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STRONGLY DAMPED WAVE EQUATION AND ITS YOSIDA APPROXIMATIONS

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Abstract. In this work we study the continuity for the family of global attractors of the equations $u_{tt} - \Delta u - \Delta u_t - \epsilon \Delta u_{tt} = f(u)$ at $\epsilon = 0$ when Ω is a bounded smooth domain of $\mathbb{R}^n$, with $n \geqslant 3$, and the nonlinearity f satisfies a subcritical growth condition. Also, we obtain an uniform bound for the fractal dimension of these global attractors.

1. Introduction

We study the continuity of global attractors of the following semilinear evolution equation of second order in time

$$\begin{cases} u_{tt} - \Delta u - \Delta u_t - \epsilon \Delta u_{tt} = f(u), \ t > 0, \\ (u(0), u_t(0)) = (u_0, v_0), \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

and we give an uniform bound for the fractal dimension of these global attractors.

We know that, for $\epsilon = 0$, this equation is the usual strongly damped wave equation, and its asymptotic dynamics - related to global atrtactors - has already been vastly explored; see for instance [6, 7, 9, 12, 15, 22, 23, 26, 27, 28]. However, for each $\epsilon > 0$ fixed, we have a special form of the improved Boussinesq equation (see [4, 19, 20, 25]) with damping $-\Delta u_t$, which, among other things, is used to describe ion-sound waves in plasma (see [20, 21]).

For each $\epsilon > 0$ fixed, this equation has been studied in [8], in terms of existence and uniqueness of solutions, existence of global attractors and asymptotic bootstrapping; in this case, the linear part of the equation (after a change of variables) is a bounded operator. Here, since we want to study the continuity of attractors at $\epsilon = 0$, we will use the properties of the limiting problem with $\epsilon = 0$ (local and global well posedness, regularity and existence of global attractors) as reported in [6, 7].

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Throughout this paper, we will assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function, respecting a growth condition with subcritical exponent; that is, there exist constants $c > 0$ and $\rho < \frac{n+2}{n-2}$ such that for all $s_1, s_2 \in \mathbb{R}$

$$|f(s_1) - f(s_2)| \leqslant c|s_1 - s_2|(1 + |s_1|^{\rho-1} + |s_2|^{\rho-1}), \tag{1.2}$$

and also, if λ_1 denotes the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions in Ω, we assume the following dissipation condition

$$\limsup_{|s|\to\infty} \frac{f(s)}{s} < \lambda_1. \tag{1.3}$$

To begin our study, we will write further A for $-\Delta$ with the Dirichlet boundary conditions. Our problem then takes the form

$$\begin{cases} u_{tt} + Au + Au_t + \epsilon A u_{tt} = f(u), \ t > 0 \\ (u(0), v(0)) = (u_0, v_0). \end{cases} \tag{1.4}$$

and it is well-known that $A : H^1_0(\Omega) \cap H^2(\Omega) \subset L^2(\Omega) \longrightarrow L^2(\Omega)$ is a closed, densely define operator which has the following properties:

- **(O1)** A is self-adjoint with compact resolvent;
- **(O2)** A is an operator of positive type;
- **(O3)** $\sigma(A) = \sigma_p(A) = \{\lambda_n\}_{n\in\mathbb{N}}$, $\lambda_1 > 0$, $\lambda_i \leq \lambda_{i+1}$, for all $i \geqslant 1$ (repeated to take into account the multiplicity), $\lambda_n \overset{n\to\infty}{\to} \infty$ and if $v_n \in L^2(\Omega)$ are unitary eigenvectors associated with λ_n then $\{v_n\}_{n\in\mathbb{N}}$ constitutes an orthonormal basis for $L^2(\Omega)$.

Remark 1.1. *We included in Appendix A the proof of the main results of functional analysis we will use, in order to make explicit the uniformity of the constants obtained for* $\epsilon \in [0, 1]$.

The key point in our analysis is the observation that the differential equation in (1.4), for $\epsilon > 0$, can be obtained from its limit, for $\epsilon = 0$, with a suitable exchange of the unbounded operator A by its *Yosida approximation* Λ_ϵ (see definition below). The techniques developed here to deal with these singular perturbation problem may be of aid to deal with other natural singular perturbation problems that appear in the literature in this form (see for example the Navier-Stokes-Voight problem in [14]).

Definition 1.2. *Let A be a closed, densely defined operator such that $\mathbb{R}^+ \subset \rho(-A)$. Then, for each $\epsilon \in [0,1]$ we define the operator $\Lambda_\epsilon : D(\Lambda_\epsilon) \subset X \to X$, given by*

$$D(\Lambda_\epsilon) = \{x \in X : \ (I + \epsilon A)^{-1} x \in D(A)\},$$

and for $x \in D(\Lambda_\epsilon)$ we set

$$\Lambda_\epsilon x = A(I + \epsilon A)^{-1} x.$$

The operators Λ_ϵ are called Yosida approximations *of A.*

In fact the differential equation in (1.4) can be rewritten as $u_{tt} + \Lambda_\epsilon u + \Lambda_\epsilon u_t = (I + \epsilon A)^{-1} f(u)$ with $\Lambda_\epsilon u_0 \overset{\epsilon \to 0}{\longrightarrow} A u_0$ for all $u_0 \in D(A)$ and $(I + \epsilon A)^{-1} u_0 \overset{\epsilon \to 0}{\longrightarrow} u_0$ for all $u_0 \in X$. We exploit this feature and a suitable change of variables to fix (independently of ϵ) the phase space to carry on our analysis.

Now, if $X \doteq L^2(\Omega)$, we will consider the double sided fractional power scales

- $\{X^\alpha, \ \alpha \in \mathbb{R}\}$, generated by (X, A);
- $\{X^\alpha_\epsilon, \ \alpha \in \mathbb{R}\}_{\epsilon \in [0,1]}$, generated by (X, Λ_ϵ) (see Definition 1.2);
- $\{\tilde{X}^\alpha_\epsilon, \ \alpha \in \mathbb{R}\}_{\epsilon \in [0,1]}$ generated by $(X, I + \epsilon A)$;

where A, Λ_ϵ and $I + \epsilon A$ have domains X^1, X^1_ϵ and $\tilde{X}^1_\epsilon$, respectively, and are positive type operators.

Now we consider the following isometric isomorphism

$$\Phi_\epsilon : X^{\frac{1}{2}} \times \tilde{X}^{\frac{1}{2}}_\epsilon \longrightarrow X \times X$$

given in its matrix form by

$$\Phi_\epsilon = \begin{bmatrix} A^{\frac{1}{2}} & 0 \\ 0 & (I + \epsilon A)^{\frac{1}{2}} \end{bmatrix},$$

for each $\epsilon \in [0,1]$.

If we apply the change of variables $\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right] = \Phi_\epsilon \left[\begin{smallmatrix} u \\ u_t \end{smallmatrix}\right]$, problem (1.4) can be rewritten as

$$\begin{cases} (I + \epsilon A)^{\frac{1}{2}} z_t + A^{\frac{1}{2}} w + A(I + \epsilon A)^{-\frac{1}{2}} z = f(A^{-\frac{1}{2}} w) \\ w_t = A^{\frac{1}{2}} (I + \epsilon A)^{-\frac{1}{2}} z, \\ (w(0), z(0)) = (A^{\frac{1}{2}} u_0, (I + \epsilon A)^{\frac{1}{2}} v_0) \end{cases} \tag{1.5}$$

or

$$
\begin{cases}
z_t + A^{\frac{1}{2}}(I+\epsilon A)^{-\frac{1}{2}} w + A(I+\epsilon A)^{-1} z = (I+\epsilon A)^{-\frac{1}{2}} f(A^{-\frac{1}{2}} w) \\
w_t = A^{\frac{1}{2}}(I+\epsilon A)^{-\frac{1}{2}} z, \\
(w(0), z(0)) = (A^{\frac{1}{2}} u_0, (I+\epsilon A)^{\frac{1}{2}} v_0).
\end{cases} \tag{1.6}
$$

The later is a first order ODE that can writen in $X \times X$ as

$$
\begin{cases}
\dfrac{d}{dt} \left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right] + \mathcal{A}_\epsilon \left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right] = \mathcal{F}_\epsilon \left(\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right]\right) \text{ in } [0,\infty) \\
(w(0), z(0)) = (w_0, z_0),
\end{cases} \tag{1.7}
$$

where $(w_0, z_0) = \Phi_\epsilon(u_0, v_0)$, in variables (t, w, z), where $\mathcal{A}_\epsilon : D(\mathcal{A}_\epsilon) \subset X \times X \to X \times X$ is a linear operator given by

$$
D(\mathcal{A}_\epsilon) = \left\{ \begin{bmatrix} w \\ z \end{bmatrix} \in X \times X_\epsilon^{\frac{1}{2}} : \ w + \Lambda_\epsilon^{\frac{1}{2}} z \in X_\epsilon^{\frac{1}{2}} \right\},
$$

and

$$
\mathcal{A}_\epsilon \begin{bmatrix} w \\ z \end{bmatrix} = \begin{bmatrix} -\Lambda_\epsilon^{\frac{1}{2}} z \\ \Lambda_\epsilon^{\frac{1}{2}} (w + \Lambda_\epsilon^{\frac{1}{2}} z) \end{bmatrix}.
$$

Of course, if $\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right] \in X_\epsilon^{\frac{1}{2}} \times X_\epsilon^1$ we have that

$$
\mathcal{A}_\epsilon \begin{bmatrix} w \\ z \end{bmatrix} = \begin{bmatrix} -\Lambda_\epsilon^{\frac{1}{2}} z \\ \Lambda_\epsilon^{\frac{1}{2}} w + \Lambda_\epsilon z \end{bmatrix} = \begin{bmatrix} 0 & -\Lambda_\epsilon^{\frac{1}{2}} \\ \Lambda_\epsilon^{\frac{1}{2}} & \Lambda_\epsilon \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix},
$$

with $X_\epsilon^{\frac{1}{2}} \times X_\epsilon^1$ being a dense subset of $D(\mathcal{A}_\epsilon)$ and a locally Lipschitz map

$$
\mathcal{F}_\epsilon \left(\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} 0 \\ f_\epsilon^e(w) \end{smallmatrix}\right], \tag{1.8}
$$

where $f_\epsilon^e(w) = (I+\epsilon A)^{-\frac{1}{2}} f(A^{-\frac{1}{2}} w)$.

Remark 1.3. *It is important to notice that for each $\epsilon > 0$, $D(\mathcal{A}_\epsilon) = X \times X$ and $\mathcal{A}_\epsilon \in \mathcal{L}(X \times X)$. The characterization above becomes important when dealing with the case $\epsilon = 0$, since $\mathcal{A}_0$ is an unbounded operator. The primary concern of our work is to deal with the uniformity in $\epsilon \in [0,1]$ of the class of problems (1.4), hence placing the problems under the same framework is crucial.*

We divide our work from now on in six sections and an appendix. In Section 2 we deal with the linear problem associated with equation (1.7). More specifically, we prove that $-\mathcal{A}_\epsilon$ generates an analytic semigroup $\{e^{-\mathcal{A}_\epsilon} : \ t \geqslant 0\}$, and we obtain convergence in the uniform norm of operators of the associated semigroups when $\epsilon \to 0^+$ as follows:

Theorem 1.4. *For any $\alpha \in [0, \frac{1}{2})$ and $\gamma \in [0,1]$ there exists a constant $C_\gamma > 0$ such that*

$$\|e^{-\mathcal{A}_\epsilon t} - e^{-\mathcal{A}_0 t}\|_{\mathcal{L}(X\times X)} \leqslant C_\gamma \epsilon^{\alpha\gamma} t^{-\gamma} e^{-\omega_1 t},$$

for all $t > 0$. In particular, $e^{-\mathcal{A}_\epsilon t} \overset{\mathcal{L}(X\times X)}{\longrightarrow} e^{-\mathcal{A}_0 t}$ as $\epsilon \to 0^+$, with uniform convergence for any interval $[T, \infty)$, $T > 0$.

In Section 3 we prove local and global well posedness results for equation (1.1) and we deal with all the cases at once. For each $\epsilon > 0$, these results are contained in Theorems 1.1 and 1.2 of [8] as for the case $\epsilon = 0$ these results are contained in the results of Section 3 of [6]. To this end, a fine analysis of the fractional powers of the operators $-\mathcal{A}_\epsilon$ is required (such analysis is done in Subsection 2.2). The main results of this section can be summarized in the results below:

Theorem 1.5. *For any initial data $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]$ lying in a bounded subset $\mathcal{B}$ of $X^{\frac{1}{2}} \times \tilde{X}_\epsilon^{\frac{1}{2}}$ there exists a number $\kappa = \kappa(\mathcal{B}, \epsilon)$ and a unique solution $[0,\kappa) \ni t \mapsto \left[\begin{smallmatrix} u_\epsilon \\ v_\epsilon \end{smallmatrix}\right](t, u_0, v_0) \in X^{\frac{1}{2}} \times \tilde{X}_\epsilon^{\frac{1}{2}}$ of (1.4) which depends continuously on its variables $(t, u_0, v_0) \in [0,\kappa) \times X^{\frac{1}{2}} \times \tilde{X}_\epsilon^{\frac{1}{2}}$ and such that, for any $s \in \left[\frac{(\rho-1)(n-2)}{4}, 1\right]$ and $\gamma \in \left(0, 1 - \frac{s}{2}\right)$,*

$$\left[\begin{smallmatrix} u_\epsilon \\ v_\epsilon \end{smallmatrix}\right](\cdot, u_0, z_0) \in C\left((0,\tau), (X^{\frac{1}{2}} \times \tilde{X}_\epsilon^{\frac{1}{2}})^\gamma\right) \cap C^1\left((0,\tau), (X^{\frac{1}{2}} \times \tilde{X}_\epsilon^{\frac{1}{2}})^{\gamma-}\right),$$

and either $\kappa = \infty$ or $\|\left[\begin{smallmatrix} u_\epsilon \\ v_\epsilon \end{smallmatrix}\right](t, u_0, v_0)\|_{X^{\frac{1}{2}} \times \tilde{X}_\epsilon^{\frac{1}{2}}} \to \infty$ as $t \to \kappa^-$.

Moreover, the solution satisfies in $X^{\frac{1}{2}} \times \tilde{X}_\epsilon^{\frac{1}{2}}$ the variation of constants formula

$$\left[\begin{smallmatrix} u_\epsilon \\ v_\epsilon \end{smallmatrix}\right](t, w_0, z_0) = e^{-\tilde{\mathcal{A}}_\epsilon t}\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] + \int_0^t e^{-\tilde{\mathcal{A}}_\epsilon (t-s)} \mathcal{G}_\epsilon\left(\left[\begin{smallmatrix} u_\epsilon \\ v_\epsilon \end{smallmatrix}\right](s, u_0, v_0)\right) ds, \quad t \in [0,\kappa),$$

where

$$\mathcal{G}_\epsilon\left(\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]\right) = \Phi_{0,\epsilon}^{-1} \mathcal{F}_\epsilon \Phi_{0,\epsilon}\left(\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]\right).$$

Theorem 1.6. *Problem (1.1) defines a C^0-semigroup $\{S_\epsilon(t) : \ t \geqslant 0\}$ on $X^{\frac{1}{2}} \times \tilde{X}_\epsilon^{\frac{1}{2}}$ for each $\epsilon \in [0,1]$, which has bounded orbits of bounded sets, defined by*

$$S_\epsilon(t)\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] = \Phi_\epsilon^{-1} T_\epsilon(t) \Phi_\epsilon \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right],$$

or equivalently

$$S_\epsilon(t)\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] = e^{-\tilde{\mathcal{A}}_\epsilon t}\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] + \int_0^t e^{-\tilde{\mathcal{A}}_\epsilon (t-s)} \mathcal{G}_\epsilon\left(S_\epsilon(s)\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]\right) ds, \ \textit{for all } t \geqslant 0.$$

In Section 4 we prove the existence of global attractors for the semigroups $\{S_\epsilon(t) : \ t \geqslant 0\}$ generated by equations (1.1), which is given by

Theorem 1.7. *The semigroup $\{S_\epsilon(t) : t \geqslant 0\}$ has a global attractor $\tilde{\mathbb{A}}_\epsilon$ in $X^{\frac{1}{2}} \times \tilde{X}_\epsilon^{\frac{1}{2}}$, for each $\epsilon \in [0,1]$.*

In [8] the authors prove the existence of global attractors for each $\epsilon > 0$ and also provide bounds (ϵ dependent) for the global attractors (see Theorem 1.3 of this reference). In [9] they prove the same for the case $\epsilon = 0$ (see the results of Subsection 4.2 in this reference); however, simply joining the results would not lead to a uniform bound for $\epsilon \in [0,1]$. We also prove the following

Theorem 1.8. *If $s \in \left[0, 1 - \frac{(\rho-1)(n-2)}{4}\right)$, then $\cup_{\epsilon\in[0,1]}\tilde{\mathbb{A}}_\epsilon$ is bounded in $X^{\frac{s+1}{2}} \times X^{\frac{s}{2}}$.*

In Section 5 we are able to prove the upper semicontinuity of the global attractors $\{\tilde{\mathbb{A}}_\epsilon\}_{\epsilon\in[0,1]}$ at $\epsilon = 0$:

Theorem 1.9. *The family $\{\tilde{\mathbb{A}}_\epsilon\}_{\epsilon\in[0,1]}$ is upper semicontinuous in $\epsilon = 0$, in $X^{\frac{1}{2}} \times X$.*

This result was also proven in [24], using a different technique, dealing with energy estimates of solutions (see Lemma 5.12 in this reference). Under (natural) additional assumptions we can also prove the lower semicontinuity

Theorem 1.10. *Assume that f is a C^2 function on $\mathbb{R}$ with f, f' and f'' bounded in $\mathbb{R}$. Also, assume that the set $\mathcal{E}$ of equilibrium points of (1.7) is finite and that each point of $\mathcal{E}$ is a hyperbolic point for (1.7) with $\epsilon = 0$. Then the family of global attractors $\{\tilde{\mathbb{A}}_\epsilon\}_{\epsilon\in[0,1]}$ is lower semicontinuous at $\epsilon = 0$.*

Lastly, in Section 6 using some further uniform estimates for the semigroup generated by equation (1.7) we obtain an uniform estimate for the fractal dimension $c(\tilde{\mathbb{A}}_\epsilon)$ of the global attractor $\tilde{\mathbb{A}}_\epsilon$.

Theorem 1.11. *There exists a number $\tau_0 > 0$ such that for any $\epsilon \in [0,1]$*

$$c(\tilde{\mathbb{A}}_\epsilon) \leqslant \tau_0.$$

In [24, Lemma 5.10], the authors prove an estimate for the fractal dimension of the global attractors using exponential attractors, but the bound depends on $\epsilon \in [0,1]$.

Remark 1.12. *We note that, most of our results are proved using techniques from functional analysis, resorting to energy estimates when is absolutely necessary. We were able to obtain some fine estimates using a bootstrapping argument in the subcritical case. This equation has been considered* *in [11], where they proved the upper semicontinuity of the global attractors of (1.1) as well as to obtain bounds for the fractal dimension of the attractors, but is not uniform in $\epsilon \in [0,1]$. Here we also prove the lower semicontinuity of the global attractors, besides recovering the upper semicontinuity and obtaining uniform (w.r.t. ϵ) bounds for the dimension using a different technique.*

2. The linear problem and the uniform convergence of the linear semigroups

In this section we study the linear problem associated with equations (1.7) in $X \times X$, given by

$$\begin{cases} \dfrac{d}{dt}\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right] + \mathcal{A}_\epsilon \left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right] = 0, \ t > 0 \\ \begin{bmatrix} w(0) \\ z(0) \end{bmatrix} = \left[\begin{smallmatrix} w_0 \\ z_0 \end{smallmatrix}\right] \in X \times X \end{cases}$$

more precisely, we will prove that the family of operators $\{\mathcal{A}_\epsilon\}_{\epsilon\in[0,1]}$ is *uniformly sectorial*; that is, we can find $\phi \in (0, \frac{\pi}{2})$, $M \geqslant 1$ and a real number ω such that the sector

$$S_{\omega,\phi} = \{\lambda \in \mathbb{C} : \phi \leqslant |\arg(\lambda - \omega)| \leqslant \pi, \lambda \neq \omega\}$$

is in the resolvent set of $\mathcal{A}_\epsilon$ for all $\epsilon \in [0,1]$ and

$$\|(\lambda - \mathcal{A}_\epsilon)^{-1}\|_{\mathcal{L}(X\times X)} \leqslant \frac{M}{|\lambda - \omega|}, \text{ for all } \lambda \in S_{\omega,\phi},$$

and moreover we will prove that we can take $\omega < 0$, which will give us an uniform exponential decay for the generated analytic semigroups $\{e^{-\mathcal{A}_\epsilon t} : \ t \geqslant 0\}_{\epsilon\in[0,1]}$.

2.1. **Uniform sectoriality.** In this subsection, our goal is to prove the uniform sectoriality of $\{\mathcal{A}_\epsilon\}_{\epsilon\in[0,1]}$ in order to obtain a convergence of the generated linear semigroups $\{e^{-\mathcal{A}_\epsilon t} : \ t \geqslant 0\}_{\epsilon\in[0,1]}$ as $\epsilon \to 0^+$.

First we begin obtaining an uniform decay in time for the generated semigroups, and to this purpose we define the notations of the inner products we will use throughout our work.

Definition 2.1. *In X we denote the usual inner product $(\cdot,\cdot)$ and in $X \times X$ we use the inner product $\langle\cdot,\cdot\rangle$ given by*

$$\langle \left[\begin{smallmatrix} w_1 \\ z_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} w_2 \\ z_2 \end{smallmatrix}\right] \rangle \doteq (w_1, w_2) + (z_1, z_2).$$

With this notation set, we are able define for each pair $(\epsilon, \beta) \in [0,1]\times[0,1]$, a map from $(X\times X)^2$ into $\mathbb{C}$ by

$$\langle \left[\begin{smallmatrix} w_1 \\ z_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} w_2 \\ z_2 \end{smallmatrix}\right] \rangle_{\epsilon,\beta} = \langle \left[\begin{smallmatrix} w_1 \\ z_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} w_2 \\ z_2 \end{smallmatrix}\right] \rangle + \frac{\beta}{2}(w_1, \Lambda_\epsilon^{-\frac{1}{2}} z_2) + \frac{\beta}{2}(z_1, \Lambda_\epsilon^{-\frac{1}{2}} w_2).$$

In what follows we will need a result of basic functional analysis, that we state below.

Proposition 2.2. *The family of operators* $\{\Lambda_\epsilon^{-\beta}\}_{(\epsilon,\beta)\in[0,1]\times[0,1]}$ *is uniformly bounded. In particular, there exists a constant* $\mu>0$ *such that*

$$(\Lambda_\epsilon^{\beta}x,x)\geqslant \mu\|x\|_X^2,\ \textit{for all } x\in X_\epsilon^1.$$

Proof: See Appendix A.

With this result we can prove a uniform equivalence between $\langle\cdot,\cdot\rangle_{\epsilon,\beta}$ and $\langle\cdot,\cdot\rangle$.

Proposition 2.3. *For all* $(\epsilon,\beta)\in[0,1]\times[0,1]$*, if we define* $\|\left[\begin{smallmatrix} w\\ z\end{smallmatrix}\right]\|_{\epsilon,\beta}^2 \doteq \langle\left[\begin{smallmatrix} w\\ z\end{smallmatrix}\right],\left[\begin{smallmatrix} w\\ z\end{smallmatrix}\right]\rangle_{\epsilon,\beta}$*, we have*

$$\left[1-\frac{\beta\mu}{2}\right]\|\left[\begin{smallmatrix} w\\ z\end{smallmatrix}\right]\|_{X\times X}^2\leqslant \|\left[\begin{smallmatrix} w\\ z\end{smallmatrix}\right]\|_{\epsilon,\beta}^2\leqslant \left[1+\frac{\beta\mu}{2}\right]\|\left[\begin{smallmatrix} w\\ z\end{smallmatrix}\right]\|_{X\times X}^2.$$

Proof: We have, since $\Lambda_\epsilon^{-\frac12}$ is self-adjoint, that

$$\langle\left[\begin{smallmatrix} w\\ z\end{smallmatrix}\right],\left[\begin{smallmatrix} w\\ z\end{smallmatrix}\right]\rangle_{\epsilon,\beta}=\langle\left[\begin{smallmatrix} w\\ z\end{smallmatrix}\right],\left[\begin{smallmatrix} w\\ z\end{smallmatrix}\right]\rangle+\beta\mathrm{Re}(w,\Lambda_\epsilon^{-\frac12}z)=\|\left[\begin{smallmatrix} w\\ z\end{smallmatrix}\right]\|_{X\times X}^2+\beta\mathrm{Re}(w,\Lambda_\epsilon^{-\frac12}z),$$

but

$$\begin{aligned}|\mathrm{Re}(w,\Lambda_\epsilon^{-\frac12}z)|&\leqslant|(w,\Lambda_\epsilon^{-\frac12}z)|\leqslant\|w\|_X\|\Lambda_\epsilon^{-\frac12}z\|_X\\&\leqslant\|\Lambda_\epsilon^{-\frac12}\|_{\mathcal{L}(X)}\|w\|_X\|z\|_X\leqslant\frac{\|\Lambda_\epsilon^{-\frac12}\|_{\mathcal{L}(X)}}{2}\|\left[\begin{smallmatrix} w\\ z\end{smallmatrix}\right]\|_{X\times X}^2,\end{aligned}$$

By Proposition 2.2, $\|\Lambda_\epsilon^{-\frac12}\|_{\mathcal{L}(X)}\leqslant\mu$ and hence

$$|\mathrm{Re}(w,\Lambda_\epsilon^{-\frac12}z)|\leqslant\frac{\mu}{2}\|\left[\begin{smallmatrix} w\\ z\end{smallmatrix}\right]\|_{X\times X}^2,$$

which concludes the proof. ∎

Corollary 2.4. *There exists* $\beta_0\in(0,1]$ *such that* $\langle\cdot,\cdot\rangle_{\epsilon,\beta}$ *is an inner product in* $X\times X$ *for all* $(\epsilon,\beta)\in[0,1]\times[0,\beta_0]$.

Proof: Almost all the properties of an inner product are easily verified; and for the coercivity it suffices to choose $\beta_0\in(0,1]$ such that $1-\frac{\beta_0\mu}{2}>0$ in the previous proposition. ∎

So far we are able to construct uniform equivalent norms in $X\times X$ and the next step is to prove that there exists a positive constant $\delta>0$ such that $\mathcal{A}_\epsilon-\delta I$ is acretive, for all $\epsilon\in[0,1]$.

Proposition 2.5. *There exist* $\beta_1\in(0,\beta_0]$ *and a constant* $\delta>0$ *such that*

$$\mathrm{Re}\,\langle(\mathcal{A}_\epsilon-\delta I)\left[\begin{smallmatrix} w\\ z\end{smallmatrix}\right],\left[\begin{smallmatrix} w\\ z\end{smallmatrix}\right]\rangle_{\epsilon,\beta_1}\geqslant 0,$$

for all $\epsilon\in[0,1]$ *and* $\left[\begin{smallmatrix} w\\ z\end{smallmatrix}\right]\in D(\mathcal{A}_\epsilon)$.

Proof: We have for $\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right] \in X_\epsilon^{\frac{1}{2}} \times X_\epsilon^1$ that

$$\left\langle \mathcal{A}_\epsilon \left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right], \left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right] \right\rangle_{\epsilon\beta} = \left\langle \left[\begin{smallmatrix} -\Lambda_\epsilon^{\frac{1}{2}} z \\ \Lambda_\epsilon^{\frac{1}{2}} w + \Lambda_\epsilon z \end{smallmatrix}\right], \left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right] \right\rangle_{\epsilon,\beta}$$
$$= -(\Lambda_\epsilon^{\frac{1}{2}} z, w) + (\Lambda_\epsilon^{\frac{1}{2}} w, z) + (\Lambda_\epsilon z, z) - \frac{\beta}{2}(z,z) + \frac{\beta}{2}(w,w) + \frac{\beta}{2}(\Lambda_\epsilon^{\frac{1}{2}} w, z),$$

which implies, since $\Lambda_\epsilon^{\frac{1}{2}}$ is self-adjoint, that

$$\mathrm{Re}\left\langle \mathcal{A}_\epsilon \left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right], \left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right] \right\rangle_\epsilon = \|\Lambda_\epsilon^{\frac{1}{2}} z\|_X^2 - \frac{\beta}{2}\|z\|_X + \frac{\beta}{2}\|w\|_X^2 + \frac{\beta}{2}\mathrm{Re}(\Lambda_\epsilon^{\frac{1}{2}} z, w).$$

But $|\mathrm{Re}(\Lambda_\epsilon^{\frac{1}{2}} z, w)| \leqslant \frac{1}{2}(\|\Lambda_\epsilon^{\frac{1}{2}} z\|_X^2 + \|w\|_X^2)$ and hence

$$\mathrm{Re}\left\langle \mathcal{A}_\epsilon \left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right], \left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right] \right\rangle_\epsilon \geqslant (1 - \frac{\beta}{4})\|\Lambda_\epsilon^{\frac{1}{2}} z\|_X^2 - \frac{\beta}{2}\|z\|_X^2 + \frac{\beta}{4}\|w\|_X^2$$
$$\geqslant \left[(1 - \frac{\beta}{4})\mu^2 - \frac{\beta}{2}\right]\|z\|_X^2 + \frac{\beta}{4}\|w\|_X^2.$$

Now we choose $\beta_1 \in (0, \beta_0]$ such that $(1 - \frac{\beta_1}{2})\mu^2 - \frac{\beta_1}{2} > 0$ and thus, by Proposition 2.3, we have

$$\mathrm{Re}\left\langle \mathcal{A}_\epsilon \left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right], \left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right] \right\rangle_{\epsilon,\beta_1} \geqslant \delta \left\langle \left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right], \left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right] \right\rangle_{\epsilon,\beta_1},$$

where $\delta = (1 + \frac{\beta_1 \mu_1}{2})^{-1} \min\{(1 - \frac{\beta_1}{2})\mu_1^2 - \frac{\beta_1}{2}, \frac{\beta}{4}\} > 0$, and therefore

$$\mathrm{Re}\left\langle (\mathcal{A}_\epsilon - \delta I) \left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right], \left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right] \right\rangle_{\epsilon,\beta_1} \geqslant 0.$$

■

From this we conclude that each operator $\delta I - \mathcal{A}_\epsilon$ generates a strongly continuous semigroup of contractions in $X \times X$ with the norm $\|\cdot\|_{\epsilon,\beta_1}$, which in turn, using Proposition 2.3, lead us to the following result:

Theorem 2.6. *There exist constants $M \geqslant 1$ and $\delta > 0$, such that*

$$\|e^{-\mathcal{A}_\epsilon t}\|_{\mathcal{L}(X \times X)} \leqslant M e^{-\delta t}, \text{ for all } t \geqslant 0 \text{ and } \epsilon \in [0,1].$$

Proof: Since $\left\|e^{(\delta I - \mathcal{A}_\epsilon)t} \left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right]\right\|_{\epsilon,\beta_1} \leqslant \|\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right]\|_{\epsilon,\beta_1}$, Proposition 2.3 imples that there exists $M \geqslant 1$ such that

$$\left\|e^{(\delta I - \mathcal{A}_\epsilon)t} \left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right]\right\|_{X \times X} \leqslant M \|\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right]\|_{X \times X},$$

and concludes the proof. ■

Corollary 2.7. *Given $\varphi \in (\frac{\pi}{2}, \pi)$, there exists a constant $M_\varphi \geqslant 1$ such that*

$$\|(\lambda - \mathcal{A}_\epsilon)^{-1}\|_{\mathcal{L}(X \times X)} \leqslant \frac{M_\varphi}{|\lambda - \delta|}, \text{ for all } \lambda \in S_{\delta,\varphi} \text{ and } \epsilon \in [0,1].$$

Proof: From the Inverse Laplace Transform, we know that

$$(\lambda - \mathcal{A}_\epsilon)^{-1} = -\int_0^\infty e^{\lambda t} e^{-\mathcal{A}_\epsilon t} dt,$$

for all $\lambda \in \mathbb{C}$ such that $\mathrm{Re}\lambda < \delta$ and therefore

$$\|(\lambda - \mathcal{A}_\epsilon)^{-1}\|_{\mathcal{L}(X\times X)} \leqslant \frac{M}{\delta - \mathrm{Re}\lambda},$$

for all $\epsilon \in [0,1]$. Now, given $\varphi \in (\pi/2, \pi)$, we have that

$$\|(\lambda - \mathcal{A}_\epsilon)^{-1}\|_{\mathcal{L}(X\times X)} \leqslant \frac{M}{|\cos\varphi|}\frac{1}{|\lambda - \delta|},$$

for all $\lambda \in S_{\delta,\varphi}$ and $\epsilon \in [0,1]$. ∎

So far we have proven that each $-\mathcal{A}_\epsilon$ generates a strongly-continuous semigroup in $X \times X$ (which for $\epsilon > 0$ is trivial, since $\mathcal{A}_\epsilon$ is bounded in $X\times X$) and furthermore we proved an uniform exponential decay for the generated semigroups for $\epsilon \in [0,1]$. But we would like to prove the convergence of $e^{-\mathcal{A}_\epsilon t}$ to $e^{-\mathcal{A}_0 t}$ in $\mathcal{L}(X\times X)$ as $\epsilon \to 0^+$, and to this purpose, we will need to work a little more.

For $\epsilon \in [0,1]$ define $D(\mathcal{B}_\epsilon) = D(\mathcal{A}_\epsilon)$, $D(\mathcal{P}_\epsilon) = X \times X$ and

$$\mathcal{B}_\epsilon \doteq \mathcal{A}_\epsilon + \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}, \ \mathcal{P}_\epsilon \doteq \begin{bmatrix} I & 0 \\ \Lambda_\epsilon^{-\frac{1}{2}} & I \end{bmatrix},$$

so that

$$\mathcal{P}_\epsilon^{-1} = \begin{bmatrix} I & 0 \\ -\Lambda_\epsilon^{-\frac{1}{2}} & I \end{bmatrix}.$$

Also, if we set $D(\mathcal{D}_\epsilon) = \left\{ \begin{bmatrix} w \\ z \end{bmatrix} \in X \times X : \mathcal{P}_\epsilon^{-1} \begin{bmatrix} w \\ z \end{bmatrix} \in D(\mathcal{B}_\epsilon) \right\}$, we can define

$$\mathcal{D}_\epsilon \doteq \mathcal{P}_\epsilon \mathcal{B}_\epsilon \mathcal{P}_\epsilon^{-1}.$$

Remark 2.8. *It is simple to see that* $D(\mathcal{D}_\epsilon) = X \times X_\epsilon^1$ *and hence*

$$\mathcal{D}_\epsilon = \begin{bmatrix} I & -\Lambda_\epsilon^{\frac{1}{2}} \\ 0 & \Lambda_\epsilon \end{bmatrix}.$$

Finally, define $D(\tilde{\mathcal{D}}_\epsilon) = D(\mathcal{D}_\epsilon) = X \times X_\epsilon^1$ and

$$\tilde{\mathcal{D}}_\epsilon \doteq \begin{bmatrix} I & 0 \\ 0 & \Lambda_\epsilon \end{bmatrix}.$$

For what follows we will need the definition and one result concerning the numerical range of an operator, which are given below.

Definition 2.9. *If $\mathcal{B} : D(B) \subset Z \to Z$ is a closed densely defined operator in a complex Hilbert space Z with inner product $\langle \cdot, \cdot \rangle$, then the* numerical range $W(B)$ *of B is the set*

$$W(B) = \{\langle Bz, z \rangle : \ z \in D(B), \ \|z\|_Z = 1\}.$$

Theorem 2.10. *Let $B : D(B) \subset Z \to Z$ be a closed densely defined operator in a complex Hilbert space Z, $W(B)$ its numerical range and Σ an open connected set in $\mathbb{C} \setminus \overline{W(B)}$. If $\Sigma \cap \rho(B) \neq \varnothing$ then $\Sigma \subset \rho(B)$ and*

$$\|(\lambda - B)^{-1}\|_{\mathcal{L}(Z)} \leqslant \frac{1}{\mathrm{d}(\lambda, \overline{W(B)})}, \ \textit{for all } \lambda \in \Sigma,$$

where $\mathrm{d}(\lambda, \overline{W(B)})$ is the distance between λ and $\overline{W(B)}$.

Proof: See Theorem 21.11 of [3].

With this result at hand, we can prove our first lemma.

Lemma 2.11. *The operators $\tilde{\mathcal{D}}_\epsilon : D(\tilde{\mathcal{D}}_\epsilon) \subset X \times X \longrightarrow X \times X$ constitute a family of uniformly sectorial operators.*

Proof: Using again Proposition 2.2, there exists $\mu > 0$ such that for all $z \in X^1_\epsilon$ we have

$$(\Lambda_\epsilon z, z) \geqslant \mu(z, z).$$

Thus, for $\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right] \in D(\tilde{\mathcal{D}}_\epsilon)$ we obtain

$$\left\langle \tilde{\mathcal{D}}_\epsilon \left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right], \left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right] \right\rangle = (w, w) + (\Lambda_\epsilon z, z) \geqslant (w, w) + \mu(z, z) \geqslant \tilde{\mu} \left\langle \left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right], \left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right] \right\rangle,$$

where $\tilde{\mu} = \min\{1, \mu\} > 0$ and therefore the numerical image $W(\tilde{\mathcal{D}}_\epsilon)$ is contained in $[\tilde{\mu}, \infty)$, for all $\epsilon \in [0, 1]$. Defining $\Sigma \doteq \mathbb{C} \setminus [\tilde{\mu}, \infty)$ we have that $0 \in \Sigma \cap \rho(\tilde{\mathcal{D}}_\epsilon)$ for all $\epsilon \in [0, 1]$ and hence, by Theorem 2.10, $\Sigma \subset \rho(\tilde{\mathcal{D}}_\epsilon)$, for all $\epsilon \in [0, 1]$, and

$$\|(\lambda - \tilde{\mathcal{D}}_\epsilon)^{-1}\|_{\mathcal{L}(X \times X)} \leqslant \frac{1}{\mathrm{d}(\lambda, W(\tilde{\mathcal{D}}_\epsilon))} \leqslant \frac{1}{\mathrm{d}(\lambda, [\tilde{\mu}, \infty))}, \text{ for all } \lambda \in \Sigma.$$

Now given $\phi \in (0, \pi/2)$, if $\lambda \in S_{\tilde{\mu}, \phi}$ we have that

$$\mathrm{d}(\lambda, [\tilde{\mu}, \infty)) \geqslant |\lambda - \tilde{\mu}| \sin \phi,$$

and hence

$$\|(\lambda - \tilde{\mathcal{D}}_\epsilon)^{-1}\|_{\mathcal{L}(X \times X)} \leqslant \frac{1}{\sin \phi |\lambda - \tilde{\mu}|}, \text{ for all } \lambda \in S_{\tilde{\mu}, \phi} \text{ and } \epsilon \in [0, 1].$$

∎

To continue, we will need well know results in functional analysis, concerning interpolation of fractional powers of an operator, which we will state below.

Definition 2.12. *Let $C \geqslant 1$. A closed densely defined linear operator $B : D(B) \subset Z \to Z$ is said an* operator of positive type with constant C *if* $[0,\infty) \in \rho(-B)$ *and*

$$(1+s)\|(s+B)^{-1}\|_{\mathcal{L}(Z)} \leqslant C, \text{ for all } s \in [0,\infty).$$

The set of all operators of positive type in Z with constant C will be denoted by $\mathscr{P}_C(Z)$.

Proposition 2.13. *Let A be an operator of positive type with constant C in X, then the Yosida approximations Λ_ϵ of A are positive type operators with constant $1+C$, for all $\epsilon \in [0,1]$.*

Proof: See Appendix A.

Theorem 2.14. *Assume that $B \in \mathscr{P}_C(Z)$ and $0 \leqslant \alpha \leqslant 1$, then there exists a constant $K > 0$ such that*

$$\|B^\alpha z\|_Z \leqslant K\|Bz\|_Z^\alpha \|z\|_Z^{1-\alpha}, \text{ for all } z \in D(B);$$

moreover, the constant K depends only on the constant C and not on the particular operator B.

Proof: See Theorem 1.4.4 of [17].

Corollary 2.15. *There exists a constant $K > 0$, independent of $\epsilon \in [0,1]$, such that if $0 \leqslant \alpha \leqslant 1$ we have*

$$\|\Lambda_\epsilon^\alpha x\|_X \leqslant K\|\Lambda_\epsilon x\|_X^\alpha \|x\|_X^{1-\alpha}, \text{ for all } x \in X_\epsilon^1.$$

Lemma 2.16. *The operators $\mathcal{D}_\epsilon : D(\mathcal{D}_\epsilon) \subset X \times X \longrightarrow X \times X$ constitute a family of uniformly sectorial operators.*

Proof: We have that $\tilde{\mathcal{D}}_\epsilon - \mathcal{D}_\epsilon = \left[\begin{smallmatrix} 0 & \Lambda_\epsilon^{\frac{1}{2}} \\ 0 & 0 \end{smallmatrix}\right]$, and hence for all $\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right] \in X \times X_\epsilon^1$

$$\begin{aligned}\|(\tilde{\mathcal{D}}_\epsilon - \mathcal{D}_\epsilon)\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right]\|_{X\times X} &= \|\Lambda_\epsilon^{\frac{1}{2}} z\|_X \leqslant K\|\Lambda_\epsilon z\|_X^{\frac{1}{2}}\|z\|_X^{\frac{1}{2}} \\ &\leqslant \frac{K\eta}{2}\|\Lambda_\epsilon z\|_X + \frac{K}{2\eta}\|z\|_X,\end{aligned}$$

for all $\eta > 0$, where $K > 0$ is the constant given in Theorem 2.14, which is independent of $\epsilon \in [0,1]$ and therefore

$$\|(\tilde{\mathcal{D}}_\epsilon - \mathcal{D}_\epsilon)\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right]\|_{X\times X} \leqslant \frac{K\eta}{2}\|\tilde{\mathcal{D}}_\epsilon\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right]\|_{X\times X} + \frac{K}{2\eta}\|\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right]\|_{X\times X}.$$

By Theorem 1.3.2 of [17] and Lemma 2.11 we have that the family $\{\mathcal{D}_\epsilon\}_{\epsilon\in[0,1]}$ is uniformly sectorial.

■

Lemma 2.17. *The operators $\mathcal{B}_\epsilon : D(\mathcal{B}_\epsilon) \subset X \times X \longrightarrow X \times X$ constitute a family of uniformly sectorial operators.*

Proof: We have for all $\lambda \in \mathbb{C}$ that

$$(\lambda - \mathcal{D}_\epsilon) = \mathcal{P}_\epsilon(\lambda - \mathcal{B}_\epsilon)\mathcal{P}_\epsilon^{-1},$$

hence $\rho(\mathcal{B}_\epsilon) = \rho(\mathcal{D}_\epsilon)$, and since the operators $\mathcal{P}_\epsilon, \mathcal{P}_\epsilon^{-1}$ are uniformly bounded in $X \times X$ (see Proposition 2.2), Lemma 2.16 implies that $\{\mathcal{B}_\epsilon\}_{\epsilon\in[0,1]}$ is uniformly sectorial. ■

Theorem 2.18. *The operators $\mathcal{A}_\epsilon : D(\mathcal{A}_\epsilon) \subset X \times X \longrightarrow X \times X$ constitute a family of uniformly sectorial operators.*

Proof: Since

$$\left\| \left[\begin{smallmatrix} 0 & 0 \\ 0 & I \end{smallmatrix}\right] \left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right] \right\|_{X\times X} \leqslant \eta \|\mathcal{B}_\epsilon \left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right]\|_{X\times X} + \|\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right]\|_{X\times X},$$

for all $\eta > 0$, Lemma 2.17 and Theorem 1.3.2 of [17] imply that $\{\mathcal{A}_\epsilon\}_{\epsilon\in[0,1]}$ is uniformly sectorial. ■

So far, with our efforts, Theorem 2.18 implies the existence of constants $M \geqslant 1$, $\omega \in \mathbb{R}$ and $\phi \in (0, \pi/2)$ such that

$$\|(\lambda - \mathcal{A}_\epsilon)^{-1}\|_{\mathcal{L}(X\times X)} \leqslant \frac{M}{|\lambda - \omega|}, \text{ for all } \lambda \in S_{\omega,\phi} \text{ and } \epsilon \in [0,1],$$

but $\omega \in \mathbb{R}$ can be a negative real number (and using the results reported in [17], we can see that the number $\omega \in \mathbb{R}$ obtained is, in fact, negative), which does not guarantee an uniform exponential decay for the generated semigroups. But these results together with Corollary 2.7 give us conditions to obtain the desired uniform sectoriality of $\{\mathcal{A}_\epsilon\}_{\epsilon\in[0,1]}$ with a uniform exponential decay:

Theorem 2.19. *There exist constants $M \geqslant 1$, $\omega > 0$ and $\varphi \in (0, \pi/2)$ such that $\rho(\mathcal{A}_\epsilon) \supset S_{\omega,\varphi}$ and*

$$\|(\lambda - \mathcal{A}_\epsilon)^{-1}\|_{\mathcal{L}(X\times X)} \leqslant \frac{M}{|\lambda - \omega|},$$

for all $\lambda \in S_{\omega,\varphi}$ and $\epsilon \in [0,1]$.

Proof: This follows from Corollary 2.7 and Theorem 2.18. ■

Corollary 2.20. *$-\mathcal{A}_\epsilon$ is the infinitesimal generator of an analytic semigroup $\{e^{-\mathcal{A}_\epsilon t} : t \geqslant 0\}$ for each $\epsilon \in [0,1]$ and*

$$e^{-\mathcal{A}_\epsilon t} = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda + \mathcal{A}_\epsilon)^{-1} d\lambda, \textit{ for all } \epsilon \in [0,1],$$

where Γ is a contour in $-S_{\delta,\omega}$ such that $\arg(\lambda) \to \pm\theta$ as $|\lambda| \to \infty$ for some $\theta \in (\frac{\pi}{2}, \pi)$.

Proof: See Theorem 1.3.4 of [17]. ■

Corollary 2.21. *Given $\omega_1 \in (0,\omega)$, there exists constant $M_{\omega_1} \geqslant 1$ such that*

$$\|\mathcal{A}_\epsilon(\lambda - \mathcal{A}_\epsilon)^{-1}\|_{\mathcal{L}(X\times X)} \leqslant M_{\omega_1}, \text{ for all } \lambda \in S_{\omega_1,\varphi} \text{ and } \epsilon \in [0,1].$$

Proof: We have that $\mathcal{A}_\epsilon(\lambda - \mathcal{A}_\epsilon)^{-1} = \lambda(\lambda - \mathcal{A}_\epsilon)^{-1} - I$ and hence

$$\|\mathcal{A}_\epsilon(\lambda - \mathcal{A}_\epsilon)^{-1}\|_{\mathcal{L}(X\times X)} \leqslant |\lambda| \|(\lambda - \mathcal{A}_\epsilon)^{-1}\|_{\mathcal{L}(X\times X)} + 1 \leqslant M\frac{|\lambda|}{|\lambda - \omega|} + 1.$$

Now, for each $\omega_1 \in (0,\omega)$, the map $S_{\omega_1,\varphi} \ni \lambda \mapsto \frac{\lambda}{\lambda-\omega}$ is bounded, hence there exists M_{ω_1} such that

$$\|\mathcal{A}_\epsilon(\lambda - \mathcal{A}_\epsilon)^{-1}\|_{\mathcal{L}(X\times X)} \leqslant M_{\omega_1}, \text{ for all } \lambda \in S_{\omega_1,\varphi}.$$

To obtain the uniform convergence of resolvents, for λ in a sector of $\mathbb{C}$, we will need the following result:

Proposition 2.22. *If A is a positive type operator and Λ_ϵ its Yosida approximation then, for all $\alpha \in [0,\frac{1}{2})$,*

$$\|\Lambda_\epsilon^{-1/2} - \Lambda_0^{-1/2}\|_{\mathcal{L}(X)} \leqslant C\epsilon^\alpha.$$

Proof: See Appendix A.

With this result and Corollary 2.21 we can prove:

Corollary 2.23. *Given $\omega_1 \in (0,\omega)$ we have that $(\lambda - \mathcal{A}_\epsilon)^{-1} \overset{\mathcal{L}(X\times X)}{\longrightarrow} (\lambda - \mathcal{A}_0)^{-1}$ as $\epsilon \to 0^+$, uniformly for $\lambda \in S_{\omega_1,\varphi}$.*

Proof: We have that

$$\mathcal{A}_\epsilon^{-1} - \mathcal{A}_0^{-1} = \begin{bmatrix} 0 & \Lambda_\epsilon^{-\frac{1}{2}} - \Lambda_0^{-\frac{1}{2}} \\ \Lambda_0^{-\frac{1}{2}} - \Lambda_\epsilon^{-\frac{1}{2}} & 0 \end{bmatrix},$$

and

$$(\lambda - \mathcal{A}_\epsilon)^{-1} - (\lambda - \mathcal{A}_0)^{-1} = \mathcal{A}_\epsilon(\lambda - \mathcal{A}_\epsilon)^{-1}(\mathcal{A}_\epsilon^{-1} - \mathcal{A}_0^{-1})\mathcal{A}_0(\lambda - \mathcal{A}_0)^{-1}.$$

Therefore Proposition 2.22 and Corollary 2.21 we have, given $\omega_1 \in (0,\omega)$ and $\alpha \in [0,\frac{1}{2})$, that

$$\|(\lambda - \mathcal{A}_\epsilon)^{-1} - (\lambda - \mathcal{A}_0)^{-1}\|_{\mathcal{L}(X\times X)} \leqslant M_{\omega_1}^2 C\epsilon^\alpha.$$

■

Remark 2.24. *If A is the negative Laplacian with Dirichlet boundary conditions, then we can take $\alpha = \frac{1}{2}$ (see Remark A.1).*

Let $w_1 \in (0,\omega)$. Given $r > 0$, Corollary 2.20 implies that we can choose the curve Γ given by $\Gamma = \Gamma_1 \cup \Gamma_r \cup \overline{\Gamma_1}$, where

$$\Gamma_1 = \{\lambda \in \mathbb{C} : \lambda = -\omega_1 + se^{i(\pi-\varphi)},\ s \geqslant r\}, \quad \Gamma_r = \{\lambda \in \mathbb{C} : \lambda = -\omega_1 + re^{i\xi},\ \xi \in [\pi-\varphi, \varphi-\pi]\},$$

such that

$$e^{-\mathcal{A}_\epsilon t} = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda + \mathcal{A}_\epsilon)^{-1} d\lambda,$$

for all $\epsilon \in [0,1]$ and $t > 0$.

Proof of Theorem 1.4: We have that

$$e^{-\mathcal{A}_\epsilon t} - e^{-\mathcal{A}_0 t} = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\left[(\lambda + \mathcal{A}_\epsilon)^{-1} - (\lambda + \mathcal{A}_0)^{-1}\right] d\lambda,$$

thus, if $\alpha \in [0, \frac{1}{2})$, then

$$\begin{aligned}
\|e^{-\mathcal{A}_\epsilon t} - e^{-\mathcal{A}_0 t}\|_{\mathcal{L}(X\times X)} &\leqslant \frac{C\epsilon^\alpha}{2\pi}\int_\Gamma e^{\mathrm{Re}\lambda t}|d\lambda| \\
&= \frac{C\epsilon^\alpha}{2\pi} e^{-\omega_1 t}\left[2\int_r^\infty e^{-st\cos\varphi} ds + r\int_{\varphi-\pi}^{\pi-\varphi} e^{rt\cos\xi} d\xi\right] \\
&\leqslant \frac{C\epsilon^\alpha}{2\pi} e^{-\omega_1 t}\left[2\frac{e^{-rt\cos\varphi}}{t\cos\varphi} + 2r(\pi-\varphi)e^{rt}\right] \\
&\leqslant \frac{C\epsilon^\alpha}{\pi}\frac{e^{-\omega_1 t}}{t\cos\varphi} + \frac{C\epsilon^\alpha}{\pi} e^{-\omega_1 t} r(\pi-\varphi),
\end{aligned}$$

for any $0 < r < \omega_1$ and therefore making $r \to 0^+$, we obtain

$$\|e^{-\mathcal{A}_\epsilon t} - e^{-\mathcal{A}_0 t}\|_{\mathcal{L}(X\times X)} \leqslant \frac{C}{\pi\cos\varphi}\epsilon^\alpha t^{-1} e^{-\omega_1 t}.$$

But $\|e^{-\mathcal{A}_\epsilon t} - e^{-\mathcal{A}_0 t}\|_{\mathcal{L}(X\times X)} \leqslant 2Me^{-\omega_1 t}$ and hence, for $\gamma \in [0,1]$ we have

$$\|e^{-\mathcal{A}_\epsilon t} - e^{-\mathcal{A}_0 t}\|_{\mathcal{L}(X\times X)} \leqslant (2M)^{1-\gamma}\left(\frac{C}{\pi\cos\varphi}\right)^\gamma \epsilon^{\alpha\gamma} t^{-\gamma} e^{-\omega_1 t}.$$

∎

Remark 2.25. *Again, if A is the negative Laplacian with Dirichlet boundary conditions, we can take $\alpha = \frac{1}{2}$.*

2.2. **Fractional powers of $\mathcal{A}_\epsilon$.** In this subsection we are interested in some properties of the fractional powers of the operators $\mathcal{A}_\epsilon$. We know that for $\epsilon > 0$ we are always working with $X \times X$ with an equivalent norm, but again, we are concerned about the uniformity in $\epsilon \in [0,1]$ for the problems (1.7), and it will be useful to have some additional properties of the fractional powers of $\mathcal{A}_\epsilon$.

Proposition 2.26. *$\mathcal{A}_\epsilon$ is a positive type operator for some constant $C \geqslant 1$.*

Proof: We know that $\delta I - \mathcal{A}_\epsilon$ is dissipative in $X \times X$ with the norm $\|\cdot\|_{\epsilon,\beta_1}$, by Proposition 2.5, and $\rho(\delta I - \mathcal{A}_\epsilon) \cap (0,\infty) \neq \varnothing$, and thus by Lumer's Theorem, we have

$$\|(\lambda + (\mathcal{A}_\epsilon - \delta I))^{-1}\|_{\mathcal{L}(X\times X), \|\cdot\|_{\epsilon,\beta_1}} \leqslant \frac{1}{\lambda}, \text{ for all } \lambda > 0.$$

Therefore, if $\mu = \lambda - \delta$,

$$\|(\mu + \mathcal{A}_\epsilon)^{-1}\|_{\mathcal{L}(X\times X), \|\cdot\|_{\epsilon,\beta_1}} \leqslant \frac{1}{\mu + \delta}, \text{ for all } \mu > -\delta,$$

and thus if $\mu > 0$ we have that

$$(1+\mu)\|(\mu + \mathcal{A}_\epsilon)^{-1}\|_{\mathcal{L}(X\times X), \|\cdot\|_{\epsilon,\beta_1}} \leqslant \frac{\mu+1}{\mu+\delta},$$

and since the map $[0,\infty) \ni \mu \mapsto \frac{\mu+1}{\mu+\delta}$ is bounded and the norms $\|\cdot\|_{\epsilon,\beta_1}$ and $\|\cdot\|_{X\times X}$ are uniformly equivalent, the result follows. ■

Now, for $\tau \in [0,\infty)$, we have that $s \in \rho(-\mathcal{A}_\epsilon)$

$$(\tau + \mathcal{A}_\epsilon)^{-1} = \frac{1}{\tau+1} \begin{bmatrix} \tau + \Lambda_\epsilon & \Lambda_\epsilon^{1/2} \\ -\Lambda_\epsilon^{1/2} & \tau \end{bmatrix} \left(\frac{\tau^2}{\tau+1} + \Lambda_\epsilon \right)^{-1},$$

and hence for $\alpha \in (0,1)$ we have (see Theorem 1.4.2 of [17]) that

$$\mathcal{A}_\epsilon^{-\alpha} = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \frac{\tau^{-\alpha}}{\tau+1} \begin{bmatrix} \tau + \Lambda_\epsilon & \Lambda_\epsilon^{1/2} \\ -\Lambda_\epsilon^{1/2} & \tau \end{bmatrix} \left(\frac{\tau^2}{\tau+1} + \Lambda_\epsilon \right)^{-1} d\tau.$$

If we set

$$\mathcal{A}_\epsilon^{-\alpha} = \begin{bmatrix} P_{1,1}(\epsilon,\alpha) & P_{1,2}(\epsilon,\alpha) \\ P_{2,1}(\epsilon,\alpha) & P_{2,2}(\epsilon,\alpha) \end{bmatrix},$$

we have that

$$
\begin{aligned}
P_{1,1}(\epsilon,\alpha) &= \frac{\sin \pi\alpha}{\pi}\int_0^\infty \frac{\tau^{-\alpha}}{\tau+1}(\tau+\Lambda_\epsilon)\left(\frac{\tau^2}{\tau+1}+\Lambda_\epsilon\right)^{-1} d\tau,\\
P_{1,2}(\epsilon,\alpha) &= \frac{\sin \pi\alpha}{\pi}\int_0^\infty \frac{\tau^{-\alpha}}{\tau+1}\Lambda_\epsilon^{\frac{1}{2}}\left(\frac{\tau^2}{\tau+1}+\Lambda_\epsilon\right)^{-1} d\tau,\\
P_{2,1}(\epsilon,\alpha) &= -B_{1,2}(\epsilon,\alpha),\\
P_{2,2}(\epsilon,\alpha) &= \frac{\sin \pi\alpha}{\pi}\int_0^\infty \frac{\tau^{-\alpha+1}}{\tau+1}\left(\frac{\tau^2}{\tau+1}+\Lambda_\epsilon\right)^{-1} d\tau.
\end{aligned}
$$

To continue, we will need the following result.

Proposition 2.27. *If A is a positive type operator with constant C then there exists a constant C_1 such that, for any $\beta\in(0,1)$ and $\epsilon\in[0,1]$*

$$\|\Lambda_\epsilon^\beta(\mu+\Lambda_\epsilon)^{-1}\|_{\mathcal{L}(X)} \leqslant \frac{C_1}{(\mu+1)^{1-\beta}},\ \textit{for all } \mu\geqslant 0.$$

Proof: See Appendix A.

And now we can state our result for the fracional powers of $\mathcal{A}_\epsilon$.

Proposition 2.28. *For each $\beta\in(0,\frac{1}{2})$ and $\alpha\in(\beta,1)$, the operators $\Lambda_\epsilon^\beta P_{1,2}(\epsilon,\alpha)$ and $\Lambda_\epsilon^\beta P_{2,2}(\epsilon,\alpha)$ are uniformly bounded for $\epsilon\in[0,1]$.*

Proof: For $P_{1,2}(\epsilon,\alpha)$ we have that

$$\Lambda_\epsilon^\beta P_{1,2}(\epsilon,\alpha) = \frac{\sin \pi\alpha}{\pi}\int_0^\infty \frac{\tau^{-\alpha}}{\tau+1}\Lambda_\epsilon^{\frac{1}{2}+\beta}\left(\frac{\tau^2}{\tau+1}+\Lambda_\epsilon\right)^{-1} d\tau,$$

and thus by Proposition 2.27 we have

$$\|\Lambda_\epsilon^\beta P_{1,2}(\epsilon,\alpha)\|_{\mathcal{L}(X)} \leqslant \frac{C_1\sin\pi\alpha}{\pi}\int_0^\infty \frac{\tau^{-\alpha}}{\tau+1}\left(\frac{\tau+1}{\tau^2+\tau+1}\right)^{\frac{1}{2}-\beta} d\tau \leqslant \frac{C_1\sin\pi\alpha}{\pi}\int_0^\infty \frac{\tau^{-\alpha}}{\tau+1}d\tau,$$

and the integral on the right side is convergent, for any $\alpha\in(0,1)$.

For $P_{2,2}(\epsilon,\alpha)$ we have that

$$\Lambda_\epsilon^\beta P_{2,2}(\epsilon,\alpha) = \frac{\sin \pi\alpha}{\pi}\int_0^\infty \frac{\tau^{-\alpha+1}}{\tau+1}\Lambda_\epsilon^{\beta}\left(\frac{\tau^2}{\tau+1}+\Lambda_\epsilon\right)^{-1} d\tau,$$

and thus by Proposition 2.27 we have

$$\|\Lambda_\epsilon^\beta P_{2,2}(\epsilon,\alpha)\|_{\mathcal{L}(X)} \leqslant \frac{C_1\sin\pi\alpha}{\pi}\int_0^\infty \frac{\tau^{-\alpha+1}}{\tau+1}\left(\frac{\tau+1}{\tau^2+\tau+1}\right)^{1-\beta} d\tau,$$

and the integral on the right side is convergent, provided that $\alpha\in(\beta,1)$. ■

3. Local and global well posedness results

3.1. **Local well posedness result.** To state the results of local well posedness of equations (1.7), and consequently of (1.4), we firstly prove the auxiliary lemma below.

Lemma 3.1. *Let $f:\mathbb{R}\to\mathbb{R}$, and A is the negative Dirichlet Laplacian in X with domain $X^1 = H^2(\Omega)\cap H^1_0(\Omega)$ and consider its closed extension to $H^{-r} = (X^{\frac{r}{2}})'$, where Y' represents the dual space of the Banach space Y, (in particular, $H^{-1} = H^1_0(\Omega)'$). Then*

$$f^e(\phi)(x) = f(\phi(x)),\ x\in\Omega,$$

defines an operator from $X^{\frac{s}{2}}$ into H^{-r} which is Lipschitz continuous in bounded sets provided that condition (1.2) holds and $r\in\left[\frac{(\rho-1)(n-2)}{4},1\right]$, $s\in[r,1]\cap\left[\frac{n}{2}-\frac{2}{\rho-1},1\right]$. If in addition, r can be taken strictly less than 1, then f^e takes bounded sets of $X^{\frac{s}{2}}$ into relatively compact sets of H^{-1}.

Proof: Let B be a bounded set in $X^{\frac{s}{2}}$ and choose arbitrary $\phi_1,\phi_2\in B$. Since condition (1.2) holds we use the Sobolev and Hölder inequalities to get

$$\begin{aligned}
\|f^e(\phi_1)-f^e(\phi_2)\|_{H^{-r}} &\leqslant C\|f^e(\phi_1)-f^e(\phi_2)\|_{L^{\frac{2n}{n+2r}}(\Omega)}\\
&\leqslant \hat{C}\|\phi_1-\phi_2\|_{L^{\frac{2n}{n-2r}}(\Omega)}\left(1+\|\phi_1\|^{\rho-1}_{L^{\frac{n(\rho-1)}{2r}}(\Omega)}+\|\phi_2\|^{\rho-1}_{L^{\frac{n(\rho-1)}{2r}}(\Omega)}\right)\\
&\leqslant \overline{C}\|\phi_1-\phi_2\|_{X^{\frac{s}{2}}}\left(1+\|\phi_1\|^{\rho-1}_{X^{\frac{s}{2}}}+\|\phi_2\|^{\rho-1}_{X^{\frac{s}{2}}}\right),
\end{aligned}$$

for any $s\in[r,1]\cap\left[\frac{n}{2}-\frac{2}{\rho-1},1\right]$. The last statement holds since H^{-r} is compact embedded in H^{-1} for $r<1$. ■

To continue, let $\mathcal{W}_\epsilon$ be the extrapolated space of $X\times X$ - which is the completion of the normed space $(X\times X, \|\mathcal{A}_\epsilon^{-1}\cdot\|_{X\times X})$ - and we consider the power scale $\{\mathcal{W}_\epsilon^\alpha\}_{\alpha\in[0,1]}$ generated by $(\mathcal{W}_\epsilon, \|\mathcal{A}_\epsilon^\alpha\cdot\|_{\mathcal{W}_\epsilon})$.

Remark 3.2. *Note that $\mathcal{W}_\epsilon^1 = X\times X$ for all $\epsilon\in[0,1]$.*

Lemma 3.3. *Let $s\in\left[\frac{(\rho-1)(n-2)}{4},1\right]$ and $\gamma\in(0,1-\frac{s}{2})$, then $\mathcal{F}_\epsilon$ (defined in (1.8)) takes $\mathcal{W}_\epsilon^1$ in $\mathcal{W}_\epsilon^\gamma$ and is Lipschitz continuous in bounded sets.*

Proof: Let $\mathcal{B}$ be a bounded subset of $\mathcal{W}^1_\epsilon$ and $\left[\begin{smallmatrix} w_1 \\ z_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} w_2 \\ z_2 \end{smallmatrix}\right] \in \mathcal{B}$. We have that

$$\|\mathcal{F}_\epsilon(\left[\begin{smallmatrix} w_1 \\ z_1 \end{smallmatrix}\right]) - \mathcal{F}_\epsilon(\left[\begin{smallmatrix} w_2 \\ z_2 \end{smallmatrix}\right])\|_{\mathcal{W}^\gamma_\epsilon} = \left\|\mathcal{A}_\epsilon^{\gamma-1}\left[\mathcal{F}_\epsilon(\left[\begin{smallmatrix} w_1 \\ z_1 \end{smallmatrix}\right]) - \mathcal{F}_\epsilon(\left[\begin{smallmatrix} w_2 \\ z_2 \end{smallmatrix}\right])\right]\right\|_{X\times X}$$
$$= \left\|\left[\begin{smallmatrix} \Lambda_\epsilon^{\frac{r}{2}} P_{1,2}(\epsilon,1-\gamma)A^{-\frac{r}{2}}(I+\epsilon A)^{\frac{r-1}{2}}(f(A^{-\frac{1}{2}}w_1)-f(A^{-\frac{1}{2}}w_2)) \\ \Lambda_\epsilon^{\frac{r}{2}} P_{2,2}(\epsilon,1-\gamma)A^{-\frac{r}{2}}(I+\epsilon A)^{\frac{r-1}{2}}(f(A^{-\frac{1}{2}}w_1)-f(A^{-\frac{1}{2}}w_2)) \end{smallmatrix}\right]\right\|_{X\times X}$$

and hence, by Proposition 2.28 along with Proposition 2.2 we have that

$$\|\mathcal{F}_\epsilon(\left[\begin{smallmatrix} w_1 \\ z_1 \end{smallmatrix}\right]) - \mathcal{F}_\epsilon(\left[\begin{smallmatrix} w_2 \\ z_2 \end{smallmatrix}\right])\|_{\mathcal{W}^\gamma_\epsilon} \leqslant const.\|f(A^{-\frac{1}{2}}w_1) - f(A^{-\frac{1}{2}}w_2)\|_{H^{-r}}.$$

Finally, Lemma 3.1 and Remark 3.2 guarantee that

$$\|\mathcal{F}_\epsilon(\left[\begin{smallmatrix} w_1 \\ z_1 \end{smallmatrix}\right]) - \mathcal{F}_\epsilon(\left[\begin{smallmatrix} w_2 \\ z_2 \end{smallmatrix}\right])\|_{\mathcal{W}^\gamma_\epsilon} \leqslant const.\|\left[\begin{smallmatrix} w_1 \\ z_1 \end{smallmatrix}\right] - \left[\begin{smallmatrix} w_2 \\ z_2 \end{smallmatrix}\right]\|_{\mathcal{W}^1_\epsilon}.$$

■

Now we can state a result of local well posedness for (1.7) in $\mathcal{W}^1_\epsilon$.

Theorem 3.4. *For any initial data* $\left[\begin{smallmatrix} w_0 \\ z_0 \end{smallmatrix}\right]$ *lying in a bounded subset* $\mathcal{B}$ *of* $\mathcal{W}^1_\epsilon$ *there exists a number* $\tau = \tau(\mathcal{B},\epsilon)$ *and a unique solution* $[0,\tau) \ni t \mapsto \left[\begin{smallmatrix} w_\epsilon \\ z_\epsilon \end{smallmatrix}\right](t,w_0,z_0) \in \mathcal{W}^1_\epsilon$ *of* (1.7) *which depends continuously on its variables* $(t,w_0,z_0) \in [0,\tau)\times\mathcal{W}^1_\epsilon$ *and such that, for any* $s \in \left[\frac{(\rho-1)(n-2)}{4}, 1\right]$ *and* $\gamma \in \left(0, 1-\frac{s}{2}\right)$,

$$\left[\begin{smallmatrix} w_\epsilon \\ z_\epsilon \end{smallmatrix}\right](\cdot,w_0,z_0) \in C\left((0,\tau),\mathcal{W}^{1+\gamma}_\epsilon\right) \cap C^1\left((0,\tau),\mathcal{W}^{1+\gamma-}_\epsilon\right),$$

and either $\tau = \infty$ *or* $\|\left[\begin{smallmatrix} w_\epsilon \\ z_\epsilon \end{smallmatrix}\right](t,w_0,z_0)\|_{\mathcal{W}^1_\epsilon} \to \infty$ *as* $t \to \tau^-$.

Moreover, the solution satisfies in $\mathcal{W}^1_\epsilon$ *the variation of constants formula*

$$\left[\begin{smallmatrix} w_\epsilon \\ z_\epsilon \end{smallmatrix}\right](t,w_0,z_0) = e^{-\mathcal{A}_\epsilon t}\left[\begin{smallmatrix} w_0 \\ z_0 \end{smallmatrix}\right] + \int_0^t e^{-\mathcal{A}_\epsilon(t-s)}\mathcal{F}_\epsilon(\left[\begin{smallmatrix} w_\epsilon \\ z_\epsilon \end{smallmatrix}\right](s,w_0,z_0))\,ds, \quad t\in[0,\tau).$$

Proof: The theorem above is a consequence of the results reported in [17]. ■

To state the result of local well posedness for (1.4), we define $\tilde{\mathcal{A}}_\epsilon : D(\tilde{\mathcal{A}}_\epsilon) \subset X^{\frac{1}{2}}\times\tilde{X}_\epsilon^{\frac{1}{2}} \longrightarrow X^{\frac{1}{2}}\times\tilde{X}_\epsilon^{\frac{1}{2}}$ by

$$D(\tilde{\mathcal{A}}_\epsilon) = \left\{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] \in X^{\frac{1}{2}}\times\tilde{X}_\epsilon^{\frac{1}{2}} :\ \Phi_\epsilon\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] \in D(\mathcal{A}_\epsilon)\right\},$$

and for $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] \in D(\tilde{\mathcal{A}}_\epsilon)$

$$\tilde{\mathcal{A}}_\epsilon\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] = \Phi_\epsilon^{-1}\mathcal{A}_\epsilon\Phi_\epsilon\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right].$$

Since $\Phi_\epsilon : X^{\frac{1}{2}}\times\tilde{X}_\epsilon^{\frac{1}{2}} \longrightarrow X\times X$ is an isometric isomorphism for all $\epsilon\in[0,1]$, we have that each $\tilde{\mathcal{A}}_\epsilon$ is a closed densely defined operator and also the following result

Proposition 3.5. *Each operator* $\tilde{\mathcal{A}}_\epsilon$ *is a positive type operator (with an uniform constant) and sectorial (with an uniform sector and uniform constants* $M\geqslant 1$, $\varphi\in(0,\frac{\pi}{2})$ *and* $\omega>0$*) in* $X^{\frac{1}{2}}\times\tilde{X}_\epsilon^{\frac{1}{2}}$.

Let $\mathcal{Z}_\epsilon$ be the extrapolated space of $X^{\frac{1}{2}} \times \tilde{X}_\epsilon^{\frac{1}{2}}$ which is the completion of the normed space $(X^{\frac{1}{2}} \times \tilde{X}_\epsilon^{\frac{1}{2}}, \|\tilde{\mathcal{A}}_\epsilon^{-1} \cdot\|_{X^{\frac{1}{2}} \times \tilde{X}_\epsilon^{\frac{1}{2}}})$ and we consider the power scale $\{\mathcal{Z}_\epsilon^\alpha\}_{\alpha\in[0,1]}$ generated by $(\mathcal{Z}_\epsilon, \|\tilde{\mathcal{A}}_\epsilon^\alpha \cdot\|_{\mathcal{Z}_\epsilon})$.

Remark 3.6. *Note that $\mathcal{Z}_\epsilon^1 = X^{\frac{1}{2}} \times \tilde{X}_\epsilon^{\frac{1}{2}}$ for all $\epsilon \in [0,1]$.*

Proof of Theorem 1.5: Let $\tilde{\mathcal{B}} = \Phi_{0,\epsilon}\mathcal{B}$ which is a bounded subset of $\mathcal{W}_\epsilon^1$. Thus, by Theorem 3.4, there exists $\tau = \tau(\tilde{\mathcal{B}}, \epsilon)$ and a solution $[0,\tau) \ni t \mapsto \left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right](t, w_0, z_0, \epsilon) \in \mathcal{W}_\epsilon^1$. Defining $\kappa = \tau$ and $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t, u_0, v_0, \epsilon) = \Phi_\epsilon \left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right](t, w_0, v_0, \epsilon)$ we obtain the desired result. ∎

3.2. **Global solutions.** We want to prove that problem (1.7) generates a strongly continuous semigroup, and conclude consequently the analogous result to (1.1). To this end, from now on we assume that $A : H_0^1(\Omega) \cap H^2(\Omega) \subset X \to X$ is the negative Laplacian with Dirichlet boundary condition (hence satisfies conditions (O1), (O2) and (O3)), and we will begin with the following lemma:

Lemma 3.7. *Under the assumptions and notation of Theorem 3.4, if A is the negative Laplacian with Dirichlet boundary condition in X, then condition (1.3) implies the existence of a constant $C > 0$, independent of $\epsilon \in [0,1]$, such that if $\left[\begin{smallmatrix} w_0 \\ z_0 \end{smallmatrix}\right] \in \mathcal{W}_\epsilon^1$, then solution of equation (1.7) given by $[0, \tau(w_0, z_0, \epsilon)) \ni t \mapsto \left[\begin{smallmatrix} w(t,w_0,z_0,\epsilon) \\ z(t,w_0,z_0,\epsilon) \end{smallmatrix}\right] \in \mathcal{W}_\epsilon^1$ fulfills the estimate*

$$\left\| \left[\begin{smallmatrix} w(t,w_0,z_0,\epsilon) \\ z(t,w_0,z_0,\epsilon) \end{smallmatrix}\right] \right\|_{X\times X}^2 \leqslant C\left(1 + \|z_0\|_X^2 + \|w_0\|_X^{\rho+1}\right).$$

Proof: We take the X scalar product $(\cdot,\cdot)$ of each side of the first equation in (1.5) with $A^{-\frac{1}{2}} w_t$ to get

$$\frac{1}{2}\frac{d}{dt}\left(\|w\|_{X_2}^2 + \|z\|_X^2\right) - (f^e(A^{-\frac{1}{2}}w), A^{-\frac{1}{2}}w_t) = -\|\Lambda_\epsilon^{\frac{1}{2}} z\|_X^2. \tag{3.1}$$

Since A is the negative Laplacian with Dirichlet boundary condition, the Poincaré inequality reads

$$\|A^{\frac{1}{2}}\phi\|_X^2 \geqslant \lambda_1 \|\phi\|_X^2, \ \phi \in X^{\frac{1}{2}},$$

which for $\phi = A^{-\frac{1}{2}}\psi$ translates into the estimate

$$\|\psi\|_X^2 \geqslant \lambda_1 \|A^{-\frac{1}{2}}\psi\|_X^2, \ \psi \in X. \tag{3.2}$$

If F is the primitive function of f in $\mathbb{R}$ we then have

$$\int_\Omega f(A^{-\frac{1}{2}}w)A^{-\frac{1}{2}}w_t dx = \frac{d}{dt}\int_\Omega F(A^{-\frac{1}{2}}w)dx.$$

We now remark that (1.3) implies the existence of constants $\mathcal{C}, \xi > 0$, for which

$$F(t) = \int_0^t f(s)ds \leqslant \frac{1}{2}(\lambda_1 - \xi)t^2 + \mathcal{C},\ \ t \in \mathbb{R}.$$

As a consequence we infer

$$(F(A^{-\frac{1}{2}}w), 1) \leqslant \frac{1}{2}(\lambda_1 - \xi)\|A^{-\frac{1}{2}}w\|_X^2 + \mathcal{C}|\Omega|,$$

which with the aid of (3.2) reads

$$\int_\Omega F(A^{-\frac{1}{2}}w)dx \leqslant \frac{1}{2}\left(1 - \frac{\xi}{\lambda_1}\right)\|w\|_X^2 + \mathcal{C}|\Omega|. \tag{3.3}$$

Connecting (3.1)-(3.3) we get for

$$\mathcal{L}(w,z) \doteq \frac{1}{2}\|w\|_X^2 + \frac{1}{2}\|z\|_X^2 - \int_\Omega F(A^{-\frac{1}{2}}w)dx \tag{3.4}$$

that

$$\frac{d}{dt}\mathcal{L}(w,z) = -\|\Lambda_\epsilon^{\frac{1}{2}} z\|_X^2 \leqslant 0 \tag{3.5}$$

and hence

$$\frac{\xi}{2\lambda_1}\|w\|_X^2 + \frac{1}{2}\|z\|_X^2 - \mathcal{C}|\Omega| \leqslant \mathcal{L}(w,z) \leqslant \mathcal{L}(w_0,z_0), \tag{3.6}$$

as long as the solution exists.

We then have

$$\frac{\xi}{2\lambda_1}\|w\|_X^2 + \frac{1}{2}\|z\|_X^2 - \mathcal{C}|\Omega| \leqslant \frac{1}{2}\|w_0\|_X^2 + \frac{1}{2}\|z_0\|_X^2 - \int_\Omega F(A^{-\frac{1}{2}}w_0)dx$$

and gives us

$$\frac{\xi}{\lambda_1}\|w\|_X^2 + \|z\|_X^2 \leqslant \|w_0\|_X^2 + \|z_0\|_X^2 - 2\int_\Omega F(A^{-\frac{1}{2}}w_0)dx + 2\mathcal{C}|\Omega|$$

where

$$\|F(A^{-\frac{1}{2}}w_0)\|_{L^1(\Omega)} \leqslant const.\left(1 + \|w_0\|_X^{\rho+1}\right),$$

with the constant independent of ϵ, since condition (1.3) implies that $|F(s)| \leqslant const.(1 + |s|^{\rho+1})$ for $s \in \mathbb{R}$. Thus

$$\|w\|_X^2 + \|z\|_X^2 \leqslant C\left(1 + \|z_0\|_X^2 + \|w_0\|_X^{\rho+1}\right).$$

■

Theorem 3.8. *Under the assumptions of Lemma 3.7, the solutions from Theorem 3.4 exist globally in time and the problem (1.7) defines a C^0-semigroup $\{T_\epsilon(t): \ t \geqslant 0\}$ on $\mathcal{W}^1_\epsilon$ for each $\epsilon \in [0,1]$, which has bounded orbits of bounded sets, defined by*

$$T_\epsilon(t)\left[\begin{smallmatrix} w_0 \\ z_0 \end{smallmatrix}\right] = e^{-\mathcal{A}_\epsilon t}\left[\begin{smallmatrix} w_0 \\ z_0 \end{smallmatrix}\right] + \int_0^t e^{-\mathcal{A}_\epsilon (t-s)}\mathcal{F}_\epsilon\left(T_\epsilon(s)\left[\begin{smallmatrix} w_0 \\ z_0 \end{smallmatrix}\right]\right)ds, \ \textit{for all } t \geqslant 0.$$

Proof of Theorem 1.6: Is a direct consequence of Theorem 3.8. ∎

4. Existence of attractors and uniform bounds

In this section our goal is to prove the existence of a global attractor $\mathbb{A}_\epsilon$ of the semigroup $\{T_\epsilon(t): \ t \geqslant 0\}$ for each $\epsilon \in [0,1]$ and to prove that $\{T_\epsilon(t) : t \geqslant 0\}$ is a gradient semigroup.

Let $\mathcal{E} = \left\{\left[\begin{smallmatrix} \phi \\ 0 \end{smallmatrix}\right] : \phi \in \mathcal{E}_1\right\}$, where $\mathcal{E}_1 = \{\phi \in X : A^{\frac{1}{2}}\phi = f(A^{-\frac{1}{2}}\phi)\}$. It is clear that $\mathcal{E}$ is the set of equilibrium points of $\{T_\epsilon(t) : t \geqslant 0\}$, for all $\epsilon \in [0,1]$.

First we need an auxiliary lemma:

Lemma 4.1. *If $X = L^2(\Omega)$ and A is the negative Laplacian with Dirichlet boundary condition and domain $X^1 = H^2(\Omega) \cap H^1_0(\Omega)$, then*

$$(A^{-\frac{1}{2}}\phi, A^{\frac{1}{2}}\psi) = \int_\Omega \phi\psi dx, \quad \phi \in L^{\frac{2n}{n+2}}(\Omega), \ \psi \in X^{\frac{1}{2}}. \tag{4.1}$$

Proof: See Lemma 2.1 of [9].

Now we can give an estimate for the bound of the equilibrium set $\mathcal{E}$.

Lemma 4.2. *$\mathcal{E}$ is bounded in $X \times X$, moreover for each $\phi \in \mathcal{E}_1$, $A^{-\frac{1}{2}}\phi \in L^\infty(\Omega)$.*

Proof: Let $\phi \in X^{\frac{1}{2}}$ such that $\left[\begin{smallmatrix} \phi \\ 0 \end{smallmatrix}\right] \in \mathcal{E}$. Thus by Lemma 4.1 we have

$$\begin{aligned} \|\phi\|^2_X = (\phi,\phi) &= (\phi, A^{-\frac{1}{2}}f(A^{-\frac{1}{2}}\phi)) = (A^{\frac{1}{2}}A^{-\frac{1}{2}}\phi, A^{-\frac{1}{2}}f(A^{-\frac{1}{2}})\phi) \\ &= \int_\Omega f(A^{-\frac{1}{2}}\phi)A^{-\frac{1}{2}}\phi dx, \end{aligned}$$

and hence, with (1.3) and the aid of the Poincaré inequality, we have that

$$\|\phi\|^2_X = \int_\Omega f(A^{-\frac{1}{2}}\phi)A^{-\frac{1}{2}}\phi dx \leqslant (\lambda_1 - \xi)\|A^{-\frac{1}{2}}\phi\|^2_X + C|\Omega| \leqslant (1-\xi\lambda_1^{-1})\|\phi\|^2_X + C|\Omega|.$$

Therefore

$$\sup_{\left[\begin{smallmatrix} \phi \\ 0 \end{smallmatrix}\right] \in \mathcal{E}} \|\phi\|^2_X \leqslant \lambda_1 \xi^{-1} C|\Omega|.$$

For the second part, if $\psi \in \mathcal{E}_1$, then $\psi = A^{-\frac{1}{2}}\phi \in X^{\frac{1}{2}}$ is a solution of the problem

$$A\psi = f(\psi),$$

and hence, since f has subcritical growth, it follows by a bootstrapping argument that $\psi \in L^\infty(\Omega)$.

∎

Corollary 4.3. *The set* $\tilde{\mathcal{E}} = \left\{ \left[\begin{smallmatrix} A^{-\frac{1}{2}}\phi \\ 0 \end{smallmatrix} \right] : \ \phi \in \mathcal{E}_1 \right\}$ *is the set of equilibrium points of* (1.1)*. Moreover, this set is uniformly bounded in* $X^{\frac{1}{2}} \times \tilde{X}_\epsilon^{\frac{1}{2}}$*, for* $\epsilon \in [0,1]$*, and* $\tilde{\mathcal{E}} \subset L^\infty(\Omega) \times L^\infty(\Omega)$*.*

Lemma 3.7 gives us also the following result:

Proposition 4.4. *Under the assumptions of Lemma 3.7 the function* $\mathcal{L} : X \times X \to \mathbb{R}$ *satisfies*

- **(i)** $\mathcal{L}(T_\epsilon(\cdot) \left[\begin{smallmatrix} w_0 \\ z_0 \end{smallmatrix} \right])$ *is bounded from below and non-increasing in* $[0,\infty)$ *for any* $\left[\begin{smallmatrix} w_0 \\ z_0 \end{smallmatrix} \right] \in X \times X$*;*
- **(ii)** *If* $\left[\begin{smallmatrix} w_0 \\ z_0 \end{smallmatrix} \right] \in X \times X$ *and* $\mathcal{L}(T_\epsilon(\cdot) \left[\begin{smallmatrix} w_0 \\ z_0 \end{smallmatrix} \right]) = const.$ *in* $[0,\infty)$ *then* $\left[\begin{smallmatrix} w_0 \\ z_0 \end{smallmatrix} \right] \in \mathcal{E}$*.*

Proof: Equations (3.5) and (3.6) show that $\mathcal{L}(T_\epsilon(\cdot) \left[\begin{smallmatrix} w_0 \\ z_0 \end{smallmatrix} \right])$ is decreasing and bounded below in $[0,\infty)$. If $\mathcal{L}(T_\epsilon(\cdot) \left[\begin{smallmatrix} w_0 \\ z_0 \end{smallmatrix} \right]) = const.$ in $[0,\infty)$, then

$$\mathcal{L}(T_\epsilon(t) \left[\begin{smallmatrix} w_0 \\ z_0 \end{smallmatrix} \right]) = \mathcal{L}(\left[\begin{smallmatrix} w_0 \\ z_0 \end{smallmatrix} \right]),$$

for all $t \in [0,\infty)$. If $T_\epsilon(t) \left[\begin{smallmatrix} w_0 \\ z_0 \end{smallmatrix} \right] = \left[\begin{smallmatrix} w_\epsilon(t) \\ z_\epsilon(t) \end{smallmatrix} \right]$ then $z_\epsilon(t) = 0$ for all $t \geqslant 0$, and in particular, $z_0 = 0$. Also $\frac{d}{dt} w_\epsilon(t) = \Lambda_\epsilon^{\frac{1}{2}} z_\epsilon(t) = 0$ for all $t > 0$, thus $w_\epsilon(t)$ is constant which implies that $w_\epsilon(t) = w_0$ for all $t \geqslant 0$. Finally, equation (1.6) implies that $A^{\frac{1}{2}} w_0 = f(A^{-\frac{1}{2}} w_0)$ and therefore $\left[\begin{smallmatrix} w_0 \\ z_0 \end{smallmatrix} \right] \in \mathcal{E}$. ∎

Proposition 4.5. *Under the assumptions of Lemma 3.7, for each* $\epsilon \in [0,1]$ *there exists a function* $\mathcal{V}_\epsilon : X^{\frac{1}{2}} \times \tilde{X}_\epsilon^{\frac{1}{2}} \to \mathbb{R}$ *satisfying*

- **(i)** $\mathcal{V}_\epsilon(S_\epsilon(\cdot) \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix} \right])$ *is bounded from below and non-increasing in* $[0,\infty)$ *for any* $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix} \right] \in X^{\frac{1}{2}} \times \tilde{X}_\epsilon^{\frac{1}{2}}$*;*
- **(ii)** *If* $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix} \right] \in X^{\frac{1}{2}} \times \tilde{X}_\epsilon^{\frac{1}{2}}$ *and* $\mathcal{V}_\epsilon(S_\epsilon(\cdot) \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix} \right]) = const.$ *in* $[0,\infty)$ *then* $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix} \right] \in \tilde{\mathcal{E}}$*.*

Proof: Just define, for each $\epsilon \in [0,1]$,

$$\mathcal{V}_\epsilon (\left[\begin{smallmatrix} u \\ v \end{smallmatrix} \right]) = \mathcal{L} \left(\Phi_{0,\epsilon} \left[\begin{smallmatrix} u \\ v \end{smallmatrix} \right] \right),$$

and this functional has the desired properties. ∎

To ensure the existence of an attractor $\mathbb{A}_\epsilon$ for the semigroup $\{T_\epsilon(t) : t \geqslant 0\}$, for each $\epsilon \in [0,1]$, it remains to show that $\{T_\epsilon(t) : t \geqslant 0\}$ is an asymptotically compact semigroup, for each $\epsilon \in [0,1]$.

Proposition 4.6. *For each $\epsilon \in [0,1]$, the semigroup $\{T_\epsilon(t) : t \geqslant 0\}$ is asymptotically compact in $X \times X$.*

Proof: Define, for each $\epsilon \in [0,1]$,

$$L_\epsilon(t) \left[\begin{smallmatrix} w_0 \\ z_0 \end{smallmatrix}\right] = e^{-\mathcal{A}_\epsilon t} \left[\begin{smallmatrix} w_0 \\ z_0 \end{smallmatrix}\right] \quad \text{and} \quad U_\epsilon(t) \left[\begin{smallmatrix} w_0 \\ z_0 \end{smallmatrix}\right] = \int_0^t e^{-\mathcal{A}_\epsilon (t-s)} \mathcal{F}_\epsilon \left(T_\epsilon(s) \left[\begin{smallmatrix} w_0 \\ z_0 \end{smallmatrix}\right] \right).$$

From Lemma 3.1, $f(A^{-\frac{1}{2}} \cdot)$ take bounded subsets of X into precompact sets of H^{-s} for $s \in \left[\frac{(\rho-1)(n-2)}{4}, 1\right)$, thus $\mathcal{F}_\epsilon$ takes bounded sets of $X \times X$ into precompact sets of $X \times X$, $T_\epsilon(t)$ is the sum of an exponentially decaying semigroup with a compact family of maps, which implies that the semigroup is asymptotically compact. ■

Theorem 4.7. *The semigroup $\{T_\epsilon(t) : t \geqslant 0\}$ has a global attractor $\mathbb{A}_\epsilon$ in $X \times X$, for each $\epsilon \in [0,1]$.*

Proof of Theorem 1.7: Define $\tilde{\mathbb{A}}_\epsilon = \Phi_{0,\epsilon}^{-1} \mathbb{A}_\epsilon$, for each $\epsilon \in [0,1]$. ■

4.1. **Uniform estimates on the global attractors.** In this subsection we are concerned with uniform estimates for the family of attractors $\{\mathbb{A}_\epsilon\}_{\epsilon \in [0,1]}$ and also for $\{\tilde{\mathbb{A}}_\epsilon\}_{\epsilon \in [0,1]}$, since this will be an essential tool to prove the upper semicontinuity for both of them at $\epsilon = 0$.

Theorem 4.8. *$\bigcup_{\epsilon \in [0,1]} \mathbb{A}_\epsilon$ is bounded in $X \times X$.*

Proof: We define, for $(\epsilon, \gamma) \in [0,1] \times [0,1]$, the functional $V_{\epsilon,\gamma} : X \times X \to \mathbb{R}$ by

$$V_{\epsilon,\gamma}(w,z) = \frac{1}{2} \left(\|w\|_X^2 + \|z\|_X^2 \right) - \int_\Omega F(A^{-\frac{1}{2}} w) dx + \gamma \mathrm{Re}((I + \epsilon A)^{\frac{1}{2}} A^{-\frac{1}{2}} w, z).$$

Now

$$\begin{aligned} V_{\epsilon,\gamma}(w,z) &\geqslant \frac{1}{2}\|w\|_X^2 + \frac{1}{2}\|z\|_X^2 - \frac{1}{2}\left(1 - \frac{\xi}{\lambda_1}\right)\|w\|_X^2 - C|\Omega| + \gamma \mathrm{Re}((I+\epsilon A)^{\frac{1}{2}} A^{-\frac{1}{2}} w, z) \\ &\geqslant \frac{\xi}{2\lambda_1}\|w\|_X^2 + \frac{1}{2}\|z\|_X^2 - C|\Omega| - \frac{\gamma}{2}(\mu_1 \|z\|_X^2 + \|w\|_X^2) \\ &= \left(\frac{\xi}{2\lambda_1} - \frac{\gamma}{2}\right)\|w\|_X^2 + \left(\frac{1}{2} - \frac{\mu\gamma}{2}\right)\|z\|_X^2 - C|\Omega|, \end{aligned}$$

and we choose $\gamma \in (0,1)$ such that $\frac{\xi}{\lambda_1} - \gamma > 0$, $1 - \mu\gamma > 0$ and $\gamma < \frac{\mu}{2}$. Now, if we take $\left[\begin{smallmatrix} w(t) \\ z(t) \end{smallmatrix}\right]$ being a solution of (1.5) we have that

$$\begin{aligned} \frac{d}{dt} V_{\epsilon,\gamma}(w,z) &= -\|w_t\|_X^2 + \gamma \mathrm{Re}((I+\epsilon A)^{\frac{1}{2}} A^{-\frac{1}{2}} w_t, z) + \gamma \mathrm{Re}((I+\epsilon A)^{\frac{1}{2}} A^{-\frac{1}{2}} w, z_t) \\ &= -\|w_t\|_X^2 + \gamma \|z\|_X^2 - \gamma \mathrm{Re}((I+\epsilon A)^{\frac{1}{2}} A^{-\frac{1}{2}} w, A^{\frac{1}{2}} (I+\epsilon A)^{-\frac{1}{2}} w) - \gamma \mathrm{Re}(w_t, w) \\ &\quad + \gamma \mathrm{Re}(f(A^{-\frac{1}{2}} w), A^{-\frac{1}{2}} w) \end{aligned}$$

hence, for each $\eta > 0$

$$\begin{aligned}\frac{d}{dt}V_{\epsilon,\gamma}(w,z) \leqslant& \left(\frac{\gamma}{2\eta}-1\right)\|w_t\|_X^2+\gamma\|z\|_X^2+\frac{\gamma\eta}{2}\|w\|_X^2\\ &-\gamma\left(1-\frac{\xi}{\lambda_1}\right)\mathrm{Re}(A^{\frac12}(I+\epsilon A)^{-\frac12}w,(I+\epsilon A)^{\frac12}A^{-\frac12}w)\\ &-\frac{\gamma\xi}{\lambda_1}\mathrm{Re}(A^{\frac12}(I+\epsilon A)^{-\frac12}w,(I+\epsilon A)^{\frac12}A^{-\frac12}w)+\gamma(\lambda_1-\xi)\|A^{-\frac12}w\|_X^2+C\gamma|\Omega|\\ \leqslant& \left(\frac{\gamma\eta}{2}-\frac{\gamma\xi}{\lambda_1}\right)\|w\|_X^2+\left(-\mu+\frac{\gamma\mu}{2\eta}+\gamma\right)\|z\|_X^2+C\gamma|\Omega|,\end{aligned}$$

hence for $\eta=\frac{\xi}{\lambda_1}$ we have

$$\frac{d}{dt}V_{\epsilon,\gamma}(w,z)\leqslant -\frac{\gamma\xi}{2\lambda_1}\|w\|_X^2-\left(\frac{\mu}{2}-\gamma\right)\|z\|_X^2+C\gamma|\Omega|.$$

Now, for any $\zeta>0$, we have

$$\begin{aligned}\frac{d}{dt}V_{\epsilon,\gamma}(w,z)\leqslant& -\frac{\gamma\xi}{2\lambda_1}\|w\|_X^2-\left(\frac{\mu}{2}-\gamma\right)\|z\|_X^2+C\gamma|\Omega|\\ &\pm\ \zeta\int_\Omega F(A^{-\frac12}w)dx\ \pm\ \gamma\zeta\mathrm{Re}((I+\epsilon A)^{\frac12}A^{-\frac12}w,z),\end{aligned}$$

and thus

$$\begin{aligned}\frac{d}{dt}V_{\epsilon,\gamma}(w,z)\leqslant& \left(-\frac{\gamma\xi}{2\lambda_1}+\frac{\zeta(\lambda_1-\xi)}{2}+\frac{\gamma\zeta}{2}\right)\|w\|_X^2+\left(\frac{\mu}{2}+\gamma+\frac{\gamma\zeta\mu}{2}\right)\|z\|_X^2\\ &+C\gamma|\Omega|+2C\zeta|\Omega|+\zeta\int_\Omega F(A^{-\frac12}w)dx-\zeta\gamma\mathrm{Re}((I+\epsilon A)^{\frac12}A^{-\frac12}w,z).\end{aligned}$$

We can choose $\zeta>0$ such that

$$-\frac{\gamma\xi}{2\lambda_1}+\frac{\zeta(\lambda_1-\xi)}{2}+\frac{\gamma\zeta}{2}<-\frac{\zeta}{2}\text{ and }\frac{\mu}{2}+\gamma+\frac{\gamma\zeta\mu}{2}<-\frac{\zeta}{2},$$

and therefore

$$\begin{aligned}\frac{d}{dt}V_{\epsilon,\gamma}(w,z)\leqslant& -\frac{\zeta}{2}\|w\|_X^2-\frac{\zeta}{2}\|z\|_X^2+\zeta\int_\Omega F(A^{-\frac12}w)dx-\gamma\zeta\mathrm{Re}((I+\epsilon A)^{\frac12}A^{-\frac12}w,z)+\tilde C\\ =& -\zeta V_{\epsilon,\gamma}(w,z)+\tilde C,\end{aligned}$$

or equivalently

$$\frac{d}{dt}V_{\epsilon,\gamma}(w,z)+\zeta V_{\epsilon,\gamma}(w,z)\leqslant\tilde C.$$

This implies that for all $t>0$ we have

$$\frac{d}{dt}\left(e^{\zeta t}V_{\epsilon,\gamma}(w,z)\right)\leqslant e^{\zeta t}\tilde C,$$

and hence

$$V_{\epsilon,\gamma}(w,z)\leqslant e^{-\zeta t}V_{\epsilon,\gamma}(w_0,z_0)+\tilde C.$$

Therefore

$$\|[\begin{smallmatrix} w \\ z \end{smallmatrix}]\|^2_{X\times X} \leqslant \hat{C}e^{-\zeta t}(1+\|z_0\|^2_X+\|w_0\|^2_X+\|w_0\|^{\rho+1}_X)+\tilde{R},$$

for all $t>0$ and given a bounded set $\mathcal{B}$ in $X\times X$, there exists $T_{\mathcal{B}}\geqslant 0$, independent of $\epsilon\in[0,1]$, such that

$$\|[\begin{smallmatrix} w \\ z \end{smallmatrix}]\|^2_{X\times X} \leqslant 2\tilde{R}, \text{ for all } t\geqslant T_{\mathcal{B}},$$

and we conclude the proof of the theorem. ■

With this uniform bound in $X\times X$, using the subcritical growth of f we are able to provide an uniform estimate in a more regular space.

Theorem 4.9. *If $s\in\Big[0,1-\frac{(\rho-1)(n-2)}{4}\Big)$, then $\cup_{\epsilon\in[0,1]}\mathbb{A}_\epsilon$ is bounded in $X^{\frac{s}{2}}\times X^{\frac{s}{2}}$.*

Proof: If $\left[\begin{smallmatrix} w(\cdot,w_0,z_0,\epsilon) \\ z(\cdot,w_0,z_0,\epsilon)\end{smallmatrix}\right]$ is a solution of (1.6) in the attractor $\mathbb{A}_\epsilon$ then

$$\left[\begin{smallmatrix} w(t,w_0,z_0,\epsilon) \\ z(t,w_0,z_0,\epsilon)\end{smallmatrix}\right] = \int_{-\infty}^t e^{-\mathcal{A}_\epsilon(t-s)}\mathcal{F}_\epsilon\left(\left[\begin{smallmatrix} w(s,w_0,z_0,\epsilon) \\ z(s,w_0,z_0,\epsilon)\end{smallmatrix}\right]\right),$$

for all $t\in\mathbb{R}$. Thus, if we take $\alpha\in(\frac{1}{2},1)$, we have

$$\left\|\left[\begin{smallmatrix} w(t,w_0,z_0,\epsilon) \\ z(t,w_0,z_0,\epsilon)\end{smallmatrix}\right]\right\|_{X^{\frac{s}{2}}\times X^{\frac{s}{2}}} \leqslant \int_{-\infty}^t \|\mathcal{A}_\epsilon^\alpha e^{-\mathcal{A}_\epsilon(t-s)}\|_{\mathcal{L}(X\times X)}\left\|\left[\begin{smallmatrix} \Lambda_\epsilon^{\frac{1}{2}}P_{1,2}(\epsilon,\alpha)A^{\frac{s-1}{2}}f(A^{-\frac{1}{2}}w(s,w_0,z_0,\epsilon)) \\ \Lambda_\epsilon^{\frac{1}{2}}P_{2,2}(\epsilon,\alpha)A^{\frac{s-1}{2}}f(A^{-\frac{1}{2}}w(s,w_0,z_0,\epsilon))\end{smallmatrix}\right]\right\|_{X\times X},$$

and there exists a constant $\tilde{M}\geqslant 1$ such that (using Proposition 2.28)

$$\left\|\left[\begin{smallmatrix} w(t,w_0,z_0,\epsilon) \\ z(t,w_0,z_0,\epsilon)\end{smallmatrix}\right]\right\|_{X^{\frac{s}{2}}\times X^{\frac{s}{2}}} \leqslant \tilde{M}\int_{-\infty}^t e^{-\omega(t-s)}(t-s)^{-\alpha}\|f(A^{-\frac{1}{2}}w(s,w_0,z_0,\epsilon))\|_{H^{s-1}},$$

and since, by Theorem 4.8, $\|w(s,w_0,z_0,\epsilon)\|_X$ is uniformly bounded in X we have that $\bigcup_{\epsilon\in[0,1]}\mathcal{A}_\epsilon$ is bounded in $X^{\frac{s}{2}}\times X^{\frac{s}{2}}$. ■

Corollary 4.10. *$\cup_{\epsilon\in[0,1]}\mathbb{A}_\epsilon$ is precompact in $X\times X$.*

Proof: It follows directly from the fact that $X^{\frac{s}{2}}\times X^{\frac{s}{2}}$ is compact embedded in $X\times X$. ■

To finish this section and give a proof of Theorem 1.8, we need the following result.

Proposition 4.11. *Let A be an operator of positive type with constant $C\geqslant 1$ in X, then the operators $I+\epsilon A: D(I+\epsilon A)\subset X\to X$ are of positive type with constant C. Moreover, the family of operators $\{(I+\epsilon A)^{-\beta}\}_{(\epsilon,\beta)\in[0,1]\times[0,1]}$ is uniformly bounded.*

Proof: See Appendix A.

Proof of Theorem 1.8: It follows from the previous theorem and Proposition 4.11. ∎

Corollary 4.12. $\cup_{\epsilon\in[0,1]}\tilde{\mathbb{A}}_\epsilon$ *is precompact in* $X^{\frac{1}{2}}\times X$*.*

5. Continuity of attractors

5.1. **Upper semicontinuity of attractors.** This section is devoted to the study of the upper semicontinuity of the family of global attractors $\{\mathbb{A}_\epsilon\}_{\epsilon\in[0,1]}$ at $\epsilon=0$ and as a consequence, the upper semicontinuity of $\{\tilde{\mathbb{A}}_\epsilon\}_{\epsilon\in[0,1]}$

To start this discussion, we have the following lemma:

Lemma 5.1. *If* $\left\{\begin{bmatrix} w_\epsilon^0 \\ z_\epsilon^0 \end{bmatrix}\right\}_{\epsilon\in(0,1]}\subset X\times X$ *is such that* $\begin{bmatrix} w_\epsilon^0 \\ z_\epsilon^0 \end{bmatrix}\stackrel{\epsilon\to0^+}{\longrightarrow}\begin{bmatrix} w_0^0 \\ z_0^0 \end{bmatrix}$ *for some* $\begin{bmatrix} w_0^0 \\ z_0^0 \end{bmatrix}\in X\times X$*, then we have*

$$\begin{bmatrix} w_\epsilon(t) \\ z_\epsilon(t) \end{bmatrix}\stackrel{\epsilon\to0^+}{\longrightarrow}\begin{bmatrix} w_0(t) \\ z_0(t) \end{bmatrix},\ \textit{for each } t\geqslant 0,$$

where $\begin{bmatrix} w_\epsilon(\cdot) \\ z_\epsilon(\cdot) \end{bmatrix}$ *is the solution of* (1.7) *with initial condition* $\begin{bmatrix} w_\epsilon(0) \\ z_\epsilon(0) \end{bmatrix}=\begin{bmatrix} w_\epsilon^0 \\ z_\epsilon^0 \end{bmatrix}$*, for each* $\epsilon\in[0,1]$*.*

Proof: We know that, for each $\begin{bmatrix} w_\epsilon^0 \\ z_\epsilon^0 \end{bmatrix}\in X\times X$, the solution of (1.7) is given by

$$\begin{bmatrix} w_\epsilon(t) \\ z_\epsilon(t) \end{bmatrix}=e^{-\mathcal{A}_\epsilon t}\begin{bmatrix} w_\epsilon^0 \\ z_\epsilon^0 \end{bmatrix}+\int_0^t e^{-\mathcal{A}_\epsilon(t-s)}\mathcal{F}_\epsilon\left(\begin{bmatrix} w_\epsilon(s) \\ z_\epsilon(s) \end{bmatrix}\right)ds,$$

for each $t\geqslant 0$. Thus we have

$$\begin{bmatrix} w_\epsilon(t) \\ z_\epsilon(t) \end{bmatrix}-\begin{bmatrix} w_0(t) \\ z_0(t) \end{bmatrix}=\underbrace{e^{-\mathcal{A}_\epsilon t}\begin{bmatrix} w_\epsilon^0 \\ z_\epsilon^0 \end{bmatrix}-e^{-\mathcal{A}_0 t}\begin{bmatrix} w_0^0 \\ z_0^0 \end{bmatrix}}_{I_1(\epsilon)}+\underbrace{\int_0^t e^{-\mathcal{A}_\epsilon(t-s)}\mathcal{F}_\epsilon\left(\begin{bmatrix} w_\epsilon(s) \\ z_\epsilon(s) \end{bmatrix}\right)-e^{-\mathcal{A}_0(t-s)}\mathcal{F}_0\left(\begin{bmatrix} w_0(s) \\ z_0(s) \end{bmatrix}\right)ds}_{I_2(\epsilon)}.$$

We analise $I_1(\epsilon)$ and $I_2(\epsilon)$ separately. First, note that

$$I_1=e^{-\mathcal{A}_\epsilon t}\left(\begin{bmatrix} w_\epsilon^0 \\ z_\epsilon^0 \end{bmatrix}-\begin{bmatrix} w_0^0 \\ z_0^0 \end{bmatrix}\right)+(e^{-\mathcal{A}_\epsilon t}-e^{-\mathcal{A}_0 t})\begin{bmatrix} w_0^0 \\ z_0^0 \end{bmatrix},$$

and the hypothesis together with Theorem 1.4 ensures that $I_1(\epsilon) \to 0$ as $\epsilon \to 0^+$. Now

$$\begin{aligned} I_2(\epsilon) = & \underbrace{\int_0^t e^{-\mathcal{A}_\epsilon(t-s)} \left[\mathcal{F}_\epsilon\left(\left[\begin{smallmatrix} w_\epsilon(s) \\ z_\epsilon(s)\end{smallmatrix}\right]\right) - \mathcal{F}_\epsilon\left(\left[\begin{smallmatrix} w_0(s) \\ z_0(s)\end{smallmatrix}\right]\right)\right] ds}_{I_2^1(\epsilon)} \\ & + \underbrace{\int_0^t \left[e^{-\mathcal{A}_\epsilon(t-s)} - e^{-\mathcal{A}_0(t-s)}\right] \mathcal{F}_\epsilon\left(\left[\begin{smallmatrix} w_0(s) \\ z_0(s)\end{smallmatrix}\right]\right) ds}_{I_2^2(\epsilon)} \\ & + \underbrace{\int_0^t e^{-\mathcal{A}_0(t-s)} \left[\mathcal{F}_\epsilon\left(\left[\begin{smallmatrix} w_0(s) \\ z_0(s)\end{smallmatrix}\right]\right) - \mathcal{F}_0\left(\left[\begin{smallmatrix} w_0(s) \\ z_0(s)\end{smallmatrix}\right]\right)\right] ds,}_{I_2^3(\epsilon)} \end{aligned}$$

and again we will analise $I_2^1(\epsilon), I_2^2(\epsilon)$ and $I_2^3(\epsilon)$ separately. For $I_2^1(\epsilon)$ we have that, given $\alpha \in (1/2, 1)$,

$$\begin{aligned} \|I_2^1(\epsilon)\|_{X\times X} & \leqslant \int_0^t \|\mathcal{A}_\epsilon^\alpha e^{-\mathcal{A}_\epsilon(t-s)}\|_{\mathcal{L}(X\times X)} \left\|\mathcal{A}_\epsilon^{-\alpha}\left[\mathcal{F}_\epsilon\left(\left[\begin{smallmatrix} w_\epsilon(s) \\ z_\epsilon(s)\end{smallmatrix}\right]\right) - \mathcal{F}_\epsilon\left(\left[\begin{smallmatrix} w_0(s) \\ z_0(s)\end{smallmatrix}\right]\right)\right]\right\|_{X\times X} ds \\ & \leqslant \int_0^t Ce^{-\omega(t-s)}(t-s)^{-\alpha} \left\|\left[\begin{smallmatrix} w_\epsilon(s) \\ z_\epsilon(s)\end{smallmatrix}\right] - \left[\begin{smallmatrix} w_0(s) \\ z_0(s)\end{smallmatrix}\right]\right\|_{X\times X} ds. \end{aligned}$$

For $I_2^2(\epsilon)$ we have that, given $s \in \left[\frac{(\rho-1)(n-2)}{4}, 1\right)$ and $\gamma \in (s, 1)$,

$$\begin{aligned} \|I_2^2(\epsilon)\|_{X\times X} & \leqslant \int_0^t \left\|e^{-\mathcal{A}_\epsilon(t-s)} - e^{-\mathcal{A}_0(t-s)}\right\|_{\mathcal{L}(X\times X)} \|(I+\epsilon A)^{-\frac12} f(A^{-\frac12} w_0(s))\|_X ds \\ & \leqslant \int_0^t MC_1 e^{-\omega(t-s)}(t-s)^{-\gamma}\epsilon^{\frac{\gamma-s}{2}} \|f(A^{-\frac12}w_0(s))\|_{H^{-s}} ds \\ & \leqslant \tilde{C}\epsilon^{\frac{\gamma-s}{2}}. \end{aligned}$$

For $I_2^3(\epsilon)$ we have that, for a given $\alpha \in (1/2, 1)$ and $s \in \left[\frac{(\rho-1)(n-2)}{4}, 1\right)$,

$$\begin{aligned} \|I_2^3(\epsilon)\|_{X\times X} & \leqslant \int_0^t \|\mathcal{A}_0^\alpha e^{-\mathcal{A}_0(t-s)}\|_{\mathcal{L}(X\times X)} \left\|\mathcal{A}_0^{-\alpha}\left[\mathcal{F}_\epsilon\left(\left[\begin{smallmatrix} w_0(s) \\ z_0(s)\end{smallmatrix}\right]\right) - \mathcal{F}_0\left(\left[\begin{smallmatrix} w_0(s) \\ z_0(s)\end{smallmatrix}\right]\right)\right]\right\|_{X\times X} ds \\ & \leqslant \int_0^t Ce^{-\omega(t-s)}(t-s)^{-\alpha}\|[A^{-\frac12}(I+\epsilon A)^{-\frac12} - A^{-\frac12}]f(A^{-\frac12}w_0(s))\|_X ds \\ & = \int_0^t Ce^{-\omega(t-s)}(t-s)^{-\alpha}\|[A^{\frac{s-1}{2}}(I+\epsilon A)^{-\frac12} - A^{\frac{s-1}{2}}]A^{-\frac{s}{2}}f(A^{-\frac12}w_0(s))\|_X ds \\ & \leqslant \tilde{C}\epsilon^{\frac{1-s}{2}}. \end{aligned}$$

Joining these estimates we proved that

$$\left\|\left[\begin{smallmatrix} w_\epsilon(t) \\ z_\epsilon(t)\end{smallmatrix}\right] - \left[\begin{smallmatrix} w_0(t) \\ z_0(t)\end{smallmatrix}\right]\right\|_{X\times X} \leqslant l(\epsilon) + \int_0^t Ce^{-\omega(t-s)}(t-s)^{-\alpha}\left\|\left[\begin{smallmatrix} w_\epsilon(s) \\ z_\epsilon(s)\end{smallmatrix}\right] - \left[\begin{smallmatrix} w_0(s) \\ z_0(s)\end{smallmatrix}\right]\right\|_{X\times X} ds,$$

where $l(\epsilon) \to 0$ as $\epsilon \to 0^+$, and using a Singular Gronwall's Lemma (Lemma 7.1.1 in [17]) we have that

$$\left\| \begin{bmatrix} w_\epsilon(t) \\ z_\epsilon(t) \end{bmatrix} - \begin{bmatrix} w_0(t) \\ z_0(t) \end{bmatrix} \right\|_{X\times X} \to 0, \text{ as } \epsilon \to 0^+, \text{ for each } t \geqslant 0.$$

■

Now, using this result together with Corollary 4.10 we can prove the following:

Lemma 5.2. *If* $\left\{ \begin{bmatrix} w_\epsilon^0 \\ z_\epsilon^0 \end{bmatrix} \right\}_{\epsilon\in(0,1]} \subset X \times X$ *is such that* $\begin{bmatrix} w_\epsilon^0 \\ z_\epsilon^0 \end{bmatrix} \in \mathbb{A}_\epsilon$ *for each* $\epsilon \in (0,1]$ *and* $\begin{bmatrix} w_\epsilon^0 \\ z_\epsilon^0 \end{bmatrix} \stackrel{\epsilon\to 0^+}{\longrightarrow} \begin{bmatrix} w_0^0 \\ z_0^0 \end{bmatrix}$ *for some* $\begin{bmatrix} w_0^0 \\ z_0^0 \end{bmatrix} \in X \times X$, *then* $\begin{bmatrix} w_0^0 \\ z_0^0 \end{bmatrix} \in \mathbb{A}_0$.

Proof: Let $\begin{bmatrix} w_\epsilon(t) \\ z_\epsilon(t) \end{bmatrix}$ be the global solution through $\begin{bmatrix} w_\epsilon^0 \\ z_\epsilon^0 \end{bmatrix}$, for each $\epsilon \in (0,1]$. Since $\begin{bmatrix} w_\epsilon(-1) \\ z_\epsilon(-1) \end{bmatrix} \in \bigcup_{\epsilon\in[0,1]} \mathbb{A}_\epsilon$, there exists a subsequence $\epsilon_{n_1} \to 0$ as $n_1 \to \infty$ and a point $\begin{bmatrix} w_0(-1) \\ z_0(-1) \end{bmatrix} \in X \times X$ such that

$$\begin{bmatrix} w_{\epsilon_{n_1}}(-1) \\ z_{\epsilon_{n_1}}(-1) \end{bmatrix} \to \begin{bmatrix} w_0(-1) \\ z_0(-1) \end{bmatrix}, \text{ as } n_1 \to \infty.$$

By Lemma 5.1,

$$\begin{bmatrix} w_\epsilon^0 \\ z_\epsilon^0 \end{bmatrix} = T_\epsilon(1) \begin{bmatrix} w_{\epsilon_{n_1}}(-1) \\ z_{\epsilon_{n_1}}(-1) \end{bmatrix} \to T_0(1) \begin{bmatrix} w_0(-1) \\ z_0(-1) \end{bmatrix},$$

and hence $T_0(1) \begin{bmatrix} w_0(-1) \\ z_0(-1) \end{bmatrix} = \begin{bmatrix} w_0^0 \\ z_0^0 \end{bmatrix}$. Inductively, if we have chosen a subsequence $\{n_k\}$ of $\{n_{k-1}\}$ and a point $\begin{bmatrix} w_0(-k) \\ z_0(-k) \end{bmatrix} \in X \times X$ such that

$$\begin{bmatrix} w_{\epsilon_{n_k}}(-k) \\ z_{\epsilon_{n_k}}(-k) \end{bmatrix} \to \begin{bmatrix} w_0(-k) \\ z_0(-k) \end{bmatrix}, \text{ as } n_k \to \infty.$$

Again, using Lemma 5.1, we have

$$\begin{bmatrix} w_\epsilon(-k+1) \\ z_\epsilon(-k+1) \end{bmatrix} = T_\epsilon(1) \begin{bmatrix} w_{\epsilon_{n_1}}(-k) \\ z_{\epsilon_{n_1}}(-k) \end{bmatrix} \to T_0(1) \begin{bmatrix} w_0(-k) \\ z_0(-k) \end{bmatrix},$$

and hence $T_0(1) \begin{bmatrix} w_0(-k) \\ z_0(-k) \end{bmatrix} = \begin{bmatrix} w_0(-k+1) \\ z_0(-k+1) \end{bmatrix}$. Now define for each $t \in \mathbb{R}$

$$\begin{bmatrix} w_0(t) \\ z_0(t) \end{bmatrix} = \begin{cases} \begin{bmatrix} w_0(-k) \\ z_0(-k) \end{bmatrix}, \text{ if } t = -k \in \mathbb{Z}_-; \\ T_0(t+k) \begin{bmatrix} w_0(-k) \\ z_0(-k) \end{bmatrix}, \text{ if } t \in (-k, -k+1); \\ \begin{bmatrix} w_0^0 \\ z_0^0 \end{bmatrix}, \text{ if } t = 0; \\ T_0(t) \begin{bmatrix} w_0^0 \\ z_0^0 \end{bmatrix}, \text{ if } t > 0. \end{cases},$$

and thus $\begin{bmatrix} w_0(t) \\ z_0(t) \end{bmatrix}$ is a bounded global solution through $\begin{bmatrix} w_0^0 \\ z_0^0 \end{bmatrix}$ of $\{T_0(t) : t \geqslant 0\}$ and therefore $\begin{bmatrix} w_0^0 \\ z_0^0 \end{bmatrix} \in \mathbb{A}_0$. ■

Lemmas 5.1 and 5.2 together with Lemma 3.2 of [5] prove the upper semicontinuity at $\epsilon = 0$ of $\{\mathbb{A}_\epsilon\}_{\epsilon\in[0,1]}$ and we have the following result:

Theorem 5.3. *The family $\{\mathbb{A}_\epsilon\}_{\epsilon\in[0,1]}$ is upper semicontinuous in $\epsilon = 0$.*

With the upper semicontinuity of the family $\{\mathbb{A}_\epsilon\}_{\epsilon\in[0,1]}$ at $\epsilon = 0$ we are one step away to prove the upper semicontinuity of the family of global attractors $\{\tilde{\mathbb{A}}_\epsilon\}_{\epsilon\in[0,1]}$ at $\epsilon = 0$. All we need is the following proposition:

Proposition 5.4. *If $s \in [0,1]$ and $x \in D(A^{\frac{s}{2}})$ then*

$$\|(I+\epsilon A)^{-\frac{1}{2}}x - x\|_X \leqslant C\epsilon^{\frac{s}{2}}\|A^{\frac{s}{2}}x\|_X.$$

Proof of Theorem 1.9: Just note that

$$\|\left[\begin{smallmatrix} u_\epsilon \\ v_\epsilon \end{smallmatrix}\right] - \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]\|_{X^{\frac{1}{2}}\times X} \leqslant \left\| \left[\begin{smallmatrix} w_\epsilon \\ (I+\epsilon A)^{-\frac{1}{2}} z_\epsilon \end{smallmatrix}\right] - \left[\begin{smallmatrix} w_0 \\ z_0 \end{smallmatrix}\right] \right\|_{X\times X} \leqslant \|(I+\epsilon A)^{-\frac{1}{2}}z_\epsilon - z_\epsilon\|_X + \|\left[\begin{smallmatrix} w_\epsilon \\ z_\epsilon \end{smallmatrix}\right] - \left[\begin{smallmatrix} w_0 \\ z_0 \end{smallmatrix}\right]\|_{X\times X},$$

for any $\left[\begin{smallmatrix} u_\epsilon \\ v_\epsilon \end{smallmatrix}\right] \in \mathbb{C}_\epsilon$. Now the result follows Proposition 5.4 and Theorem 5.3. ∎

5.2. **Lower semicontinuity of attractors.** The study of lower semicontinuity of attractors is a harder deal than the upper semicontinuity and requires a fine study of the local structures in the global attractors; that is, we need to study the continuity of the local unstable manifolds of the linearized problems around each equilibrium point $\left[\begin{smallmatrix} \phi \\ 0 \end{smallmatrix}\right] \in \mathcal{E}$ (recall Section 4), which is given by

$$(P_\epsilon) \qquad \frac{d}{dt}\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right] + \mathcal{A}_{\epsilon,\phi}\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right] = \mathcal{F}_{\epsilon,\phi}(\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right]),$$

where $\mathcal{A}_{\epsilon,\phi} = \mathcal{A}_\epsilon - D\mathcal{F}_\epsilon(\left[\begin{smallmatrix} \phi \\ 0 \end{smallmatrix}\right])$ and $\mathcal{F}_{\epsilon,\phi}(\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right]) = \mathcal{F}_\epsilon(\left[\begin{smallmatrix} w+\phi \\ z \end{smallmatrix}\right]) - \mathcal{F}_\epsilon(\left[\begin{smallmatrix} \phi \\ 0 \end{smallmatrix}\right]) - D\mathcal{F}_\epsilon(\left[\begin{smallmatrix} \phi \\ 0 \end{smallmatrix}\right])\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right]$.

From now on we will make the following assumption:

(LS1) ϕ is an non-degenerate equilibrium for $A^{\frac{1}{2}}u = f^e \circ A^{-\frac{1}{2}}(u)$; that is $1 \in \rho(A^{-\frac{1}{2}}D(f^e \circ A^{-\frac{1}{2}})(\phi))$ and hence $I - A^{-\frac{1}{2}}D(f^e \circ A^{-\frac{1}{2}})(\phi)$ is invertible.

It is easy to see that

$$D\mathcal{F}_\epsilon(\left[\begin{smallmatrix} \phi \\ 0 \end{smallmatrix}\right]) = \begin{bmatrix} 0 & 0 \\ (I+\epsilon A)^{-\frac{1}{2}}D(f^e \circ A^{-\frac{1}{2}})(\phi) & 0 \end{bmatrix},$$

We now will study the convergence of the linear local unstable manifolds of the problems (P_ϵ), and to begin we discuss the generation of analytic semigroups by $-\mathcal{A}_{\epsilon,\phi}$.

Proposition 5.5. *Using the notations of Lemma 4.2, if $\phi \in \mathcal{E}_1$ then $D(f^e \circ A^{-\frac{1}{2}})(\phi)$ is a bounded linear operator in X.*

Proof: We know that, for each $\phi \in \mathcal{E}_1$ and $\eta \in X$,

$$(D(f^e \circ A^{-\frac{1}{2}})(\phi)\eta)(x) = f'(A^{-\frac{1}{2}}\phi(x))A^{-\frac{1}{2}}\eta(x),$$

and hence

$$\|D(f^e \circ A^{-\frac{1}{2}})(\phi)\eta\|_X^2 = \int_\Omega |f'(A^{-\frac{1}{2}}\phi(x))A^{-\frac{1}{2}}\eta(x)|^2 dx.$$

But since $A^{-\frac{1}{2}}\phi \in L^\infty(\Omega)$ (by Lemma 4.2) and $|f'(s)| \leqslant c(1+|s|^{\rho-1})$, $f'(A^{-\frac{1}{2}}\phi(\cdot)) \in L^\infty(\Omega)$ and thus

$$\|D(f^e \circ A^{-\frac{1}{2}})(\phi)\eta\|_X \leqslant K\|\eta\|_X.$$

■

Corollary 5.6. *$\{D\mathcal{F}_\epsilon([\begin{smallmatrix}\phi\\0\end{smallmatrix}])\}_{\epsilon\in[0,1]}$ is an uniformly bounded linear family of operators in $X \times X$.*

Corollary 5.7. *$\{\mathcal{A}_{\epsilon,\phi}\}_{\epsilon\in[0,1]}$ is an uniformly sectorial family of operators in $X\times X$, hence each $-\mathcal{A}_{\epsilon,\phi}$ generates an analytic semigroup $\{e^{-\mathcal{A}_{\epsilon,\phi}} :\ t \geqslant 0\}$ and there exist constants $M \geqslant 1$, $\omega \in \mathbb{R}$ such that*

$$\|e^{-\mathcal{A}_{\epsilon,\phi}t}\|_{\mathcal{L}(X\times X)} \leqslant Me^{-\omega t},\ \textit{for all } t \geqslant 0 \textit{ and all } \epsilon \in [0,1],$$

also there exists a $\varphi \in (0,\frac{\pi}{2})$ such that

$$\|(\lambda - \mathcal{A}_{\epsilon,\phi})^{-1}\| \leqslant \frac{M}{|\lambda-\omega|},\ \textit{for all } \lambda \in S_{\omega,\varphi} \textit{ and all } \epsilon \in [0,1].$$

It is by a simple calculation, and recalling that $0 \in \rho(\mathcal{A}_\epsilon)$ for all $\epsilon \in [0,1]$, that we can see that

$$\mathcal{A}_{\epsilon,\phi} = \mathcal{A}_\epsilon(I - \mathcal{A}_\epsilon^{-1}D\mathcal{F}_\epsilon([\begin{smallmatrix}\phi\\0\end{smallmatrix}])) = \mathcal{A}_\epsilon B,$$

where B is the invertible linear bounded operator given by

$$B = \begin{bmatrix} I - A^{-\frac{1}{2}}D(f^e\circ A^{-\frac{1}{2}})(\phi) & 0 \\ 0 & I \end{bmatrix}.$$

Therefore, using the assumption (LS1), we have that $0 \in \rho(\mathcal{A}_{\epsilon,\phi})$ and $\mathcal{A}_{\epsilon,\phi}^{-1} = B^{-1}\mathcal{A}_\epsilon^{-1}$ which gives

$$\|\mathcal{A}_{\epsilon,\phi}^{-1} - \mathcal{A}_{0,\phi}^{-1}\|_{\mathcal{L}(X)} \leqslant \|B^{-1}\|_{\mathcal{L}(X)}\|\mathcal{A}_\epsilon^{-1} - \mathcal{A}_0^{-1}\| \leqslant C\epsilon.$$

Now let $K \subseteq \mathbb{C}$ be a compact set and assume that $K \subseteq \rho(\mathcal{A}_{0,\phi})$. Since $\mathcal{A}_{0,\phi}(\lambda - \mathcal{A}_{0,\phi})^{-1}$ and $(\lambda - \mathcal{A}_{0,\phi})\mathcal{A}_{0,\phi}$ are in $\mathcal{L}(X)$ and they are inverse with each other, we have that $\lambda\mathcal{A}_{0,\phi}^{-1} - I = (\lambda - \mathcal{A}_{0,\phi})\mathcal{A}_{0,\phi}$ is an invertible operator and since

$$(\lambda\mathcal{A}_{\epsilon,\phi}^{-1} - I) - (\lambda\mathcal{A}_{0,\phi}^{-1} - I) = \lambda(\mathcal{A}_{\epsilon,\phi}^{-1} - \mathcal{A}_{0,\phi}^{-1}),$$

we have that, for ϵ sufficiently small, $(\lambda\mathcal{A}_{\epsilon,\phi}^{-1} - I)$ is invertible and

$$(\lambda\mathcal{A}_{\epsilon,\phi}^{-1} - I)^{-1} - (\lambda\mathcal{A}_{0,\phi}^{-1} - I)^{-1} \overset{\mathcal{L}(X)}{\to} 0,$$

as $\epsilon \to 0^+$, uniformly for $\lambda \in K$. Thus $(\lambda - \mathcal{A}_{\epsilon,\phi})$ is invertible for $\lambda \in K$ and ϵ sufficiently small, and

$$(\lambda - \mathcal{A}_{\epsilon,\phi})^{-1} - (\lambda - \mathcal{A}_{0,\phi})^{-1} \overset{\mathcal{L}(X)}{\to} 0,$$

as $\epsilon \to 0^+$, and we have proved the following result

Proposition 5.8. *Given $K \subseteq \mathbb{C}$ a compact set such that $K \subseteq \rho(\mathcal{A}_{0,\phi})$, there exists $\epsilon_0 \in [0,1]$ such that $K \subseteq \rho(\mathcal{A}_{\epsilon,\phi})$ for all $\epsilon \in [0,\epsilon_0]$ and*

$$\sup_{\lambda\in K} \|(\lambda - \mathcal{A}_{\epsilon,\phi})^{-1} - (\lambda - \mathcal{A}_{0,\phi})^{-1}\|_{\mathcal{L}(X\times X)} \to 0, \text{ as } \epsilon \to 0^+.$$

This lead us to the following result:

Proposition 5.9. *If $\left[\begin{smallmatrix}\phi\\0\end{smallmatrix}\right] \in \mathcal{E}$ is a hyperbolic equilibrium point for the problem (P_0) then there exists $\epsilon_0 \in (0,1]$ such that $\left[\begin{smallmatrix}\phi\\0\end{smallmatrix}\right]$ it is a hyperbolic equilibrium point for the problems (P_ϵ), for each $\epsilon \in [0,\epsilon_0]$.*

Proof: Since $\left[\begin{smallmatrix}\phi\\0\end{smallmatrix}\right]$ is a hyperbolic equilibrium point for (P_0), $\sigma(\mathcal{A}_{0,\phi})$ is separated from the imaginary axis; hence there exists a rectangle $K = \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \in [-a,a] \text{ and } \mathrm{Im}\lambda \in [-b,b]\}$ with $a,b > 0$ such that $\sigma(\mathcal{A}_{0,\phi}) \cap K = \varnothing$, and by Corollary 5.7, we can choose K such that it split $\mathbb{C} \setminus S_{\omega,\varphi}$ into two separated sets.

Then Proposition 5.8 implies that there exists $\epsilon_0 \in (0,1]$ such that $\sigma(\mathcal{A}_{\epsilon,\phi}) \cap K = \varnothing$ for all $\epsilon \in [0,\epsilon_0]$ and therefore $\left[\begin{smallmatrix}\phi\\0\end{smallmatrix}\right]$ is a hyperbolic equilibrium point for (P_ϵ). ∎

Now let $\sigma^+ = \sigma(-\mathcal{A}_{\epsilon,\phi}) \cap \{\mathrm{Re}\lambda > 0\}$ and Γ_+ be a closed simple curve in $\rho(-\mathcal{A}_{\epsilon,\phi})$ enclosing σ^+. We know that the associated linear unstable manifold U_ϵ of problem (P_ϵ) is given as the image of the projection Π_ϵ^+ defined by

$$\Pi_\epsilon^+ = \frac{1}{2\pi i}\int_{\Gamma^+} (\lambda + \mathcal{A}_{\epsilon,\phi})^{-1} d\lambda,$$

and Proposition 5.8 implies that

$$\|\Pi_\epsilon^+ - \Pi_0^+\|_{\mathcal{L}(X\times X)} \to 0, \text{ as } \epsilon \to 0^+.$$

Now that we have the convergence of the linear unstable manifolds, we study unstable manifolds of problem (P_ϵ), and to this end we begin with the following lemma.

Lemma 5.10. *If $f:\mathbb{R}\to\mathbb{R}$ is a C^2 function with f, f' and f'' bounded in $\mathbb{R}$, there exists $\zeta\in(0,1)$ such that*

$$\|f^e(A^{-\frac{1}{2}}u)-f^e(A^{-\frac{1}{2}}v)-f'(A^{-\frac{1}{2}}v)A^{-\frac{1}{2}}(u-v)\|_X\leqslant c\|u-v\|_X^{1+\zeta},\ \textit{for all}\ u,v\in X.$$

Proof: First we set $g(u,v)=f^e(A^{-\frac{1}{2}}u)-f^e(A^{-\frac{1}{2}}v)-f'(A^{-\frac{1}{2}}v)A^{-\frac{1}{2}}(u-v)$, and we can see that

$$\begin{aligned}|g(u,v)|&=|f^e(A^{-\frac{1}{2}}u)-f^e(A^{-\frac{1}{2}}v)-f'(A^{-\frac{1}{2}}v)A^{-\frac{1}{2}}(u-v)|\\&=|[f'(\theta A^{-\frac{1}{2}}u+(1-\theta)A^{-\frac{1}{2}}v)-f'(A^{-\frac{1}{2}}v)]A^{-\frac{1}{2}}(u-v)|\\&=|f''(\eta(\theta A^{-\frac{1}{2}}u+(1-\theta)A^{-\frac{1}{2}}v)+(1-\eta)A^{-\frac{1}{2}}v)||\theta||A^{-\frac{1}{2}}(u-v)|^2,\end{aligned}$$

and it is easy to see that there exist constants $c_1,c_2>0$ such that

$$\|g(u,v)\|_{L^1(\Omega)}\leqslant c_1\|A^{-\frac{1}{2}}(u-v)\|^2_{L^{\frac{2n}{n-2}}(\Omega)}$$

and

$$\|g(u,v)\|_{L^{\frac{2n}{n-2}}(\Omega)}\leqslant c_2\|A^{-\frac{1}{2}}(u-v)\|_{L^{\frac{2n}{n-2}}(\Omega)}.$$

In this way there exists $\zeta\in(0,1)$ such that $\frac{1}{2}=\zeta+(1-\zeta)\frac{n-2}{2n}$ and

$$\|g(u,v)\|_X\leqslant\|g(u,v)\|^{1-\zeta}_{L^1(\Omega)}\|g(u,v)\|^{\zeta}_{L^{\frac{2n}{n-2}}(\Omega)}\leqslant c_1^{1-\zeta}c_2^{\zeta}\|A^{-\frac{1}{2}}(u-v)\|^{1+\zeta}_{L^{\frac{2n}{n-2}}(\Omega)},$$

which concludes the proof, since $H^1_0(\Omega)\hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$. ∎

Corollary 5.11. *If $f:\mathbb{R}\to\mathbb{R}$ is a C^2 function with f, f' and f'' bounded in $\mathbb{R}$, there exists a $\zeta\in(0,1)$ such that*

$$\|\mathcal{F}_{\epsilon,\phi}(\left[\begin{smallmatrix}w_1\\z_1\end{smallmatrix}\right])-\mathcal{F}_{\epsilon,\phi}(\left[\begin{smallmatrix}w_2\\z_2\end{smallmatrix}\right])\|_{X\times X}\leqslant c\|\left[\begin{smallmatrix}w_1\\v_1\end{smallmatrix}\right]-\left[\begin{smallmatrix}w_2\\v_2\end{smallmatrix}\right]\|^{1+\zeta}_{X\times X},$$

for each $\epsilon\in[0,1]$.

Proposition 5.12. *In the conditions above, for each $\epsilon\in[0,\epsilon_0]$ there exists a local unstable manifold $W^{\mathrm{u},\epsilon}_{loc}(\left[\begin{smallmatrix}\phi\\0\end{smallmatrix}\right])$ which is a graph over a ball $B_r(0)$ of U_ϵ. Moreover, the family of local unstable manifolds $\{W^{\mathrm{u},\epsilon}_{loc}(\left[\begin{smallmatrix}\phi\\0\end{smallmatrix}\right])\}_{\epsilon\in[0,\epsilon_0]}$ is continuous at $\epsilon=0$.*

Proof: This is a consequence of Corollary 5.11 and the results reported in [18]. ∎

Theorem 5.13. *Suppose that all the conditions above are satisfied and assume also that the set of equilibrium points $\mathcal{E}$ and each $\left[\begin{smallmatrix}\phi\\0\end{smallmatrix}\right]\in\mathcal{E}$ is a hyperbolic equilibrium point for (P_0), then family of global attractors $\{\mathbb{A}_\epsilon\}_{\epsilon\in[0,1]}$ is lower semicontinuous at $\epsilon=0$.*

Proof: Proposition 5.9 implies that $\mathcal{E}$ consists of hyperbolic points of (P_ϵ) for each $\epsilon \in [0, \epsilon_0]$ and by Proposition 5.12, the family of local unstable manifolds $\{W^{u,\epsilon}_{loc}(\left[\begin{smallmatrix}\phi\\0\end{smallmatrix}\right])\}_{\epsilon\in[0,\epsilon_0]}$ is continuous at $\epsilon = 0$. Finally, Proposition 4.4 implies, in particular, that

$$\mathbb{A}_0 = \bigcup_{\left[\begin{smallmatrix}\phi\\0\end{smallmatrix}\right]\in\mathcal{E}} W^{u,0}_{loc}(\left[\begin{smallmatrix}\phi\\0\end{smallmatrix}\right]),$$

and the result follows from the results reported in [2, 16]. ∎

<u>Proof of Theorem 1.10:</u> It is analogous to the proof of Theorem 1.9 , using Theorem 5.13 instead of Theorem 5.3.

∎

6. Fractal dimension of attractors and entropy numbers

In this section, we are interested in giving uniform bounds for the fractal dimension of the global attractors $\mathbb{A}_\epsilon$ of the semigroups $\{T_\epsilon(t) : t \geqslant 0\}$ generated by equation (1.7). To begin, let us recall the definitions of fractal dimension and entropy numbers.

Definition 6.1. *Let $\mathcal{Z}$ be a metric space and $\mathcal{K}$ a compact subset of Z. For each $r > 0$ let $\mathrm{N}_{\mathcal{Z}}(r, \mathcal{K})$ be the minimum number of balls of radius r necessary to cover $\mathcal{K}$. The fractal dimension of $\mathcal{K}$ is defined by*

$$c(\mathcal{K}) \leqslant \limsup_{r\to 0^+} \frac{\ln \mathrm{N}_{\mathcal{Z}}(r, \mathcal{K})}{\ln(\frac{1}{r})}.$$

Definition 6.2. *Let $\mathcal{Z}$ and $\mathcal{W}$ two Banach spaces such that $\mathcal{Z}$ is compactly embedded in $\mathcal{W}$. We define the entropy numbers e_k of $\mathcal{Z}$ in $\mathcal{W}$ by*

$$e_k = \inf \left\{ \eta > 0 : \ B^{\mathcal{Z}}_1(0) \subset \bigcup_{j=1}^{2^{k-1}} B^{\mathcal{W}}_\eta(w_j), \ w_j \in \mathcal{W} \text{ for } 1 \leqslant j \leqslant 2^{k-1} \right\}.$$

Roughly speaking, e_k is the solution of the equation $\mathrm{N}_{\mathcal{W}}(\eta, B^{\mathcal{Z}}_1(0)) = 2^{k-1}$.

Firstly, using Theorem 4 of [10], we are able to estimate the fractal dimension of the global attractors of (1.7). To this end, we prove two auxiliary lemmas.

Lemma 6.3. *For any $\gamma \in (0, 1)$, there exists a continuous function $h_\gamma : \mathbb{R} \to \mathbb{R}$ such that*

$$\|T_\epsilon(t) \left[\begin{smallmatrix}w_0\\z_0\end{smallmatrix}\right] - T_\epsilon(t) \left[\begin{smallmatrix}w_1\\z_1\end{smallmatrix}\right]\|_{H^{-\gamma}\times H^{-\gamma}} \leqslant h_\gamma(t) \, \|\left[\begin{smallmatrix}w_0\\z_0\end{smallmatrix}\right] - \left[\begin{smallmatrix}w_1\\z_1\end{smallmatrix}\right]\|_{H^{-\gamma}\times H^{-\gamma}},$$

for all $\left[\begin{smallmatrix}w_0\\z_0\end{smallmatrix}\right], \left[\begin{smallmatrix}w_1\\z_1\end{smallmatrix}\right] \in X \times X$.

Proof: Using the variation of constants formula, we have that for $\alpha \in (\frac{1}{2}, 1)$

$$\|T_\epsilon(t)\left[\begin{smallmatrix} w_0 \\ z_0 \end{smallmatrix}\right] - T_\epsilon(t)\left[\begin{smallmatrix} w_1 \\ z_1 \end{smallmatrix}\right]\|_{H^{-\gamma}\times H^{-\gamma}} \leqslant Me^{-\omega t}\|\left[\begin{smallmatrix} w_0 \\ z_0 \end{smallmatrix}\right] - \left[\begin{smallmatrix} w_1 \\ z_1 \end{smallmatrix}\right]\|_{H^{-\gamma}\times H^{-\gamma}}$$
$$+ \int_0^t Me^{-\omega(t-s)}(t-s)^{-\alpha}\left\|\mathcal{A}_\epsilon^{-\alpha}\left[\mathcal{F}_\epsilon\left(T_\epsilon(s)\left[\begin{smallmatrix} w_0 \\ z_0 \end{smallmatrix}\right]\right) - \mathcal{F}_\epsilon\left(T_\epsilon(s)\left[\begin{smallmatrix} w_1 \\ z_1 \end{smallmatrix}\right]\right)\right]\right\|_{H^{-\gamma}\times H^{-\gamma}} ds$$
$$\leqslant Me^{-\omega t}\|\left[\begin{smallmatrix} w_0 \\ z_0 \end{smallmatrix}\right] - \left[\begin{smallmatrix} w_1 \\ z_1 \end{smallmatrix}\right]\|_{H^{-\gamma}\times H^{-\gamma}} + M_\gamma\int_0^t e^{-\omega(t-s)}(t-s)^{-\alpha}\|T_\epsilon(s)\left[\begin{smallmatrix} w_0 \\ z_0 \end{smallmatrix}\right] - T_\epsilon(s)\left[\begin{smallmatrix} w_1 \\ z_1 \end{smallmatrix}\right]\|_{H^{-\gamma}\times H^{-\gamma}} ds,$$

and the result follows from a singular version of Grownwall's Lemma (Lemma 7.1.1 in [17]). ■

Lemma 6.4. *There exists $\gamma \in (0,1)$ and a continuous function $k : \mathbb{R} \to \mathbb{R}$ such that*

$$\|T_\epsilon(t)\left[\begin{smallmatrix} w_0 \\ z_0 \end{smallmatrix}\right] - T_\epsilon(t)\left[\begin{smallmatrix} w_1 \\ z_1 \end{smallmatrix}\right]\|_{X\times X} \leqslant Me^{-\omega t}\|\left[\begin{smallmatrix} w_0 \\ z_0 \end{smallmatrix}\right] - \left[\begin{smallmatrix} w_1 \\ z_1 \end{smallmatrix}\right]\|_{X\times X} + k(t)\|\left[\begin{smallmatrix} w_0 \\ z_0 \end{smallmatrix}\right] - \left[\begin{smallmatrix} w_1 \\ z_1 \end{smallmatrix}\right]\|_{H^{-\gamma}\times H^{-\gamma}},$$

for all $\left[\begin{smallmatrix} w_0 \\ z_0 \end{smallmatrix}\right], \left[\begin{smallmatrix} w_1 \\ z_1 \end{smallmatrix}\right] \in X \times X$.

Proof: We can write $T_\epsilon(t) = L_\epsilon(t) + U_\epsilon(t)$ where

$$L_\epsilon(t)\cdot = e^{-\mathcal{A}_\epsilon t}\cdot \ \text{ and } U_\epsilon(t)\cdot = \int_0^t e^{-\mathcal{A}_\epsilon(t-s)}\mathcal{F}_\epsilon(T_\epsilon(s)\cdot)ds.$$

It is easy to see that

$$\|L_\epsilon(t)\left[\begin{smallmatrix} w_0 \\ z_0 \end{smallmatrix}\right] - L_\epsilon(t)\left[\begin{smallmatrix} w_1 \\ z_1 \end{smallmatrix}\right]\|_{X\times X} \leqslant Me^{-\omega t}\|\left[\begin{smallmatrix} w_0 \\ z_0 \end{smallmatrix}\right] - \left[\begin{smallmatrix} w_1 \\ z_1 \end{smallmatrix}\right]\|_{X\times X}.$$

Also, if we choose $\alpha \in (\frac{1}{2}, 1)$ and $\gamma \in (0,1)$, we have that

$$\|U_\epsilon(t)\left[\begin{smallmatrix} w_0 \\ z_0 \end{smallmatrix}\right] - U_\epsilon(t)\left[\begin{smallmatrix} w_1 \\ z_1 \end{smallmatrix}\right]\|_{X\times X} \leqslant M\int_0^t e^{-\omega(t-s)}(t-s)^{-\alpha}\|\mathcal{A}_\epsilon^{-\alpha}[\mathcal{F}_\epsilon(T_\epsilon(s)\left[\begin{smallmatrix} w_0 \\ z_0 \end{smallmatrix}\right]) - \mathcal{F}_\epsilon(T_\epsilon(s)\left[\begin{smallmatrix} w_1 \\ z_1 \end{smallmatrix}\right])]\|_{X\times X}ds$$
$$\leqslant M_\gamma\int_0^t e^{-\omega(t-s)}(t-s)^{-\alpha}\|T_\epsilon(s)\left[\begin{smallmatrix} w_0 \\ z_0 \end{smallmatrix}\right] - T_\epsilon(s)\left[\begin{smallmatrix} w_1 \\ z_1 \end{smallmatrix}\right]\|_{H^{-\gamma}\times H^{-\gamma}}ds,$$

and by Lemma 6.3, there exists a function $k : \mathbb{R} \to \mathbb{R}$ such that

$$\|U_\epsilon(t)\left[\begin{smallmatrix} w_0 \\ z_0 \end{smallmatrix}\right] - U_\epsilon(t)\left[\begin{smallmatrix} w_1 \\ z_1 \end{smallmatrix}\right]\|_{X\times X} \leqslant k(t)\|\left[\begin{smallmatrix} w_0 \\ z_0 \end{smallmatrix}\right] - \left[\begin{smallmatrix} w_1 \\ z_1 \end{smallmatrix}\right]\|_{H^{-\gamma}\times H^{-\gamma}}.$$

■

Theorem 6.5. *Let $t_0 \geqslant 0$ such that $\lambda \doteq Me^{-\omega t_0} < \frac{1}{2}$ and define $K \doteq k(t_0)$, where k is the continuous function given in Lemma 6.4. Then for any $\nu \in (0, \frac{1}{2} - \lambda)$ we have that*

$$c(\mathbb{A}_\epsilon) \leqslant \frac{\ln \mathrm{N}_{H^{-\gamma}\times H^{-\gamma}}\left(\frac{\nu}{K}, B_1^{X\times X}(0)\right)}{\ln\left(\frac{1}{2(\lambda+\nu)}\right)}.$$

Proof: Is a direct consequence of Theorem 4 of [10]. ■

Now, using the results of Section 3.3.2 of [13], we can see that there exists a constant $c > 0$ such that, for the spaces $X = L^2(\Omega)$ and $H^{-\gamma}$, we have

$$e_k \leqslant ck^{-\frac{\gamma}{n}},$$

and therefore, taking k_0 sufficiently large so that $ck^{-\frac{\gamma}{n}} \leqslant \frac{\nu}{K}$, for $k \geqslant k_0$, we have that

$$\mathrm{N}_{H^{-\gamma}\times H^{-\gamma}}\left(\frac{\nu}{K}, B_1^{X\times X}(0)\right) \leqslant 2^{2k-2},$$

which implies that

$$\log \mathrm{N}_{H^{-\gamma}\times H^{-\gamma}}\left(\frac{\nu}{K}, B_1^{X\times X}(0)\right) \leqslant 2\ln 2\frac{(\frac{\nu}{cK})^{-\frac{n}{\gamma}}-1}{-\ln\left(2(\lambda+\nu)\right)}.$$

Defining $g(\nu) = \frac{(\frac{\nu}{cK})^{-\frac{n}{\gamma}}-1}{-\ln(2(\lambda+\nu))}$ we can see that

$$\lim_{\nu\to 0^+} g(\nu) = +\infty \text{ and } \lim_{\nu\to(\frac{1}{2}-\lambda)^-} g(\nu) = +\infty,$$

which means that $g(\nu)$ has a minimum ν_0 in the interval $\left(0, \frac{1}{2}-\lambda\right)$ and hence

$$c(\mathbb{A}_\epsilon) \leqslant 2\ln 2 g(\nu_0),$$

which proves the following result:

Theorem 6.6. *For any $\epsilon \in [0,1]$ we have that*

$$c(\mathbb{A}_\epsilon) \leqslant 2\ln 2 g(\nu_0).$$

And as a direct consequence, we have

Proof of Theorem 1.11: The result follows noting that $\Phi_\epsilon : X^{\frac{1}{2}} \times \tilde{X}_\epsilon^{\frac{1}{2}} \longrightarrow X\times X$ is an isometric isomorphism and $\tilde{\mathbb{A}}_\epsilon = \Phi_\epsilon^{-1}\mathbb{A}_\epsilon$, thus $c(\tilde{\mathbb{A}}_\epsilon) = c(\mathbb{A}_\epsilon)$ and taking $\tau_0 = 2\ln 2g(\nu_0)$. ■

Remark 6.7. *It is worthwhile to point out that is this last result, the fractal dimension must be rightly interpreted. The fractal dimension $c(\tilde{\mathbb{A}}_\epsilon)$ is obtained using $X^{\frac{1}{2}}\times\tilde{X}_\epsilon^{\frac{1}{2}}$ as the metric base space while $c(\mathbb{A}_\epsilon)$ is obtained using $X\times X$.*

Appendix A. Results on Functional Analysis

In this appendix we prove the basic results of functional analysis we used throughout our work.

Proof of Proposition 2.2: By Proposition 2.13 we have that for $\beta \in (0,1)$

$$\Lambda_\epsilon^{-\beta} = \frac{\sin \pi\beta}{\pi} \int_0^\infty s^{-\beta}(s+\Lambda_\epsilon)^{-1} ds,$$

and hence

$$\begin{aligned} \|\Lambda_\epsilon^{-\beta}\|_{\mathcal{L}(X)} &\leqslant (1+C)\frac{\sin \pi\beta}{\pi} \int_0^\infty \frac{s^{-\beta}}{s+1} ds \\ &\leqslant (1+C)\frac{\sin \pi\beta}{\pi} \left[\frac{1}{1-\beta} + \frac{1}{\beta}\right], \end{aligned}$$

and the cases $\beta = 0,1$ are trivial. Iin particular, there exists a constant $\mu > 0$ such that

$$(\Lambda_\epsilon^\beta x, x) = (\Lambda_\epsilon^{\frac{\beta}{2}} x, \Lambda_\epsilon^{\frac{\beta}{2}} x) \geqslant \mu \|x\|_X^2,$$

for all $\epsilon \in [0,1]$ and $\beta \in [0,1]$. ∎

Proof of Proposition 2.13: Let $s \in [0,\infty)$. We have that

$$\begin{aligned} s + \Lambda_\epsilon = s + A(I+\epsilon A)^{-1} &= [sI + (\epsilon s + 1)A](I+\epsilon A)^{-1} \\ &= (\epsilon s+1)\left(\frac{s}{\epsilon s+1} + A\right)(I+\epsilon A)^{-1}, \end{aligned}$$

thus $s + \Lambda_\epsilon$ is invertible and

$$\begin{aligned} (s+\Lambda_\epsilon)^{-1} &= \frac{1}{\epsilon s+1}(I+\epsilon A)\left(\frac{s}{\epsilon s+1} + A\right)^{-1} \\ &= \frac{\epsilon}{\epsilon s+1} I + \frac{1}{(\epsilon s+1)^2}\left(\frac{s}{\epsilon s+1} + A\right)^{-1}. \end{aligned}$$

Therefore

$$(1+s)\|(s+\Lambda_\epsilon)^{-1}\|_{\mathcal{L}(X)} \leqslant 1 + C,$$

for all $s \in [0,\infty)$ and $\epsilon \in [0,1]$. ∎

Proof of Proposition 2.27: By Theorem 2.14 we have that

$$\|\Lambda_\epsilon^\beta(\mu+\Lambda_\epsilon)^{-1}x\|_X \leqslant K\|\Lambda_\epsilon(\mu+\Lambda_\epsilon)^{-1}x\|_X^\beta \|(\mu+\Lambda_\epsilon)^{-1}x\|_X \leqslant \frac{K(1+C)^\beta C^{1-\beta}}{(\mu+1)^{1-\beta}}\|x\|_X.$$

∎

Proof of Proposition 2.22: We know that, for any given $\alpha \in [0, \frac{1}{2})$,

$$\Lambda_\epsilon^{-\frac{1}{2}} - \Lambda_0^{-\frac{1}{2}} = \frac{1}{\pi}\int_0^\infty s^{-\frac{1}{2}}(s+\Lambda_\epsilon)^{-1}(\Lambda_0 - \Lambda_\epsilon)(s+\Lambda_0)^{-1}ds$$
$$= \frac{1}{\pi}\int_0^\infty s^{-\frac{1}{2}}\Lambda_\epsilon^{\alpha}(s+\Lambda_\epsilon)^{-1}\Lambda_\epsilon^{1-\alpha}(\Lambda_\epsilon^{-1} - \Lambda_0^{-1})\Lambda_0(s+\Lambda_0)^{-1}ds,$$

and therefore

$$\|\Lambda_\epsilon^{-\frac{1}{2}} - \Lambda_0^{-\frac{1}{2}}\|_{\mathcal{L}(X)} \leqslant \frac{C\epsilon^\alpha}{\pi}\int_0^\infty \frac{s^{-\frac{1}{2}}}{(s+1)^{1-\alpha}}ds,$$

and the integral above is convergent for $\alpha \in [0, \frac{1}{2})$. ■

Remark A.1. *In the general case of a positive type operator, we cannot obtain the decay rate of $\epsilon^{\frac{1}{2}}$ with the technique of the last proposition. However, when we work with specific properties of a given operator, we may be able to obtain such rate. For instance, if A is the negative Laplacian with Dirichlet boundary conditions, we are able to prove the previous result with $\alpha = \frac{1}{2}$ as follows: let $v_n \in X$ an unitary eigenvector of A associated with λ_n, then*

$$(\Lambda_\epsilon^{-1/2} - \Lambda_0^{-1/2})v_n = \left[\frac{(1+\epsilon\lambda_n)^{1/2}}{\lambda_n^{1/2}} - \frac{1}{\lambda_n^{1/2}}\right]v_n = \frac{\epsilon\lambda_n^{1/2}}{1+(1+\epsilon\lambda_n)^{1/2}}v_n.$$

Therefore

$$\|(\Lambda_\epsilon^{-1/2} - \Lambda_0^{-1/2})v_n\|_X \leqslant \epsilon^{1/2},$$

and since the eigenfunctions constitute an orthonormal basis of X, we obtain the desired result.

Proof of Proposition 4.11: Let $s > 0$ and $\epsilon \in [0,1]$. If $\epsilon = 0$ then

$$(s + I + \epsilon A)^{-1} = (s+I)^{-1} = (s+1)^{-1}I,$$

and if $\epsilon \in (0,1]$ we have

$$(s + I + \epsilon A)^{-1} = \frac{1}{\epsilon}\left(\frac{s+1}{\epsilon} + A\right)^{-1},$$

which proves in both cases that $(0,\infty) \subset \rho(-(I+\epsilon A))$. Also, it is now easy to see that

$$(s+1)\|(s+I+\epsilon A)^{-1}\|_{\mathcal{L}(X)} \leqslant C,$$

and this proves that $I + \epsilon A$ is a positive type operator with constant C. For the last statement, we know that

$$(I+\epsilon A)^{-\beta} = \frac{\sin \pi\beta}{\pi}\int_0^\infty s^{-\beta}(s+I+\epsilon A)^{-1}ds,$$

and thus

$$\begin{aligned}\|(I+\epsilon A)^{-\beta}\|_{\mathcal{L}(X)} &\leqslant \frac{C}{\pi}\int_0^\infty \frac{s^{-\beta}}{s+1}ds \\ &\leqslant \frac{C\sin\pi\beta}{\pi}\left[\int_0^1 s^{-\beta}ds + \int_1^\infty s^{-\beta-1}ds\right] \\ &= \frac{C\sin\pi\beta}{\pi}\left[\frac{1}{1-\beta}+\frac{1}{\beta}\right],\end{aligned}$$

which proves the result. ■

Proof of Proposition 5.4: We have that

$$\begin{aligned}(I+\epsilon A)^{-\frac{1}{2}}x - x &= \frac{1}{\pi}\int_0^\infty t^{-\frac{1}{2}}(t+I+\epsilon A)^{-1}x\cdot dt - \frac{1}{\pi}\int_0^\infty t^{-\frac{1}{2}}(t+I)^{-1}x\cdot dt \\ &= \frac{1}{\pi}\int_0^\infty t^{-\frac{1}{2}}[(t+I+\epsilon A)^{-1}-(t+I)^{-1}]x\cdot dt \\ &= -\frac{1}{\pi}\int_0^\infty \frac{\epsilon t^{-\frac{1}{2}}}{t+1}A(t+I+\epsilon A)^{-1}x\cdot dt \qquad \text{(A.1)} \\ &= -\frac{1}{\pi}\int_0^\infty \frac{t^{-\frac{1}{2}}}{t+1}A\left(\frac{t+1}{\epsilon}+A\right)^{-1}x\cdot dt \\ &= -\frac{1}{\pi}\int_0^\infty \frac{t^{-\frac{1}{2}}}{t+1}A^{1-\frac{s}{2}}\left(\frac{t+1}{\epsilon}+A\right)^{-1}A^{\frac{s}{2}}x\cdot dt,\end{aligned}$$

and therefore, by Proposition 2.27, we have

$$\|(I+\epsilon A)^{-\frac{1}{2}}x - x\|_X \leqslant \frac{1}{\pi}\int_0^\infty K\epsilon^{\frac{s}{2}}\frac{t^{-\frac{1}{2}}}{(t+1)^{1+\frac{s}{2}}}\|A^{\frac{s}{2}}x\|_X dt \leqslant C\epsilon^{\frac{s}{2}}\|A^{\frac{s}{2}}x\|dt.$$

■

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