t) \simeq \sum_{j\geq 0} f_j(t) x^j, \quad a(x) + i\frac{p}{q}x \simeq i\frac{p}{q}x \quad \textit{and,} \quad v(x,t) \simeq \sum_{j \geq 0} v_j(t) x^j.$$

Hence, $Lv - f = O(|x|^N)$ leads to

$$v_j'(t) + i\frac{pj}{q} v_j(t) = f_j(t), \quad \textit{if} \quad 0 \leq j \leq N.$$

For $m = 0, 1, \ldots, j_0$, $j_0 = \max\{j \in \mathbb{Z} : jq \le N\}$, conditions (5) and (9) are given by

$$\int_0^{2\pi} f_{mq}(s)e^{imps}ds = 0; \tag{10}$$

consequently, we have

$$v_{mq}(t) = \int_0^t f_{mq}(s)e^{imp(s-t)}ds.$$

Moreover, for $j \notin q\mathbb{Z}_+$, v_j is given by formula (8). Hence, for $f \in C^\infty(\Omega_\epsilon)$ satisfying (10) we can find $v \in C^\infty(\Omega_\epsilon)$ such that $Lv - f = O(|x|^N)$.

□

Remark 2.3. *Conditions (10) are in line with conditions presented in* [7], *where the function a is considered identically zero.*

Example 2.4. *For $n \in \mathbb{Z}_+$, consider the complex vector field*

$$L_n = \partial/\partial t + \left(\alpha x^{nq+1} + i\frac{p}{q}x\right)\partial/\partial x,$$

defined on Ω_ϵ, where $\alpha \in \mathbb{R} \setminus \{0\}$, $p, q \in \mathbb{Z}_+$ and $\gcd(p, q) = 1$.

Let $f \in C^\infty(\Omega_\epsilon)$ and let N be an integer greater than $nq+1$. We will seek $v \in C^\infty(\Omega_\epsilon)$ such that $Lv-f = O(|x|^N)$. By using formal Taylor expansion, we can write

$$f(x,t) \simeq \sum_{j\ge 0} f_j(t)x^j, \quad v(x,t) \simeq \sum_{j\ge 0} v_j(t)x^j.$$

Hence, $Lv - f = O(|x|^N)$ leads to

$$v_j'(t) + i\frac{pj}{q}v_j(t) = f_j(t), \quad \text{if} \quad 0 \le j < nq+1$$

and

$$v_j'(t) + i\frac{pj}{q}v_j(t) = f_j(t) - \alpha(j-nq)v_{j-nq}(t), \quad \text{if} \quad nq+1 \le j \le N.$$

First, for $m = 0, 1, \ldots, n$ conditions (5) and (9) are given by

$$\int_0^{2\pi} f_{mq}(s)e^{imps}ds = 0, \tag{11}$$

so that we have

$$v_{mq}(t) = \int_0^t f_{mq}(s)e^{imp(s-t)}ds. \tag{12}$$

Hence, if $j_0 = \max\{j \in \mathbb{Z} : jq \le N\}$ is such that $j_0 = n$ then (11) are the compatibility conditions to find v.

Now, if $j_0 \ge n+1$, , for $m = n+1, \ldots, j_0$, conditions (9) are reduced to

$$\int_0^{2\pi} f_{mq}(s)e^{imps}ds = \alpha(m-n)q\int_0^{2\pi} v_{(m-n)q}(s)e^{imps}ds; \tag{13}$$

hence, we can find v_{mq} *given by*

$$v_{mq}(t) = \int_0^t \left(f_{mq}(s) - \alpha(m-n)qv_{(m-n)q}(s)\right) e^{imp(s-t)}ds.$$

Let $r_1 = \min\{j_0, 2n\}$*. Then, for* $m = n+1, \ldots, r_1$*, using (12), Fubini's theorem and (11) we obtain*

$$\begin{aligned}\int_0^{2\pi} e^{imps} v_{(m-n)q}(s)ds &= \int_0^{2\pi}\int_0^s f_{(m-n)q}(r)e^{i(m-n)p(r-s)}e^{imps}drds \\ &= \int_0^{2\pi} f_{(m-n)q}(r)e^{i(m-n)pr}\int_r^{2\pi} e^{inps}dsdr \\ &= \frac{1}{inp}\int_0^{2\pi} f_{(m-n)q}(r)e^{i(m-n)pr}(1-e^{inpr})dr \\ &= -\frac{1}{inp}\int_0^{2\pi} f_{(m-n)q}(r)e^{impr}dr.\end{aligned}$$

Therefore, for $m = n+1, \ldots, r_1$*, (13) is equivalent to*

$$\int_0^{2\pi} f_{mq}(s)e^{imps}ds + \frac{\alpha(m-n)q}{inp}\int_0^{2\pi} f_{(m-n)q}(s)e^{imps}ds = 0. \tag{14}$$

Hence, if $j_0 \leq 2n$ *then the compatibility conditions to find* v *are given by (11) and (14).*

Finally, if $j_0 \geq (k-1)n+1$*, with* $k \geq 3$*, let* $r_{k-1} = \min\{j_0, kn\}$*. Then, for* $(k-1)n+1 \leq m \leq r_{k-1}$*, we can prove by induction that (13) is equivalent to*

$$\int_0^{2\pi} e^{imps} f_{mq}(s)ds = \sum_{l=1}^{k-1}\frac{(-1)^{l-1}(\alpha q)^l\prod_{j=1}^{l}(m-jn)}{l!(npi)^l}\int_0^{2\pi}\sum_{j=0}^{l}\beta_{j,l}e^{i(m-jn)ps}f_{(m-nl)q}(s)ds, \tag{15}$$

where $\beta_{j,l} \in \mathbb{R}$ *are determined by formulae*

$$\beta_{j,l} = -\frac{l!}{l-j}\beta_{j,l-1}\,, 0 \leq j < l$$

and

$$\beta_{l,l} = -\sum_{j=0}^{l-1}\beta_{j,l},$$

from $\beta_{0,1} = -1$ *and* $\beta_{1,1} = 1$*.*

Therefore, for $f \in C^\infty(\Omega_\epsilon)$ *satisfying the compatibility conditions above we can find* $v \in C^\infty(\Omega_\epsilon)$ *such that* $Lv - f = O(|x|^N)$*.*

□

Proposition 2.5. *Let* L *be given by* (4)*. Let* p *and* q *be positive integer numbers such that* $b_0(0) = p/q$ *and* $\gcd(p,q) = 1$*. For each fixed* $k \in \mathbb{Z}_+$ *there exists* $N = N(k) \in \mathbb{Z}_+$ *such that given* $g \in C^\infty(\Omega_\epsilon)$*, satisfying* $g(x) =$

$O(|x|^N)$, there exists $w \in C^k(\Omega_\epsilon)$ solution of the equation $Lw = g$, in a neighborhood of Σ.

Proof: Define $Z : \Omega_\epsilon \to \mathbb{C}$ by

$$Z(x,t) = \begin{cases} e^{-\int_x^\epsilon \frac{b_0(y)}{y(a_0^2(y)+b_0^2(y))}dy} \cdot e^{-i\left(t+\int_x^\epsilon \frac{a_0(y)}{y(a_0^2(y)+b_0^2(y))}dy\right)}, & x>0 \\ 0, & x=0 \\ e^{\int_{-\epsilon}^x \frac{b_0(y)}{y(a_0^2(y)+b_0^2(y))}dy} \cdot e^{-i\left(t-\int_{-\epsilon}^x \frac{a_0(y)}{y(a_0^2(y)+b_0^2(y))}dy\right)}, & x<0 \end{cases}. \tag{16}$$

Denote $\Omega_\epsilon^+ = (0,\epsilon)\times S^1$, $\Omega_\epsilon^- = (-\epsilon,0)\times S^1$ and $\Omega_\epsilon^\pm = \Omega_\epsilon^+ \cup \Omega_\epsilon^-$.

We have that $Z \in C^\infty(\Omega_\epsilon^\pm)$, $Z(\Omega_\epsilon^+) = Z(\Omega_\epsilon^-) = D(0,1)\setminus\{0\}$. Moreover, by a simple calculation,

$$\mathsf{L}Z = 0 \quad \text{and} \quad \mathsf{L}\overline{Z} = \frac{2ib_0(x)}{b_0(x)+ia_0(x)}\overline{Z}.$$

Now, consider the function $F : (-\epsilon,\epsilon) \to \mathbb{C}$ defined by

$$F(x) = |Z(x,t)| = \begin{cases} e^{-\int_x^\epsilon \frac{b_0(y)}{y(a_0^2(y)+b_0^2(y))}dy}, & x>0 \\ 0, & x=0 \\ e^{\int_{-\epsilon}^x \frac{b_0(y)}{y(a_0^2(y)+b_0^2(y))}dy}, & x<0 \end{cases}.$$

By using Taylor's formula we can write

$$\frac{b_0(x)}{a_0^2(x)+b_0^2(x)} = \frac{q}{p} + O(|x|);$$

consequently, it follows that

$$F(x) = \begin{cases} \left(\frac{x}{\epsilon}\right)^{\frac{q}{p}} e^{-\int_x^\epsilon \frac{O(|y|)}{y}dy}, & x>0 \\ 0, & x=0 \\ \left(\frac{-x}{\epsilon}\right)^{\frac{q}{p}} e^{\int_{-\epsilon}^x \frac{O(|y|)}{y}dy}, & x<0 \end{cases}. \tag{17}$$

Hence, $F \in C^\infty((-\epsilon,\epsilon)\setminus\{0\}) \cap C^0(-\epsilon,\epsilon)$. Moreover, F is injective in $(-\epsilon,0)$ and $(0,\epsilon)$. Thus if $x \neq 0$ we have $x = F^{-1}(|z|)$, for some $z \in D(0,1)$.

From (17) we can find $\alpha,\beta > 0$ such that

$$\alpha|Z(x,t)|^{\frac{p}{q}} \leq |x| \leq \beta|Z(x,t)|^{\frac{p}{q}};$$

equivalently,

$$\alpha|z|^{\frac{p}{q}} \leq | F^{-1}(|z|) | \leq \beta|z|^{\frac{p}{q}}. \tag{18}$$

Let $g = x^N h$, where $h \in C^\infty(\Omega_\epsilon)$. The pushforward of the equations

$$Lw = g, \quad \text{in} \quad \Omega_\epsilon^\pm,$$

via the map Z are given by

$$\frac{2ib_0(F^{-1}(|z|))}{b_0(F^{-1}(|z|))+ia_0(F^{-1}(|z|))}\overline{z}\frac{\partial\tilde{w}^\pm}{\partial\overline{z}} = \tilde{g}^\pm \quad \text{in} \quad D(0,1)\setminus\{0\},$$

where $\tilde{w}^{\pm}$ and $\tilde{g}^{\pm}$ are the pushforward of functions w and g in Ω_ϵ^+ and Ω_ϵ^-, respectively. Taking $z = |z|e^{i\theta}$, we can write

$$\frac{\partial \tilde{w}^{\pm}}{\partial \overline{z}} = \frac{[b_0(F^{-1}(|z|)) + ia_0(F^{-1}(|z|))]e^{i\theta}}{2ib_0(F^{-1}(|z|))}\frac{\tilde{g}^{\pm}}{|z|};$$

equivalently,

$$\frac{\partial \tilde{w}^{\pm}}{\partial \overline{z}} = \frac{[b_0(F^{-1}(|z|)) + ia_0(F^{-1}(|z|))]e^{i\theta}(F^{-1}(|z|))^N \tilde{h}^{\pm}}{2ib_0(F^{-1}(|z|))|z|}, \tag{19}$$

where $\tilde{h}^{\pm}$ are the pushforward of h in Ω_ϵ^+ and Ω_ϵ^-.

By (18) we have that

$$H(z) = \frac{[b_0(F^{-1}(|z|)) + ia_0(F^{-1}(|z|))]e^{i\theta}(F^{-1}(|z|))^N \tilde{h}^{\pm}}{2ib_0(F^{-1}(|z|))|z|} \in C^r(D(0,1)),$$

where r is the bigger integer less than or equal to $\dfrac{Np}{q} - 1$.

Hence, the solutions

$$\tilde{w}^{\pm}(z) = \frac{1}{2\pi i}\iint_{D(0,1)} \frac{H(\zeta)}{\zeta - z}\, d\zeta \wedge d\overline{\zeta}$$

belong to $C^{r+1}(D(0,1))$ (see, for instance, chapter III of [15]). Thus, for fixed $\ell \in \mathbb{Z}_+$ such that $\ell < r - k$, we can write

$$\tilde{w}^{\pm}(z) = \sum_{0 \le j \le \ell - 1} c_j^{\pm} z^j + |z|^{\ell}\tilde{v}^{\pm}(z),$$

where $\tilde{v}^{\pm}(z)$ belongs to $C^{r-\ell+1}(D(0,1))$. Note that $|z|^{\ell}\tilde{v}^+(z)$ and $|z|^{\ell}\tilde{v}^-(z)$ also satisfy (19).

Define $w : \Omega_\epsilon \to \mathbb{C}$ by

$$w(x,t) = \begin{cases} |Z(x,t)|^{\ell}\tilde{v}^+(Z(x,t)), & x > 0 \\ 0, & x = 0 \\ |Z(x,t)|^{\ell}\tilde{v}^-(Z(x,t)), & x < 0 \end{cases};$$

that is,

$$w(x,t) = \begin{cases} \left(\frac{x}{\epsilon}\right)^{\frac{\ell q}{p}} \mathrm{e}^{-\ell\int_x^{\epsilon}\frac{O(|y|)}{y}dy}\tilde{v}^+(Z(x,t)), & x > 0 \\ 0, & x = 0 \\ \left(\frac{-x}{\epsilon}\right)^{\frac{\ell q}{p}} \mathrm{e}^{\ell\int_{-\epsilon}^{x}\frac{O(|y|)}{y}dy}\tilde{v}^-(Z(x,t)), & x < 0 \end{cases}.$$

By construction we have $Lw = g$, in a neighborhood of Σ. Therefore, it is enough to choose N and ℓ sufficiently large to obtain $w \in C^k(\Omega_\epsilon)$. ∎

Finally, we are ready to state our main result:

Theorem 2.6. *Let L be given by* (4). *Let p and q be positive integer numbers such that $b_0(0) = p/q$ and $\gcd(p,q) = 1$. For each fixed $k \in \mathbb{Z}_+$ there exists $N = N(k) \in \mathbb{Z}_+$ such that given $f \in C^\infty(\Omega_\epsilon)$, satisfying (5) and conditions involving the derivatives of f of order up to $j_0 q$, where $j_0 = \max\{j \in \mathbb{Z}$:*

$jq \leq N\}$, there exists $u \in C^k(\Omega_\epsilon)$ solution of the equation $Lu = f$, in a neighborhood of Σ.

Proof: Fixed $k \geq 1$ choose N given by Proposition 2.5. Hence, by Proposition 2.1, given $f \in C^\infty(\Omega_\epsilon)$, satisfying (5) and conditions involving the derivatives of f of order up to j_0q, where $j_0 = \max\{j \in \mathbb{Z} : jq \leq N\}$, there exists $v \in C^\infty(\Omega_\epsilon)$ such that $Lv - f = O(|x|^N)$.

Let $g = Lv - f$. Now, applying again Proposition 2.5 we can find $w \in C^k$ solution of the equation $Lw = g$, in a neighborhood of Σ. Finally, define $u = v - w$. We have that $u \in C^k$ and $Lu = Lv - Lw = f + g - g = f$, in a neighborhood of Σ. ■

In the next result we will show that for each fixed $N \in \mathbb{Z}_+$, there exists $f \in C^\infty(\Omega_\epsilon)$, satisfying $f = O(|x|^N)$, such that the equation $Lu = f$ does not have C^∞ solution in any neighborhood of Σ. More precisely, we will show that there is no C^∞ function u defined in Ω_ϵ and satisfying $Lu = f$ in Ω_ϵ^+.

Theorem 2.7. *Let L be given by (4). Let p and q be positive integer numbers such that $b_0(0) = p/q$ and $\gcd(p,q) = 1$. Assume that $b_0(0)^{-1} \notin \mathbb{Z}$. Then for each fixed $N \in \mathbb{Z}_+$, there exists $f = O(|x|^N)$ of C^∞ class in Ω_ϵ such that there is no $u \in C^\infty(\Omega_\epsilon)$ satisfying $Lu = f$ in Ω_ϵ^+.*

Proof: The proof is an adaption of the arguments presented by Bergamasco and Meziani in [3] (see Theorem 3.2).

Let

$$\sum_{m=0}^{\infty} \alpha_{pm+1} z^{pm+1} \tag{20}$$

be a series in one complex variable, with radius of convergence equal to zero. By using Borel's theorem we can construct $g \in C^\infty(D(0,1))$ whose Taylor series at $z = 0$ is given by (20). Since, for each $M \in \mathbb{Z}_+$, we can write

$$g(z) = \sum_{m=0}^{M} \alpha_{pm+1} z^{pm+1} + O(|z|^{pM+1})$$

we have that

$$\frac{\partial g}{\partial \overline{z}}(z) = O(|z|^{pM+1}), \quad \forall\, M \in \mathbb{Z}_+.$$

Hence, the function $\dfrac{\partial g}{\partial \overline{z}}$ belongs to $C^\infty(D(0,1))$ and is flat at $z = 0$.

Define $f : \Omega_\epsilon \to \mathbb{C}$ by

$$f(x,t) = \begin{cases} \frac{2ib_0(x)}{b_0(x)+ia_0(x)} \overline{Z}(x,t) \frac{\partial g}{\partial \overline{z}}(Z(x,t)), & x > 0 \\ 0, & x \leq 0 \end{cases},$$

where Z is given by (16). Note that $f \in C^\infty(\Omega_\epsilon)$ and is flat along to Σ.

Suppose, by contradiction, that there is $u \in C^\infty(\Omega_\epsilon)$ solution of the equation $Lu = f$ in Ω_ϵ^+.

The pushforward of $Lu = f$ in Ω_ϵ^+, via the map Z, yields

$$\frac{2ib_0(F^{-1}(|z|))}{b_0(F^{-1}(|z|)) + ia_0(F^{-1}(|z|))}\bar{z}\frac{\partial \tilde{u}^+}{\partial \bar{z}} = \frac{2ib_0(F^{-1}(|z|))}{b_0(F^{-1}(|z|)) + ia_0(F^{-1}(|z|))}\bar{z}\frac{\partial g}{\partial \bar{z}}(z)$$

in $D(0,1) \setminus \{0\}$; hence, $\tilde{u}^+$ is a solution of the CR-equation

$$\frac{\partial \tilde{u}^+}{\partial \bar{z}} = \frac{\partial g}{\partial \bar{z}}(z), \quad \text{in} \quad D(0,1) \setminus \{0\}.$$

Therefore,

$$\tilde{u}^+ = g + h,$$

where h is a holomorphic function defined in $D(0,1)$. Let (c_m) be a sequence of complex numbers such that

$$h(z) = \sum_{j=0}^{\infty} c_j z^j.$$

Since (20) has radius of convergence equal to zero, there exists $m_0 \in \mathbb{Z}_+$ such that $\alpha_{pm_0+1} + c_{pm_0+1} \neq 0$. Take $k \in \mathbb{Z}_+$ such that $k > pm_0 + 1$. From $\tilde{u}^+ = g + h$ we have

$$\tilde{u}^+(z) = \sum_{j=0}^{k}(\alpha_j + c_j)z^j + O(|z|^k),$$

where $\alpha_j = 0$ if $j - 1 \notin p\mathbb{Z}$.

Hence, for $x > 0$ we have

$$u(x,t) = \sum_{j=0}^{k}(\alpha_j + c_j)Z^j(x,t) + O\left(\left(\frac{x}{\epsilon}\right)^{\frac{kq}{p}} e^{-k\int_x^\epsilon \frac{O(|y|)}{y}dy}\right),$$

which is a contradiction since, for $k_0 = pm_0 + 1$,

$$Z^{k_0} = \begin{cases} \left(\frac{x}{\epsilon}\right)^{\frac{qk_0}{p}} e^{-k_0\left[\int_x^\epsilon \frac{O(|y|)}{y}dy - i\left(t + \int_x^\epsilon \frac{a_0(y)}{y(a_0^2(y)+b_0^2(y))}dy\right)\right]}, & x > 0 \\ 0, & x = 0 \\ \left(\frac{-x}{\epsilon}\right)^{\frac{qk_0}{p}} e^{k_0\left[\int_{-\epsilon}^x \frac{O(|y|)}{y}dy - i\left(t - \int_{-\epsilon}^x \frac{a_0(y)}{y(a_0^2(y)+b_0^2(y))}dy\right)\right]}, & x < 0 \end{cases}$$

is no C^∞ in Ω_ϵ. ■

Remark 2.8. *A slight modification of the arguments in the proof of Theorem 2.7 allow us to prove a version for the case where the Meziani number λ, given by (2), satisfies $\lambda^{-1} \in \mathbb{C} \setminus \mathbb{Z}$.*

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