# Fast implementation for a presentation matrix associated to discriminant of co-rank one singularities from $\mathbb{C}^n$ to $\mathbb{C}^n$.

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**Abstract**

In this article we present a fast implementation of an algorithm for constructing a presentation of the $\mathcal{O}_n$-module $f_*\mathcal{O}_{(\Sigma(f),0)}$ associated to any finite co-rank 1 map germ $f \in \mathcal{O}(n,n)$, where $(\Sigma(f),0)$ denotes the germ of the critical locus of $f$. Moreover we implement this construction using the software Maple, showing explicitly how to compute the elements of the presentation matrix for such maps. Then we show how to apply this construction to obtain invariants associated to the stable singularities of such germs.

## 1 Introduction

A presentation of a finitely presented $R$-module $M$ ($R$, a commutative ring with unit) is an exact sequence

$$R^p \xrightarrow{\lambda} R^q \xrightarrow{\alpha} M \longrightarrow 0 \tag{1}$$

of $R$-modules. When such a presentation exists, we call $\lambda$ the matrix of relations among the generators of the module.

In general it is not easy to construct a presentation, but when $(X,x)$ is the multi-germ of a Cohen-Macaulay variety of dimension $n$, Mond and Pellikan in [4] showed an algorithm for constructing the presentation of the $\mathcal{O}_{n+1}$-module $\mathcal{O}_{(X,x)}$ for finite analytic map germs $f:(X,x)\to(\mathbb{C}^{n+1},0)$, which is one of the key tools to compute invariants associated to the singularities of any stabilization of $f$.

In this article we show how to construct the presentation of the $\mathcal{O}_n$-module $f_*\mathcal{O}_{(\Sigma(f),0)}$ associated to any finite co-rank 1 map germ $f$ in the ring of holomorphic map germs $\mathcal{O}(n,n)$, where $(\Sigma(f),0)$ denotes the germ of the critical locus of $f$. Moreover we implement this construction using the software Maple, showing explicitly how to compute the elements of the matrix $\lambda$ for such maps. Then we show how to apply this construction to obtain some invariants associated to the singularities of stable map germs in this ring.

## 2 The presentation

We describe here the algorithm given by Mond and Pellikaan in [4, Section 2.2] to construct the presentation of any $\mathcal{O}_{n+1}$-module $f_*\mathcal{O}_X$.

Let $(X,x)$ be the multi-germ of a Cohen-Macaulay variety of dimension $n$ and $f:(X,x)\to(\mathbb{C}^{n+1},0)$ be the germ of an finite analytic map, by the Weierstrass preparation theorem it follows that $\mathcal{O}_{(X,x)}$ is a finite $\mathcal{O}_{n+1}$-module via the function $f^*$.

To compute the matrix $\lambda$, we remark that if the classes of $g_1, g_2, \ldots, g_h$ in $\frac{\mathcal{O}_{(X,x)}}{f^*\mathfrak{m}_0}$ generate it as a vector space over $\mathbb{C}$, then $g_1, g_2, \ldots, g_h$ generate $\mathcal{O}_{(X,x)}$ as an $\mathcal{O}_{n+1}$-module, therefore it is enough to obtain the relations among the $g_i$, $i=1,\ldots,h$ over $\mathcal{O}_{n+1}$.

**Procedure to obtain the matrix $\lambda$:** Choose a projection $\pi:(\mathbb{C}^{n+1},0)\to(\mathbb{C}^n,0)$ such that $\widetilde{f}=\pi\circ f$ is also finite. After a change of coordinates we suppose that $f(x)=(\widetilde{f}(x), f_{n+1}(x))$ and as $\mathcal{O}_{X,x}$ is Cohen-Macaulay, then $\mathcal{O}_{X,x}$ is a free $\mathcal{O}_n$-module via $\widetilde{f}^*$.

Then there exist unique $\alpha^i_j \in \mathcal{O}_n$, $1\le i,j\le h$ such that

$$g_j f_n = \sum (\alpha^i_j \circ \widetilde{f}) g_i. \tag{2}$$

As the germs $g_i$ generate $\mathcal{O}_{X,x}$ over $\mathcal{O}_{n+1}$ via $f^*$, then

$$\lambda^i_j = \alpha^i_j \circ \pi, \quad i \neq j$$

$$\lambda^i_i = \alpha^i_i \circ \pi - X_{n+1},$$

since $f_{n+1} = X_{n+1} \circ f$ and $(X_1, \ldots, X_{n+1})$ denotes the coordinates of $\mathbb{C}^{n+1}$ in the target.

Therefore one has the matrix $\lambda = \lambda_{(i,j)}$ for the exact sequence of the $\mathcal{O}_{n+1}$-module $\mathcal{O}_{X,x}$.

$$\mathcal{O}^p_{n+1} \xrightarrow{\lambda} \mathcal{O}^q_{n+1} \xrightarrow{\alpha} \mathcal{O}_{X,x} \longrightarrow 0 \tag{3}$$

If $g_1, g_2, \ldots, g_h$ generate $\mathcal{O}_{X,x}$, we can consider $g_1 = 1$, $q = p = h$ and the matrix $\alpha$ is given in such a way that $\alpha(e_i) = g_i$, where $e_i$ is the $i^{th}$ element of the usual basis.

# 3 The presentation for co-rank 1 map germs in $\mathcal{O}(n, n)$

As $f$ is co-rank 1 finite map germ in $\mathcal{O}(n, n)$, then $(\Sigma(f), 0)$ is a $(n-1)$-dimensional Cohen-Macaulay variety. Therefore we can use the results of Mond-Pellikaan.

Let $f \in \mathcal{O}(n, n)$ be a finite map germ of co-rank 1. Choosing linearly adapted coordinates, we can write $f(x, z) = (x_1, \ldots, x_{n-1}, g(x, z))$, where $x = (x_1, \ldots, x_{n-1}) \in \mathbb{C}^{n-1}$, $z \in \mathbb{C}$ and $g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is a polynomial that can be written in the form

$$g(x, z) = z^{k+1} + h(x, z), \text{ with } h(x, z) = h_{k-1}(x)z^{k-1} + h_{k-2}(x)z^{k-2} + \cdots h_1(x)z + h_0(x),$$

with $h_i(0) = 0$ for all $h_i : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, $i = 0, \ldots, k-1$.

**Theorem 3.1** *For any map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ as above, the presentation matrix of $\mathcal{O}_{(\Sigma(f),0)}$ over $\mathcal{O}_n$ is given by:*

$$\lambda = \begin{bmatrix} X_n + H_{1,1}(\overline{X}) & H_{1,2}(\overline{X}) & \cdots & H_{1,k}(\overline{X}) \\ H_{2,1}(\overline{X}) & X_n + H_{2,2}(\overline{X}) & & H_{2,k}(\overline{X}) \\ \vdots & & \ddots & \vdots \\ H_{k,1}(\overline{X}) & & \cdots & X_n + H_{k,k}(\overline{X}) \end{bmatrix}_{k \times k}$$

*where, $H_{i,j} : (\mathbb{C}^{n-1}, 0) \to (\mathbb{C}, 0)$ are polynomials and $(\overline{X}, X_n) = (X_1, \ldots, X_{n-1}, X_n)$ are the target variables.*

**Proof:** The Jacobian determinant of the matrix of the derivatives of $f$ at any point $(x, z)$ is

$$J(f) = (k+1)z^k + h_z(x, z),$$

where $h_z(x, z)$ denotes the derivative of $h(x, z)$ with respect to the variable $z$.

From the fact that $f$ is of co-rank one it follows that the local algebra $\frac{\mathcal{O}_{(\Sigma(f),0)}}{f^*\mathfrak{m}_{\mathbb{C}^n,0}}$ is isomorphic to $\frac{\mathbb{C}[z]}{<z^k>}$, or in other words, $\mathcal{O}_{(\Sigma(f),0)}$ is minimally generated as $\mathcal{O}_n$-module (target) via $f|^*_{(\Sigma(f),0)}$ and in this case a system of generators $\{g_0, g_1, \ldots, g_{k-1}\}$ is given by $\{1, z, z^2, \ldots, z^{k-1}\}$.

To obtain the matrix $\lambda$ as in (1) we need to obtain $k$ relations between the variables target $X_i = f_i$, $i = \{1, \ldots, n\}$ and the set of generators $\{1, z, z^2, \ldots, z^{k-1}\}$, module the jacobian ideal $J(f)$.

For the first relation, one has

$$g(x, z).1 = z^{k+1} + h_{k-1}(x_1, \ldots, x_{n-1})z^{k-1} + \cdots + h_1(x_1, \ldots, x_{n-1})z + h_0(x_1, \ldots, x_{n-1}).1$$

Let $X_i := f_i(x,z) = x_i$, $i = \{1,\dots,n-1\}$ and $X_n := g(x,z)$, then

$$X_n.1 = z^{k+1} + H_{k-1}(X_1,\dots,X_{n-1})z^{k-1} + \cdots + H_1(X_1,\dots,X_{n-1})z + H_0(X_1,\dots,X_{n-1}).1$$

$$X_n.1 = z^{k+1} + H_{k-1}(\overline{X})z^{k-1} + \cdots + H_1(\overline{X})z + H_0(\overline{X}).1$$

$$z^{k+1} = (X_n - H_0(\overline{X})).1 - H_1(\overline{X})z - \cdots - H_{k-1}(\overline{X})z^{k-1} \tag{4}$$

where $H_j$ is just $h_j$, $j = 0,\dots,k-1$, substituting $x_i$ by $X_i$, $i = 1,\dots,n-1$.

On the other hand, $(k+1)z^k + h_z(x,z) = 0$ (in $\mathcal{O}_{(\Sigma(f),0)}$). Then

$$(k+1)z^k = -(k-1)h_{k-1}(x)z^{k-2} - \cdots - 2h_2(x)z - h_1(x).1$$

$$z^k = -\left(\frac{k-1}{k+1}\right)H_{k-1}(\overline{X})z^{k-2} - \cdots - \left(\frac{2}{k+1}\right)H_2(\overline{X})z - \left(\frac{1}{k+1}\right)H_1(\overline{X}).1 \tag{5}$$

From equality (4) and equality (5) multiplied by $z$, results

$$\left(X_n - H_0(\overline{X})\right).1 - \left(\frac{k}{k+1}\right)H_1(\overline{X}).z - \left(\frac{k-1}{k+1}\right)H_2(\overline{X}).z^2 - \cdots - \left(\frac{2}{k+1}\right)H_{k-1}(\overline{X}).z^{k-1} = 0 \tag{6}$$

Now, denote $H_{1,1}(\overline{X}) = -H_0(\overline{X})$, $H_{1,j+1}(\overline{X}) = -\left(\frac{k+1-j}{k+1}\right)H_j(\overline{X})$, $j = 1,\dots,k-1$ and the equation 6 shows that

$$(X_n + H_{1,1}(\overline{X})).g_0 + \sum_{i=1}^{k-1} H_{1,i+1}(\overline{X})g_i = 0. \tag{7}$$

Therefore, the first relation between the variable target $X_i := f_i$, $i = \{1,\dots,n\}$, and the set of generators $\{1, z, z^2, \dots, z^{k-1}\}$, module jacobian ideal $J(f)$ is given and the first line of the matrix $\lambda$ is

$$\begin{bmatrix} X_n + H_{1,1}(\overline{X}) & H_{1,2}(\overline{X}) & \cdots & H_{1,k}(\overline{X}) \end{bmatrix}_{1\times k}.$$

To obtain the second line of the matrix $\lambda$, we multiply the equation (7) by $g_1 := z$, then

$$X_n.g_1 + H_{1,1}(\overline{X}).g_1 + \sum_{i=1}^{k-1} H_{1,i+1}(\overline{X})g_i.g_1 = 0.$$

As $g_i.g_j = g_{i+j}$, $i + j < k$,

$$X_n.g_1 + H_{1,1}(\overline{X}).g_1 + H_{1,2}(\overline{X}).g_2 + \cdots + H_{1,k-1}(\overline{X}).g_{k-1} + H_{1,k}(\overline{X}).z^k = 0. \tag{8}$$

Substituting the right hand of the equation (5) in the equation (8), regrouping and renaming the terms

$$H_{2,1}(\overline{X}).g_0 + X_n.g_1 + H_{2,2}(\overline{X}).g_1 + H_{2,3}(\overline{X}).g_2 + \cdots + H_{2,k-1}(\overline{X}).g_{k-2} + H_{2,k}(\overline{X}).g_{k-1} = 0.$$

And the second line of the matrix is given by

$$\begin{bmatrix} H_{2,1}(\overline{X}) & X_n + H_{2,2}(\overline{X}) & \cdots & H_{2,k}(\overline{X}) \end{bmatrix}_{1\times k}.$$

Observe that the equation (5) does not depend of the variable $X_n$.

Proceeding in this way, to obtain the $r^{th}$ line multiply the $(r-1)^{th}$ line by $z$ or the first line by $z^{r-1}$ and use the equations (4) and (5). The process is algorithmic and finishs after $k$ lines, where $k$ is the number of generators. To conclude, by [4, Section 2.2] , the relations among the $g_i$ obtained above generate the module Ker $\alpha$. Therefore we have the desired matrix. □

**Remark 3.2** Note that the determinant of the matrix $\lambda$ is of the form $X_n^k + B$, where $B$ is a polynomial in the variables $X_1,\dots,X_n$, with $B(0,\dots 0, X_n) = 0$.

# 4 Implementation

We present in this section the source code to obtain the matrix of the presentation given in the Theorem 3.1 by using computational methods. The implementation is obtained using the software Maple. To write the source code below we follow directly the proof of the theorem.

```
Presentation_CnCn_Cor1:=proc(f)

local
Iz,Izz,B,J,G,t,i,b,j,V,V1,nf,vf,vec,zz,F,vec2,e,Xf;
global lambda,varf;
nf:=nops(f); varf:=indets(f); e:=F; vf:=nops(varf);
 if nf <> vf then
   ERROR("Dimensions of source and target are different.");
 fi:
  vec:=seq(X[i],i=1..nf);
  vec2:=seq(X[i],i=1..(nf-1));
  F:=f;
    for i to nf-1 do
      F:=subs(F[i]=X[i],F):
    od;
        for i to nf-1 do
          e[nf]:=subs(X[i]=0,e[nf]):
            if e[nf]=0 then print("f is not finite.");
               ERROR("Verify input!");
            fi;
        od;

 zz:=indets(e[nf])[]; vec2:=vec2,zz;
 J:=det(jacobian(F,[vec2])):
 b:=tcoeff(J,{vec}):
 G:=[seq(zz^i,i=1..degree(b))]; nGer:=nops(G);
 Iz:=(J-lcoeff(tcoeff(J,{vec}),{zz},'n')*n)*
     (-1/lcoeff(tcoeff(J,{vec}),{zz},'n')):
 Izz:=expand(Iz*zz):
 t:=X[nf]-F[nf];
 t:=simplify(subs(zz^(degree(e[nf]))=Izz,t)):

   if degree(t,zz) > (degree(b)-0) then
     ERROR("No result! Verify input.");
   fi;

   for j from 1 to nGer do
     V[j]:=t;
        for i from 1 to nops(t) do
           if divide(op(i,t),zz^(degree(b))) then
              t:=simplify(subs(zz^(degree(b))=Iz,t)):
```

```
              V[j]:=t;
              break:
           fi:
        od:
     t:=expand(t*zz);
   od:
--------------------- Viewing the matrix -----------------------
lambda:=Matrix(1..nGer,1..nGer):
  for j from 1 to nGer do
    V1:=expand(zz*V[j]);
      for i from 1 to nGer do
        lambda[j,i]:=  coeff(V1,G[i]):
      od:
  od:
print('lambda'=lambda);
end:
```

**Remark 4.1** 1. If dimensions of the source and target are different then, the process stop and a message is showed to user.

2. If $\mathcal{O}_{(\Sigma(f),0)}$ is not finitely generated as $\mathcal{O}_n$-module via $f^*$ then, the process stop and a message is showed to user.

# 5 Fitting ideals and multiple points sets

The Fitting ideals associated to the matrix of the presentation are one of the key tools to compute the stable singularities associated to some map germ.

**Definition 5.1** *Let $M$ be a finitely presented $R$-module (where $R$ is a commutative ring with unit) and let*

$$R^p \xrightarrow{\lambda} R^q \xrightarrow{\alpha} M \longrightarrow 0$$

*be a presentation. The $k^{th}$ Fitting ideal of $M$, $\mathcal{F}_k(M)$, is defined to be the ideal in $R$ generated by all $(q-k)\times(q-k)$ minors of the matrix $\lambda$, for $q > k \geq q-p$. $\mathcal{F}_k(M) = R$, for $k \geq q$, and $\mathcal{F}_k(M) = 0$ for $k < q-p$.*

According to [4, Proposition 1.5], $V(\mathcal{F}_k(\mathcal{O}_{(\Sigma(f),0)}))$ is the set of points in the target whose pre-image consists of $(k+1)$ or more points in $\Sigma(f)$, counting multiplicities.

In this section we apply the algorithm described here to compute the presentation matrix in some examples and to calculate the $k^{th}$ Fitting ideals of the matrix $\lambda$, for $0 \leq k \leq 2$, we use the software Singular using the command $fitting(\lambda, k)$. We also compare this method with the known method called *Elimination of variables* used to obtain the defining equation of the discriminant of $f$ (or its $0^{th}$ Fitting ideal), showing that our method is much faster.

In the examples $\lambda_f$ denotes the matrix of the presentation with respect to $\mathcal{O}_{(\Sigma(f),0)}$.

**Example 5.2** Let $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ be defined by $f(x, y, z) = (x, y, z^6 + xz + yz^2)$. The germ $f$ is of co-rank 1 and $\frac{\mathcal{O}_{(\Sigma(f),0)}}{f^*\mathfrak{m}_{(\mathbb{C}^3,0)}} \cong \frac{\mathbb{C}[z]}{<z^5>}$, i. e., $\mathcal{O}_{(\Sigma(f),0)}$ is finitely generated as $\mathcal{O}_3$-module via $f^*$, and a presentation matrix of $f$ is given by

$$\lambda_f = \begin{bmatrix} X_3 & -\frac{5}{6}X_1 & -\frac{2}{3}X_2 & 0 & 0 \\ 0 & X_3 & -\frac{5}{6}X_1 & -\frac{2}{3}X_2 & 0 \\ 0 & 0 & X_3 & -\frac{5}{6}X_1 & -\frac{2}{3}X_2 \\ \frac{1}{9}X_1X_2 & \frac{2}{9}X_2^2 & 0 & X_3 & -\frac{5}{6}X_1 \\ \frac{5}{36}X_1^2 & \frac{7}{18}X_1X_2 & \frac{2}{9}X_2^2 & 0 & X_3 \end{bmatrix}_{5\times 5}.$$

To obtain this matrix using our implementation, in Maple software just enter with the command:

$$\text{Presentation\_CnCn\_Cor1}([x, y, z^6 + xz + yz^2]);$$

Now, we shall show how to calculate the defining ideal of the multiple points set (in the target) of the deformation of $f$ given by $f_t(x, y, z) = (x, y, z^6 + xz + yz^2 + tz^4)$.

We remark that the use of computational methods does not allow us to consider the parameter $t$ as a coefficient in the space $\mathcal{O}_n$. To avoid this problem to calculate this ideal we need to consider the following unfolding of $f$ given by

$$F(t, x, y, z) = (t, x, y, z^6 + xz + yz^2 + tz^4)$$

and first compute its presentation matrix using the command Presentation_CnCn_Cor1. Then we show how to translate this result to interpret the presentation matrix of $f_t$. Note that $\frac{\mathcal{O}_{(\Sigma(F),0)}}{F^*\mathfrak{m}_{(\mathbb{C}^4,0)}} \cong \frac{\mathbb{C}[z]}{<z^5>}$.

Using the command Presentation_CnCn_Cor1([F]) we obtain the $5\times 5$ matrix $\lambda_F$ for the presentation

$$\mathcal{O}_3^5 \xrightarrow{\lambda_F} \mathcal{O}_3^5 \xrightarrow{\alpha} \mathcal{O}_{\Sigma(F),0} \to 0.$$

Where $\lambda_F$ is given by

$$\begin{bmatrix} X_4 & -\frac{5}{6}X_2 & -\frac{2}{3}X_3 & 0 & -\frac{1}{3}X_1 \\ \frac{1}{18}X_1X_2 & X_4 + \frac{1}{9}X_1X_3 & -\frac{5}{6}X_2 & -\frac{2}{3}X_3 + \frac{2}{9}X_1^2 & 0 \\ 0 & \frac{1}{18}X_1X_2 & X_4 + \frac{1}{9}X_1X_3 & -\frac{5}{6}X_2 & -\frac{2}{3}X_3 + \frac{2}{9}X_1^2 \\ \frac{1}{9}X_2(X_3 - \frac{1}{3}X_1^2) & \frac{2}{9}X_3^2 - \frac{2}{27}X_1^2X_3 & \frac{1}{18}X_1X_2 & X_4 + \frac{5}{9}X_1X_3 - \frac{4}{27}X_1^3 & -\frac{5}{6}X_2 \\ \frac{5}{36}X_2^2 & \frac{7}{18}X_2X_3 - \frac{1}{27}X_1^2X_2 & \frac{2}{9}X_3^2 - \frac{2}{27}X_1^2X_3 & \frac{11}{18}X_1X_2 & X_4 + \frac{5}{9}X_1X_3 - \frac{4}{27}X_1^3 \end{bmatrix}.$$

Observe that this matrix is obtained by using the command Presentation_CnCn_Cor1([F]) for $F$ from $(\mathbb{C}\times\mathbb{C}^3, 0)$ to $(\mathbb{C}\times\mathbb{C}^3, 0)$, so that, we have the correspondence $t \leftrightarrow X_1$, $x \leftrightarrow X_2$, $y \leftrightarrow X_3$ and $z^6 + xz + yz^2 + tz^4 \leftrightarrow X_4$.

Then, to obtain the presentation matrix for $f_t(x, y, z) = (x, y, z^6 + xz + yz^2 + tz^4)$, rename the variables, that is, make the correspondence $X_4 := X_3$, $X_3 := X_2$, $X_2 := X_1$ and $X_1 := t$. Therefore the presentation matrix $\lambda_{f_t}$ is

$$\begin{bmatrix} X_3 & -\frac{5}{6}X_1 & -\frac{2}{3}X_2 & 0 & -\frac{1}{3}t \\ \frac{1}{18}tX_1 & X_3 + \frac{1}{9}tX_2 & -\frac{5}{6}X_1 & -\frac{2}{3}X_2 + \frac{2}{9}t^2 & 0 \\ 0 & \frac{1}{18}tX_1 & X_3 + \frac{1}{9}tX_2 & -\frac{5}{6}X_1 & -\frac{2}{3}X_2 + \frac{2}{9}t^2 \\ \frac{1}{9}X_1(X_2 - \frac{1}{3}t^2) & \frac{2}{9}X_2^2 - \frac{2}{27}t^2X_2 & \frac{1}{18}tX_1 & X_3 + \frac{5}{9}tX_2 - \frac{4}{27}t^3 & -\frac{5}{6}X_1 \\ \frac{5}{36}X_1^2 & \frac{7}{18}X_1X_2 - \frac{1}{27}t^2X_1 & \frac{2}{9}X_2^2 - \frac{2}{27}t^2X_2 & \frac{11}{18}tX_1 & X_3 + \frac{5}{9}tX_2 - \frac{4}{27}t^3 \end{bmatrix}.$$

Observe that $\lambda_{f_t} = \lambda_f$ for $t = 0$.

We remark that the time of execution to obtain this matrix in a computer equipped with 2 *Ghz* Core 2 duo processor and 4 *Gb* of RAM memory is less than 0.1 second.

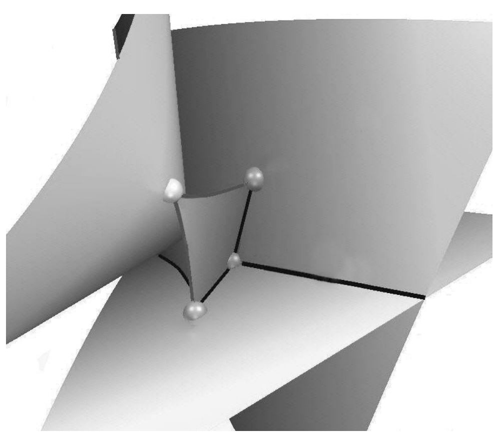
Using the command $fitting(\lambda, k)$ in the software Singular we compute the corresponding Fitting ideals.

$\mathcal{F}_0(f) = 46656X_3^5 + 3125X_1^6 + 22500X_1^4X_2X_3 + 43200X_1^2X_2^2X_3^2 + 32400tX_1^2X_3^3 + 13824X_2^3X_3^3 + 62208tX_2X_3^4 + 2000tX_1^4X_2^2 + 256X_1^2X_2^5 - 1500t^2X_1^43 + 10560tX_1^2X_2^3X_3 + 1024X_2^6X_3 - 6480t^2X_1^2X_2X_3^2 + 9216tX_2^4X_3^2 + 17280t^2X_2^2X_3^3 - 13824t^3X_3^4 - 900t^3X_1^4X_2 - 128t^2X_1^2X_2^4 - 4816t^3X_1^2X_2^2X_3 - 512t^2X_2^5X_3 - 192t^4X_1^2X_3^2 - 4352t^3X_2^3X_3^2 - 9216t^4X_2X_3^3 + 108t^5X_1^4 + 16t^4X_1^2X_2^3 + 576t^5X_1^2X_2X_3 + 64t^4X_2^4X_3 + 512t^5X_2^2X_3^2 + 1024t^6X_3^3$.

$\mathcal{F}_1(f) = 1875X_1^4 + 5400X_1^2X_2X_3 + 1728X_2^2X_3^2 + 1296tX_3^3 + 1440X_1^2X_2^2 + 256X_2^5 - 1080t^2X_1^2X_3 + 1344tX_2^3X_3 + 432t^2X_2X_3^2 - 616t^3X_1^2X_2 - 128t^2X_2^4 - 592t^3X_2^2X_3 - 192t^4X_3^2 + 72t^5X_1^2 + 16t^4X_2^3 + 64t^5X_2X_3$, $1125X_1^3X_3 + 2160X_1X_2X_3^2 - 100tX_1^3X_2 - 64X_1X_2^4 + 576tX_1X_2^2X_3 - 756t^2X_1X_3^2 + 15t^3X_1^3 + 32t^2X_1X_2^3 - 344t^3X_1X_2X_3 - 4t^4X_1X_2^2 + 48t^5X_1X_3$, $2700X_1^2X_3^2 + 2592X_2X_3^3 + 125tX_1^4 + 160X_1^2X_2^3 + 180tX_1^2X_2X_3 + 384X_2^4X_3 + 1728tX_2^2X_3^2 - 864t^2X_3^3 - 88t^2X_1^2X_2^2 + 44t^3X_1^2X_3 - 224t^2X_2^3X_3 - 960t^3X_2X_3^2 + 12t^4X_1^2X_2 + 32t^4X_2^2X_3 + 128t^5X_3^2$, $1620X_1X_3^3 - 100X_1^3X_2^2 + 150tX_1^3X_3 - 336X_1X_2^3X_3 + 216tX_1X_2X_3^2 + 65t^2X_1^3X_2 + 228t^2X_1X_2^2X_3 + 48t^3X_1X_3^2 - 9t^4X_1^3 - 32t^4X_1X_2X_3$, $3888X_3^4 + 250X_1^4X_2 + 1080X_1^2X_2^2X_3 + 540tX_1^2X_3^2 + 576X_2^3X_3^2 + 3024tX_2X_3^3 - 75t^2X_1^4 + 16tX_1^2X_2^3 - 216t^2X_1^2X_2X_3 + 64tX_2^4X_3 + 144t^2X_2^2X_3^2 - 576t^3X_3^3 - 4t^3X_1^2X_2^2 - 24t^4X_1^2X_3 - 16t^3X_2^3X_3 - 64t^4X_2X_3^2$.

$\mathcal{F}_2(f) = 75tX_1^2 + 96X_2^3 + 72tX_2X_3 - 56t^2X_2^2 - 24t^3X_3 + 8t^4X_2$, $375X_1^3 + 360X_1X_2X_3 + 80tX_1X_2^2 + 12t^2X_1X_3 - 12t^3X_1X_2$, $75X_1^2X_2 + 54tX_3^2 - 25t^2X_1^2 + 8tX_2^3 + 36t^2X_2X_3 - 2t^3X_2^2 - 8t^4X_3$, $60X_1X_2^2 - 45tX_1X_3 - 43t^2X_1X_2 + 6t^4X_1$, $225X_1^2X_3 - 20tX_1^2X_2 + 3t^3X_1^2$, $90X_1X_2X_3 - 4tX_1X_2^2 - 33t^2X_1X_3 + t^3X_1X_2$, $144X_2^2X_3 + 5t^2X_1^2 - 96t^2X_2X_3 + 16t^4X_3$, $540X_1X_3^2 + 25tX_1^3 + 36tX_1X_2X_3 + 8t^3X_1X_3$, $216X_2X_3^2 + 135tX_1^2X_2 + 32X_2^4 + 144tX_2^2X_3 - 33t^3X_1^2 - 8t^2X_2^3 - 32t^3X_2X_3$, $324X_3^3 + 20X_1^2X_2^2 + 15tX_1^2X_3 + 48X_2^3X_3 + 216tX_2X_3^2 - 5t^2X_1^2X_2 - 12t^2X_2^2X_3 - 48t^3X_3^2$, $192X_2^4 - 108t^2X_3^2 + 45t^3X_1^2 - 128t^2X_2^3 - 24t^3X_2X_3 + 20t^4X_2^2$.

With these ideals and using for example the Surfex program, we can visualize the multiple points set in the target in real time moving the parameter $t$. The Figure 5 shows the discriminant of $f_t$ for a fixed real value $t = t_0$.



Real part of the discriminant of $f_t(x, y, z) = (x, y, z^6 + xz + yz^2 + tz^4)$.

**Example 5.3** Using this algorithm we can compute the defining equation of the discriminant, or the ideal $\mathcal{F}_0(f)$ much faster than other methods, for example we compare with the *"Elimination of Variables method"*, which gives this defining equation using the command *eliminate* in the softwares Maple or Singular.

We see a description of this method in the proof of the Lemma 1.3. of [1] for the case of map germs $g : \mathbb{C}^2 \to \mathbb{C}^2$.

We give a description of this method for the case of map germs $f : \mathbb{C}^4 \to \mathbb{C}^4$.

Given a map germ $f = (f_1, f_2, f_3, f_4)$ do:

1. In the variables $\{x, y, z, w, X, Y, Z, W\}$, define the ideal $I$ generated by $\{X - f_1, Y - f_2, Z - f_3, W - f_4, Jf\}$, where $Jf$ denotes the determinant of matrix of the partial derivatives of $f$, and compute its standard basis.
2. In the standard basis of this ideal, find the unique function which depends of the variables $(X, Y, Z, W)$, denoted $G(X, Y, Z, W)$, this is the defining equation of $\Delta(f)$ in $\mathbb{C}^4$.

The maim problem of this algorithm is that it uses a huge amount of memory to compute the standard basis of the ideal $I$ and even for simple cases, using very powerful computers, the process stops.

For example, if we consider the map germ $f$ from $\mathbb{C}^4$ to $\mathbb{C}^4$ given by

$$f(x, y, z, w) = (x, y, z, w^{33} + z^{13}w^3 + y^8 w + x^3 w^8),$$

by using a computer equipped with 2.8 *Ghz* Intel Core *I7* processor with 8 *Gb* of RAM memory, the total time to find the matrix $\lambda_f$ of its presentation using the command Presentation_CnCn_Cor1([F]) is less than 0.1 second. To compute its $\mathcal{F}_0(f)$ Fitting ideal, the total time was 2.82 seconds, while using the comand *eliminate* in the software Maple, the total time spent is 460 seconds. Using the software Singular, in this computer it is not possible to obtain this ideal because there is no enough memory.

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