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CLASSIFCATION OF THE FAMILY OF QUADRATIC DIFFERENTIAL SYSTEMS POSSESSING INVARIANT ELLIPSES

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Classification of the family of quadratic differential systems possessing invariant ellipses

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Abstract

Consider the class **QS** of all non-degenerate quadratic systems. Note that each quadratic polynomial differential system can be identified with a point of $\mathbb{R}^{12}$ through its coefficients. In this paper we provide necessary and sufficient conditions for a system in **QS**, in term of its coefficients, to have at least one invariant ellipse. Let **QSE** be the whole class of non-degenerate planar quadratic differential systems possessing at least one invariant ellipse. For the class **QSE**, we give the global "bifurcation" diagram which indicates where an ellipse is present or absent and in case it is present, the diagram indicates if the ellipse is or not a limit cycle. The diagram is expressed in terms of affine invariant polynomials and it is done in the 12-dimensional space of parameters. This diagram is also an algorithm for determining for each quadratic system if it possesses an invariant ellipse and whether or not this ellipse is a limit cycle.

Key-words: Quadratic vector fields, affine invariant polynomials, invariant algebraic curve, invariant ellipse, limit cycle.

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1 Introduction and statement of main results

Consider the differential systems of the form

$$\frac{dx}{dt} = P(x, y), \qquad \frac{dy}{dt} = Q(x, y), \tag{1}$$

where P, $Q \in \mathbb{R}[x, y]$, i.e. P and Q are polynomials in x and y over $\mathbb{R}$, and their associated vector field

$$X = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y}.$$

We define the *degree* of the differential system (1) as the integer $m = \max(\deg P, \deg Q)$. A *quadratic* differential system is a system of the (1) where $m = 2$.

From now on we assume that the polynomials P and Q are coprime, otherwise, up to a rescaling of the time, systems (1) can be reduced to linear or constant systems. We call the quadratic systems under this assumption *non-degenerate quadratic systems*. Let **QS** be the whole class of real non-degenerate quadratic differential systems.

The motivation for studying the class of quadratic systems came from their usefulness in many applications, as well as for theoretical reasons, as discussed by Schlomiuk and Vulpe in the introduction of [21]. They appear when modeling natural phenomena, for instance in mathematical biology, in chemistry or in physics.

We say that a non-constant differentiable function $H : V \to \mathbb{R}$ is a first integral of system (1) on V (an open and dense subset of $\mathbb{R}^2$), if $H(x(t), y(t))$ is constant for all the values of t for which $(x(t), y(t))$ is a solution of this system contained in V. In other words, H is a first integral of systems (1) if and only if

$$X(H) = P\frac{\partial H}{\partial x} + Q\frac{\partial H}{\partial y} = 0,$$

for all $(x, y) \in V$. We say that system (1) is integrable on V if it has a first integral on V.

If we know a first integral of a given planar differential system, we can draw its phase portrait. Therefore, it is of particular interest in planar differential systems to investigate about its existence.

We say that the curve $f(x, y) = 0$ ($f \in \mathbb{C}[x, y]$) is an *invariant algebraic curve* of system (1) if there exists $K \in \mathbb{C}[x, y]$ such that

$$P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf.$$

The polynomial K is called *cofactor* of the invariant algebraic curve $f = 0$. We see that, when $K = 0$, f is a polynomial first integral.

Quadratic systems with an invariant algebraic curve have been studied by several authors, for instance Schlomiuk and Vulpe in [20, 21] have investigated quadratic systems with invariant straight lines, quadratic systems having an ellipse as a limit cycle was investigated by Qin Yuan-xum [14]; the necessary and sufficient conditions for existence and uniqueness of an invariant algebraic curve of second degree in terms of the coefficients of quadratic systems was presented by Druzhkova [10]; Cairó and Llibre in [4] have investigated the Darboux integrability of the quadratic systems having invariant algebraic conics, and Oliveira, Rezende and Vulpe [15] provided necessary and sufficient conditions for a system in **QS** to have at least one invariant hyperbola in terms of its coefficients.

In this paper we investigate non-degenerate quadratic systems having invariant ellipses applying the invariant theory. On the class **QS** acts the group of real affine transformations and time rescaling and then, modulo this group action, quadratic systems ultimately depend on five parameters. This group also acts on **QSE** and, modulo this action, the systems in this class depend on three parameters.

As we want this study to be intrinsic, independent of the normal form given to the systems, we use here invariant polynomials and geometric invariants for the classification.

If a polynomial differential system has an invariant algebraic curve $f(x,y)=0$, where $f(x,y)\in\mathbb{C}[x,y]$ is of degree n,

$$f(x,y)=a_0+a_{10}x+a_{01}y+\ldots+a_{n0}x^n+a_{n-1,1}x^{n-1}y+\ldots+a_{0n}y^n,$$

with $\hat{a}=(a_0,\ldots,a_{0n})\in\mathbb{C}^N$, where $N=(n+1)(n+2)/2$, then the equation $\lambda f(x,y)=0$, where $\lambda\in\mathbb{C}^*$, and $\mathbb{C}^*=\mathbb{C}\backslash\{0\}$, yields the same locus of complex points in the plane as the locus induced by $f(x,y)=0$. Therefore, each curve of degree n defined by $\hat{a}$ can be identified with a point $[\hat{a}]=[a_0:a_{10}:\ldots:a_{0n}]$ in $P_{N-1}(\mathbb{C})$.

Our interest here is in real polynomial differential equations. But, to each such a system of equations there corresponds the complex system with the same coefficients to which we can apply the theory of Darboux using complex invariant algebraic curves. Some of these curves may be with real coefficients. In this case we also have invariant algebraic curves in $\mathbb{R}^2$ of the real differential system.

In this work we are interested in systems possessing an invariant ellipse. The conics $f(x,y)=0$ with $f(x,y)\in\mathbb{R}[x,y]$ are classified via the group action of real affine transformation. The conics for which $f(x,y)$ is an irreducible polynomial over $\mathbb{C}$ can be brought by a real affine transformation to one the following four forms: 1) $x^2+y^2-1=0$ (ellipses); 2) $x^2-y^2-1=0$ (hyperbolas); 3) $y-x^2=0$ (parabolas); 4) $x^2+y^2+1=0$, these are empty in $\mathbb{R}^2$ with points only in $\mathbb{C}^2$. Some authors call these conics *complex ellipses* (see [4]). These complex ellipses will play a helpful role in our classification problem. So by an ellipse we will mean a conic $f(x,y)=0$ with real coefficients which can be brought by an affine transformation to an equation $x^2+y^2+a=0$ with $a=-1$ (an ordinary ellipse) or $a=1$ (a complex ellipse).

Our main results are stated in the following theorem.

Main Theorem. *Consider a non-degenerate quadratic system.*

(*A*) *The conditions $\widehat{\gamma}_1=\widehat{\gamma}_2=0$ and either $\eta<0$ or $C_2=0$ are necessary for this system to possess at least one invariant ellipse. Assume that the condition $\widehat{\gamma}_1=\widehat{\gamma}_2=0$ is satisfied for this system.*

- **(*A*$_1$)** *If $\eta<0$ and $\widetilde{N}\neq 0$, then the system could possess at most one invariant ellipse. Moreover, the necessary and sufficient conditions for the existence of such an ellipse are given in* DIAGRAM *1, where we can also find the conditions for the ellipse to be real or complex.*
- **(*A*$_2$)** *If $\eta<0$ and $\widetilde{N}=0$, then the system either has no invariant ellipse or it has an infinite family of invariant ellipses. Moreover, the necessary and sufficient conditions for the existence of a family of invariant ellipses are given in* DIAGRAM *1, where we can also find the conditions for the ellipses to be real or/and complex. In addition, this system possesses a real invariant line and the positions of the invariant ellipses with respect to this line are presented in* FIGURE *1.*
- **(*A*$_3$)** *If $C_2=0$, then the system either has no invariant ellipse or it has an infinite family of invariant ellipses. Moreover, the necessary and sufficient conditions for the existence*

of a family of invariant ellipses are given in DIAGRAM *2, where we can also find the conditions for the ellipses to be real or/and complex. In addition, this system possesses a real invariant line and the positions of the invariant ellipses with respect to this line are presented in* FIGURE *2.*

(*B*) *A non-degenerate quadratic system possesses an algebraic limit cycle, which is an ellipse, if and only if* $\widehat{\gamma}_1 = \widehat{\gamma}_2 = 0$, $\eta < 0$, $\mathcal{T}_3\mathcal{F} < 0$, $\widehat{\beta}_1\widehat{\beta}_2 \neq 0$, *and one of the following sets of conditions is satisfied:*

(*B*₁) $\theta \neq 0$, $\widehat{\beta}_3 \neq 0$, $\widehat{\mathcal{R}}_1 < 0$;

(*B*₂) $\theta \neq 0$, $\widehat{\beta}_3 = 0$, $\widehat{\gamma}_3 = 0$, $\widehat{\mathcal{R}}_1 < 0$;

(*B*₃) $\theta = 0$, $\widehat{\gamma}_6 = 0$, $\widehat{\mathcal{R}}_5 < 0$.

Moreover, we see in DIAGRAM *1 how these limit cycles are displayed in the 12-parameter space.*

(*C*) *The* DIAGRAMS *1 and 2 actually contain the global "bifurcation" diagram in the 12-dimensional space of parameters of non-degenerate systems which possess at least one invariant ellipse. The corresponding conditions are given in terms of 36 invariant polynomials with respect to the group of affine transformations and time rescaling.*

Remark 1. *We place the word bifurcation in quotation marks because the diagram does not split the space according to distinct topological properties but according to algebraic properties such as presence or absence of an ellipse. The Diagram also shows some topological features whenever the ellipse turns out to be a limit cycle*

The invariants and comitants of differential equations used for proving our main result are obtained following the theory of algebraic invariants of polynomial differential systems, developed by Sibirsky and his disciples (see for instance [2, 6, 17, 23, 25]).

2 Preliminaries

Consider real quadratic systems of the form

$$\begin{aligned} \frac{dx}{dt} &= p_0 + p_1(x,y) + p_2(x,y) \equiv P(x,y), \\ \frac{dy}{dt} &= q_0 + q_1(x,y) + q_2(x,y) \equiv Q(x,y) \end{aligned} \tag{2}$$

with homogeneous polynomials p_i and q_i $(i = 0,1,2)$ of degree i in x, y:

$$\begin{aligned} p_0 = a_{00}, \quad p_1(x,y) = a_{10}x + a_{01}y, \quad p_2(x,y) = a_{20}x^2 + 2a_{11}xy + a_{02}y^2, \\ q_0 = b_{00}, \quad q_1(x,y) = b_{10}x + b_{01}y, \quad q_2(x,y) = b_{20}x^2 + 2b_{11}xy + b_{02}y^2. \end{aligned}$$

It is known that on the set quadratic systems acts the group $\mathit{Aff}(2,\mathbb{R})$ of affine transformations on the plane (cf. [18]). For every subgroup $G \subseteq \mathit{Aff}(2,\mathbb{R})$ we have an induced action of G on **QS** . We can identify the set **QS** of systems (2) with a subset of $\mathbb{R}^{12}$ via the map $\mathbf{QS} \longrightarrow \mathbb{R}^{12}$ which associates to

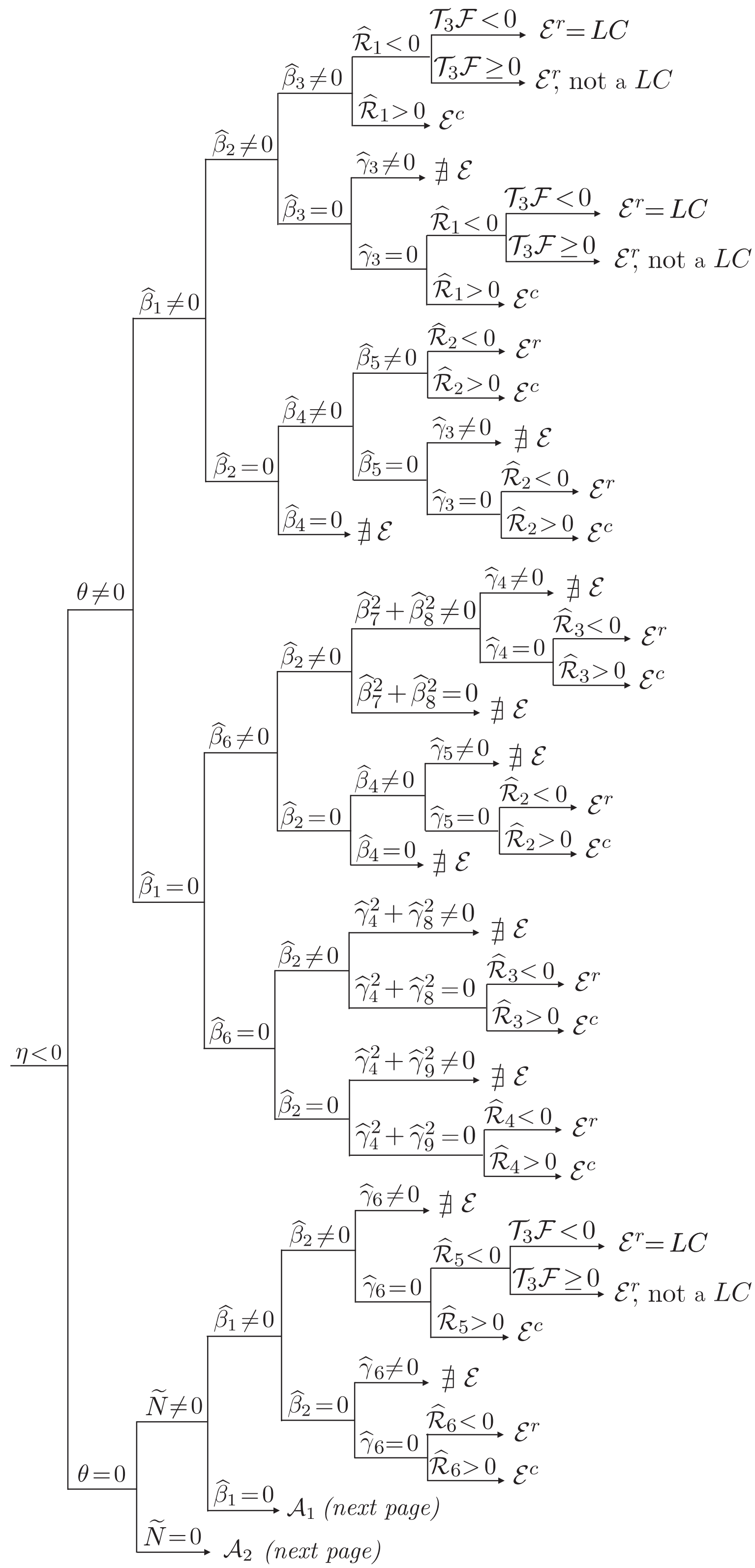

DIAGRAM 1: **The existence of invariant ellipse: the case $\boldsymbol{\eta<0}$**

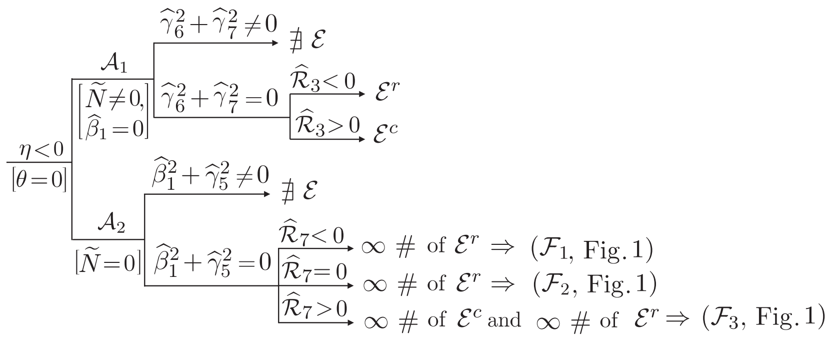

Diagram 1 (*cont.*): **The existence of invariant ellipse: the case $\boldsymbol{\eta < 0}$.**

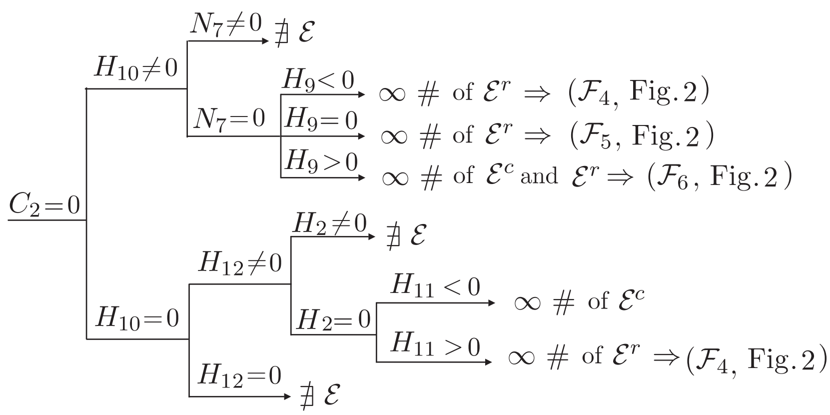

Diagram 2: **The existence of invariant ellipse: the case $\boldsymbol{C_2 = 0}$**

each system (2) the 12–tuple $\tilde{a} = (a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02})$ of its coefficients. We associate to this group action polynomials in x, y and parameters which behave well with respect to this action, the GL–comitants (GL–invariants), the T–comitants (affine invariants) and the CT–comitants. For their definitions as well as their detailed constructions we refer the reader to the paper [18] (see also [1]).

2.1 Main invariant polynomials associated with invariant ellipses

We single out the following five polynomials, basic ingredients in constructing invariant polynomials for systems (2):

$$\begin{aligned} C_i(\tilde{a}, x, y) &= y p_i(x, y) - x q_i(x, y), \ \ (i = 0, 1, 2)\\ D_i(\tilde{a}, x, y) &= \frac{\partial p_i}{\partial x} + \frac{\partial q_i}{\partial y}, \ \ (i = 1, 2). \end{aligned} \tag{3}$$

As it was shown in [23] these polynomials of degree one in the coefficients of systems (2) are GL-comitants of these systems. Let $f,\ g \in \mathbb{R}[\tilde{a}, x, y]$ and

$$(f, g)^{(k)} = \sum_{h=0}^{k} (-1)^h \binom{k}{h} \frac{\partial^k f}{\partial x^{k-h} \partial y^h} \frac{\partial^k g}{\partial x^h \partial y^{k-h}}.$$

The polynomial $(f, g)^{(k)} \in \mathbb{R}[\tilde{a}, x, y]$ is called *the transvectant of index k of (f, g)* (cf. [11,16]).

Lemma 1 ([25]). *Any GL-comitant of systems* (2) *can be constructed from the elements* (3) *by using the operations: $+$, $-$, $\times$, and by applying the differential operation $(*,*)^{(k)}$.*

Remark 2. We point out that the elements (3) generate the whole set of GL-comitants and hence also the set of affine comitants as well as the set of T-comitants.

We construct the following GL-comitants of the second degree with respect to the coefficients of the initial systems

$$\begin{aligned}
&T_1 = (C_0, C_1)^{(1)}, \quad T_2 = (C_0, C_2)^{(1)}, \quad T_3 = (C_0, D_2)^{(1)},\\
&T_4 = (C_1, C_1)^{(2)}, \quad T_5 = (C_1, C_2)^{(1)}, \quad T_6 = (C_1, C_2)^{(2)},\\
&T_7 = (C_1, D_2)^{(1)}, \quad T_8 = (C_2, C_2)^{(2)}, \quad T_9 = (C_2, D_2)^{(1)}.
\end{aligned} \tag{4}$$

Using these GL-comitants as well as the polynomials (3) we construct the additional invariant polynomials. In order to be able to calculate the values of the needed invariant polynomials directly for every canonical system we shall define here a family of T-comitants expressed through C_i ($i = 0, 1, 2$) and D_j ($j = 1, 2$):

$$\begin{aligned}
\hat{A} =\ & (C_1, T_8 - 2T_9 + D_2^2)^{(2)}/144,\\
\widehat{D} =\ & \big[2C_0(T_8 - 8T_9 - 2D_2^2) + C_1(6T_7 - T_6 - (C_1, T_5)^{(1)}\\
& + 6D_1C_1D_2 - T_5) - 9D_1^2C_2\big]/36,\\
\widehat{E} =\ & [D_1(2T_9 - T_8) - 3(C_1, T_9)^{(1)} - D_2(3T_7 + D_1D_2)]/72,\\
\widehat{F} =\ & [6D_1^2(D_2^2 - 4T_9) + 4D_1D_2(T_6 + 6T_7) + 48C_0(D_2, T_9)^{(1)}\\
& - 9D_2^2T_4 + 288D_1\widehat{E} - 24(C_2, \widehat{D})^{(2)} + 120(D_2, \widehat{D})^{(1)}\\
& - 36C_1(D_2, T_7)^{(1)} + 8D_1(D_2, T_5)^{(1)}]/144,\\
\widehat{B} =\ & \Big\{16D_1(D_2, T_8)^{(1)}(3C_1D_1 - 2C_0D_2 + 4T_2) + 32C_0(D_2, T_9)^{(1)}(3D_1D_2 - 5T_6 + 9T_7)\\
& + 2(D_2, T_9)^{(1)}\big(27C_1T_4 - 18C_1D_1^2 - 32D_1T_2 + 32(C_0, T_5)^{(1)}\big)\\
& + 6(D_2, T_7)^{(1)}\Big[8C_0(T_8 - 12T_9) - 12C_1(D_1D_2 + T_7) + D_1(26C_2D_1 + 32T_5)\\
& + C_2(9T_4 + 96T_3)\Big] + 6(D_2, T_6)^{(1)}\Big[32C_0T_9 - C_1(12T_7 + 52D_1D_2) - 32C_2D_1^2\Big]\\
& + 48D_2(D_2, T_1)^{(1)}(2D_2^2 - T_8) - 32D_1T_8(D_2, T_2)^{(1)} + 9D_2^2T_4(T_6 - 2T_7)\\
& - 16D_1(C_2, T_8)^{(1)}(D_1^2 + 4T_3) + 12D_1(C_1, T_8)^{(2)}(C_1D_2 - 2C_2D_1)\\
& + 6D_1D_2T_4(T_8 - 7D_2^2 - 42T_9) + 12D_1(C_1, T_8)^{(1)}(T_7 + 2D_1D_2) + 96D_2^2[D_1(C_1, T_6)^{(1)}\\
& + D_2(C_0, T_6)^{(1)}] - 16D_1D_2T_3(2D_2^2 + 3T_8) - 4D_1^3D_2(D_2^2 + 3T_8 + 6T_9)\\
& + 6D_1^2D_2^2(7T_6 + 2T_7)252D_1D_2T_4T_9\Big\}/(2^83^3),\\
\widehat{K} =\ & (T_8 + 4T_9 + 4D_2^2)/72, \quad \widehat{H} = (8T_9 - T_8 + 2D_2^2)/72.
\end{aligned}$$

These polynomials in addition to (3) and (4) will serve as bricks in constructing affine invariant polynomials for systems (2).

In paper [3] it was proved that the minimal polynomial basis of affine invariants up to degree 12 contain 42 elements, denoted by $A_1, \dots, A_{42}$. Here, using the above bricks, we present some of these basic elements which are necessary for the construction of needed invariant polynomials.

$$
\begin{aligned}
&A_1 = \hat{A}, \qquad A_2 = (C_2, \widehat{D})^{(3)}/12, \qquad A_3 = \left[\left[C_2, D_2\right)^{(1)}, D_2\right)^{(1)}, D_2\big)^{(1)}/48,\\
&A_4 = (\widehat{H}, \widehat{H})^{(2)}, \qquad A_5 = (\widehat{H}, \widehat{K})^{(2)}/2, \qquad A_6 = (\widehat{E}, \widehat{H})^{(2)}/2,
\end{aligned}
$$

$$
\begin{aligned}
&A_7 = \left[\left[C_2, \widehat{E}\right)^{(2)}, D_2\right)^{(1)}/8, \qquad A_8 = \left[\left[\widehat{D}, \widehat{H}\right)^{(2)}, D_2\right)^{(1)}/48, \qquad A_9 = \left[\left[\widehat{D}, D_2\right)^{(1)}, D_2\right)^{(1)},\\
&A_{10} = \left[\left[\widehat{D}, \widehat{K}\right)^{(2)}, D_2\right)^{(1)}/8, \qquad A_{11} = (\widehat{F}, \widehat{K})^{(2)}/4, \qquad A_{12} = (\widehat{F}, \widehat{H})^{(2)}/4,\\
&A_{13} = \left[\left[C_2, \widehat{H}\right)^{(1)}, \widehat{H}\right)^{(2)}, D_2\big)^{(1)}/24, \qquad A_{14} = (\widehat{B}, C_2)^{(3)}/36, \qquad A_{15} = (\widehat{E}, \widehat{F})^{(2)}/4,\\
&A_{17} = \left[\left[\widehat{D}, \widehat{D}\right)^{(2)}, D_2\right)^{(1)}, D_2\big)^{(1)}/64, \qquad A_{18} = \left[\left[\widehat{D}, \widehat{F}\right)^{(2)}, D_2\right)^{(1)}/16,\\
&A_{19} = \left[\left[\widehat{D}, \widehat{D}\right)^{(2)}, \widehat{H}\right)^{(2)}/16, \qquad A_{20} = \left[\left[C_2, \widehat{D}\right)^{(2)}, \widehat{F}\right)^{(2)}/16, \qquad A_{21} = \left[\left[\widehat{D}, \widehat{D}\right)^{(2)}, \widehat{K}\right)^{(2)}/16,\\
&A_{22} = \frac{1}{1152}\left[\left[C_2, \widehat{D}\right)^{(1)}, D_2\right)^{(1)}, D_2\big)^{(1)}, D_2\big)^{(1)} D_2\big)^{(1)}, \qquad A_{23} = \left[\left[\widehat{F}, \widehat{H}\right)^{(1)}, \widehat{K}\right)^{(2)}/8,\\
&A_{24} = \left[\left[C_2, \widehat{D}\right)^{(2)}, \widehat{K}\right)^{(1)}, \widehat{H}\big)^{(2)}/32, \qquad A_{31} = \left[\left[\widehat{D}, \widehat{D}\right)^{(2)}, \widehat{K}\right)^{(1)}, \widehat{H}\big)^{(2)}/64,\\
&A_{32} = \left[\left[\widehat{D}, \widehat{D}\right)^{(2)}, D_2\right)^{(1)}, \widehat{H}\big)^{(1)}, D_2\big)^{(1)}/64, \qquad A_{33} = \left[\left[\widehat{D}, D_2\right)^{(1)}, \widehat{F}\right)^{(1)}, D_2\big)^{(1)}, D_2\big)^{(1)}/128,\\
&A_{34} = \left[\left[\widehat{D}, \widehat{D}\right)^{(2)}, D_2\right)^{(1)}, \widehat{K}\big)^{(1)}, D_2\big)^{(1)}/64, \qquad A_{38} = \left[\left[C_2, \widehat{D}\right)^{(2)}, \widehat{D}\right)^{(2)}, \widehat{D}\big)^{(1)}, \widehat{H}\big)^{(2)}/64,\\
&A_{39} = \left[\left[\widehat{D}, \widehat{D}\right)^{(2)}, \widehat{F}\right)^{(1)}, \widehat{H}\big)^{(2)}/64, \qquad A_{41} = \left[\left[C_2, \widehat{D}\right)^{(2)}, \widehat{D}\right)^{(2)}, \widehat{F}\big)^{(1)}, D_2\big)^{(1)}/64,\\
&A_{42} = \left[\left[\widehat{D}, \widehat{F}\right)^{(2)}, \widehat{F}\right)^{(1)}, D_2\big)^{(1)}/16.
\end{aligned}
$$

In the above list, the double bracket "$[[$" is used in order to avoid placing the otherwise necessary up to five parentheses "(".

Using the elements of the minimal polynomial basis given above we construct the affine invariant polynomials

$$
\begin{aligned}
\widehat{\gamma}_1(\tilde{a}) = {} & A_1^2(3A_6 + 2A_7) - 2A_6(A_8 + A_{12}),\\
\widehat{\gamma}_2(\tilde{a}) = {} & 9A_1^2A_2(23252A_3 + 23689A_4) - 1440A_2A_5(3A_{10} + 13A_{11})\\
& - 1280A_{13}(2A_{17} + A_{18} + 23A_{19} - 4A_{20}) - 320A_{24}(50A_8 + 3A_{10}\\
& + 45A_{11} - 18A_{12}) + 120A_1A_6(6718A_8 + 4033A_9 + 3542A_{11}\\
& + 2786A_{12}) + 30A_1A_{15}(14980A_3 - 2029A_4 - 48266A_5)\\
& - 30A_1A_7(76626A_1^2 - 15173A_8 + 11797A_{10} + 16427A_{11} - 30153A_{12})\\
& + 8A_2A_7(75515A_6 - 32954A_7) + 2A_2A_3(33057A_8 - 98759A_{12})\\
& - 60480A_1^2A_{24} + A_2A_4(68605A_8 - 131816A_9 + 131073A_{10} + 129953A_{11})\\
& - 2A_2(141267A_6^2 - 208741A_5A_{12} + 3200A_2A_{13}),\\
\widehat{\gamma}_3(\tilde{a}) = {} & 843696A_5A_6A_{10} + A_1(-27(689078A_8 + 419172A_9 - 2907149A_{10}\\
& - 2621619A_{11})A_{13} - 26(21057A_3A_{23} + 49005A_4A_{23} - 166774A_3A_{24}\\
& + 115641A_4A_{24})),
\end{aligned}
$$

$$
\begin{aligned}
\widehat{\gamma}_4(\tilde{a}) &= -488A_2^3A_4 + A_2(12(4468A_8^2 + 32A_9^2 - 915A_{10}^2 + 320A_9A_{11} - 3898A_{10}A_{11} \\
&\quad - 3331A_{11}^2 + 2A_8(78A_9 + 199A_{10} + 2433A_{11})) + 2A_5(25488A_{18} \\
&\quad - 60259A_{19} - 16824A_{21}) + 779A_4A_{21}) + 4(7380A_{10}A_{31} \\
&\quad - 24(A_{10} + 41A_{11})A_{33} + A_8(33453A_{31} + 19588A_{32} - 468A_{33} - 19120A_{34}) \\
&\quad + 96A_9(-A_{33} + A_{34}) + 556A_4A_{41} - A_5(27773A_{38} + 41538A_{39} \\
&\quad - 2304A_{41} + 5544A_{42})), \\
\widehat{\gamma}_5(\tilde{a}) &= A_{22}, \\
\widehat{\gamma}_6(\tilde{a}) &= A_1(64A_3 - 541A_4)A_7 + 86A_8A_{13} + 128A_9A_{13} - 54A_{10}A_{13} \\
&\quad - 128A_3A_{22} + 256A_5A_{22} + 101A_3A_{24} - 27A_4A_{24}, \\
\widehat{\gamma}_7(\tilde{a}) &= A_2\big[2A_3(A_8 - 11A_{10}) - 18A_7^2 - 9A_4(2A_9 + A_{10}) + 22A_8A_{22} + 26A_{10}A_{22}, \\
\widehat{\gamma}_8(\tilde{a}) &= A_6, \\
\widehat{\gamma}_9(\tilde{a}) &= 12A_1^2 + 12A_8 + 5A_{10} + 17A_{11}, \\
\widehat{\beta}_1(\tilde{a}) &= 3A_1^2 - 2A_8 - 2A_{12}, \\
\widehat{\beta}_2(\tilde{a}) &= 2A_{13}, \\
\widehat{\beta}_3(\tilde{a}) &= 8A_3 + 27A_4 - 54A_5, \\
\widehat{\beta}_4(\tilde{a}) &= A_4, \\
\widehat{\beta}_5(\tilde{a}) &= 8A_5 - 5A_4, \\
\widehat{\beta}_6(\tilde{a}) &= A_3, \\
\widehat{\beta}_7(\tilde{a}) &= 24A_3 + 11A_4 + 20A_5, \\
\widehat{\beta}_8(\tilde{a}) &= 41A_8 + 44A_9 + 32A_{10}, \\
\widehat{\mathcal{R}}_1(\tilde{a}) &= \theta A_6\big[5A_6(A_{10} + A_{11}) - 2A_7(12A_1^2 + A_8 + A_{12}) - 2A_1(A_{23} - A_{24}) \\
&\quad + 2A_5(A_{14} + A_{15}) + A_6(9A_8 + 7A_{12})\big], \\
\widehat{\mathcal{R}}_2(\tilde{a}) &= \widehat{\beta}_4\widehat{\beta}_6(2A_{10} - A_8 - A_9), \\
\widehat{\mathcal{R}}_3(\tilde{a}) &= \widehat{\beta}_2\big[A_2(80A_3 - 3A_4 - 54A_5) - 80A_{22} + 708A_{23} - 324A_{24}\big], \\
\widehat{\mathcal{R}}_4(\tilde{a}) &= T_{11}, \\
\widehat{\mathcal{R}}_5(\tilde{a}) &= 12A_1^2 + 12A_8 + 5A_{10} + 17A_{11}, \\
\widehat{\mathcal{R}}_6(\tilde{a}) &= 2A_{10} - A_8 - A_9, \\
\widehat{\mathcal{R}}_7(\tilde{a}) &= 4A_8 - 3A_9, \\
\widetilde{N}(\tilde{a}, x, y) &= (D_2^2 + T_8 - 2T_9)/9, \\
\theta(\tilde{a}) &= 2A_5 - A_4 \equiv \text{Discrim}[\widetilde{N}, x]/(16y^2), \\
\mathcal{F}(\tilde{a}) &= A_7, \\
\mathcal{T}_3(\tilde{a}) &= 8A_{15} - 4A_1A_2,
\end{aligned}
$$

$$
\begin{aligned}
H_2(\tilde{a},x,y) &= \big(C_1, -8\widehat{H}-\widetilde{N}\big)^{(1)} - 2D_1\widetilde{N},\\
H_9(\tilde{a}) &= -\big[\big[\widehat{D},\widehat{D}\big)^{(2)},\widehat{D}\big)^{(1)},\widehat{D}\big)^{(3)},\\
H_{10}(\tilde{a}) &= \big[\big[\widehat{D},\widetilde{N}\big)^{(2)},D_2\big)^{(1)},\\
H_{11}(\tilde{a},x,y) &= -32\widehat{H}\Big[\big(C_2,\widehat{D}\big)^{(2)} + 8\big(\widehat{D},D_2\big)^{(1)}\Big] + 3\Big[\big(C_1,-8\widehat{H}-\widetilde{N}\big)^{(1)} - 2D_1\widetilde{N}\Big]^2,\\
H_{12}(\tilde{a},x,y) &= \big(\widehat{D},\widehat{D}\big)^{(2)},\\
N_7(\tilde{a}) &= 12D_1\big(C_0,D_2\big)^{(1)} + 2D_1^3 + 9D_1\big(C_1,C_2\big)^{(2)} + 36\big[\big[C_0,C_1\big)^{(1)},D_2\big)^{(1)}.
\end{aligned}
$$

We remark that the last six invariant polynomials H_2, H_9 to H_{12}, and N_7 are constructed in [22], whereas $\mathcal{F}$ and $\mathcal{T}_3$ are defined in [24].

2.2 Preliminary results involving polynomial invariants

Considering the GL-comitant $C_2(\tilde{a},x,y) = yp_2(\tilde{a},x,y) - xq_2(\tilde{a},x,y)$ as a cubic binary form of x and y we calculate

$$\eta(\tilde{a}) = \mathrm{Discrim}[C_2,\xi], \quad M(\tilde{a},x,y) = \mathrm{Hessian}[C_2],$$

where $\xi = y/x$ or $\xi = x/y$. According to [19] we have the following lemma.

Lemma 2 ([19]). *The number of infinite singularities (real and imaginary) of a quadratic system in QS is determined by the following conditions:*

(i) 3 *real if* $\eta > 0$*;*

(ii) 1 *real and* 2 *imaginary if* $\eta < 0$*;*

(iii) 2 *real if* $\eta = 0$ *and* $M \neq 0$*;*

(iv) 1 *real if* $\eta = M = 0$ *and* $C_2 \neq 0$*;*

(v) ∞ *if* $\eta = M = C_2 = 0$.

Moreover, for each one of these cases the quadratic systems (2) *can be brought via a linear transformation to one of the following canonical systems:*

$$(\mathbf{S}_I)\quad \begin{cases} \dot{x} = a + cx + dy + gx^2 + (h-1)xy,\\ \dot{y} = b + ex + fy + (g-1)xy + hy^2; \end{cases}$$

$$(\mathbf{S}_{II})\quad \begin{cases} \dot{x} = a + cx + dy + gx^2 + (h+1)xy,\\ \dot{y} = b + ex + fy - x^2 + gxy + hy^2; \end{cases}$$

$$(\mathbf{S}_{III})\quad \begin{cases} \dot{x} = a + cx + dy + gx^2 + hxy,\\ \dot{y} = b + ex + fy + (g-1)xy + hy^2; \end{cases}$$

$$(\mathbf{S}_{IV}) \quad \begin{cases} \dot{x} = a + cx + dy + gx^2 + hxy, \\ \dot{y} = b + ex + fy - x^2 + gxy + hy^2; \end{cases}$$

$$(\mathbf{S}_{V}) \quad \begin{cases} \dot{x} = a + cx + dy + x^2, \\ \dot{y} = b + ex + fy + xy. \end{cases}$$

Lemma 3. *If a quadratic system* (6) *possessing an invariant irreducible conic which is not a parabola, then the conditions* $\widehat{\gamma}_1 = \widehat{\gamma}_2 = 0$ *hold.*

Proof: According to [8] a system (6) possessing an invariant irreducible conic which is not a parabola as an algebraic particular integral can be written in the form

$$\dot{x} = a\Phi(x,y) + \Phi'_y(gx + hy + k), \quad \dot{y} = b\Phi(x,y) - \Phi'_x(gx + hy + k),$$

where a, b, g, h, k are real parameters and $\Phi(x,y)$ defines a conic

$$\Phi(x,y) \equiv p + qx + ry + sx^2 + 2txy + uy^2 = 0. \tag{5}$$

Straightforward calculations give $\widehat{\gamma}_1 = \widehat{\gamma}_2 = 0$ for the above systems and this completes the proof. ■

Assume that a conic (5) is an affine algebraic invariant curve for a quadratic system (2), which we rewrite in the form:

$$\begin{aligned} \frac{dx}{dt} &= a + cx + dy + gx^2 + 2hxy + ky^2 \equiv P(x,y), \\ \frac{dy}{dt} &= b + ex + fy + lx^2 + 2mxy + ny^2 \equiv Q(x,y). \end{aligned} \tag{6}$$

Remark 3. Following [12] we construct the determinant

$$\Delta = \begin{vmatrix} s & t & q/2 \\ t & u & r/2 \\ q/2 & r/2 & p \end{vmatrix},$$

associated to the conic (5). By [12] this conic is irreducible (i.e. the polynomial Φ defining the conic is irreducible over $\mathbb{C}$) if and only if $\Delta \neq 0$.

According to [9] (see also [5]) we have the next lemma.

Lemma 4. *Suppose that a polynomial system* (1) *of degree* n *has the invariant algebraic curve* $f(x,y) = 0$ *of degree* m. *Let* P_n, Q_n *and* f_m *be the homogeneous components of* P, Q *and* f *of degree* n *and* m, *respectively. Then the irreducible factors of* f_m *must be factors of* $yP_n - xQn$.

According to definition of an invariant curve (see page 2) considering the cofactor $K = Ux + Vy + W \in \mathbb{C}[x,y]$ the following identity holds:

$$\frac{\partial\Phi}{\partial x}P(x,y) + \frac{\partial\Phi}{\partial y}Q(x,y) = \Phi(x,y)(Ux + Vy + W).$$

This identity yields a system of 10 equations for determining the 9 unknown parameters p, q, r, s, t, u, U, V, W:

$$\begin{aligned}
Eq_1 &\equiv s(2g-U)+2lt=0,\\
Eq_2 &\equiv 2t(g+2m-U)+s(4h-V)+2lu=0,\\
Eq_3 &\equiv 2t(2h+n-V)+u(4m-U)+2ks=0,\\
Eq_4 &\equiv u(2n-V)+2kt=0,\\
Eq_5 &\equiv q(g-U)+s(2c-W)+2et+lr=0,\\
Eq_6 &\equiv r(2m-U)+q(2h-V)+2t(c+f-W)+2(ds+eu)=0,\\
Eq_7 &\equiv r(n-V)+u(2f-W)+2dt+kq=0,\\
Eq_8 &\equiv q(c-W)+2(as+bt)+er-pU=0,\\
Eq_9 &\equiv r(f-W)+2(bu+at)+dq-pV=0,\\
Eq_{10} &\equiv aq+br-pW=0.
\end{aligned} \tag{7}$$

3 The proof of the Main Theorem: statement $(\boldsymbol{A})$

Assuming that a quadratic system (6) in **QS** has the an invariant conic (5) which is an ellipse. Since the discriminant of the quadratic homogeneous part of the an ellipse must be negative we conclude that this system must possess either two complex distinct infinite singularities or the infinite line filled up with singularities. So according to Lemmas 2 and 3 the conditions $\widehat{\gamma}_1=\widehat{\gamma}_2=0$ and either $\eta<0$ or $C_2=0$ have to be fulfilled. In what follows we examine each one of these cases.

3.1 Systems with $\eta<0$

Supposing that the condition $\widehat{\gamma}_1=\widehat{\gamma}_2=0$ holds, we shall examine the unique family of quadratic systems (6), corresponding to the condition $\eta<0$. So, by Lemma 2, we consider the following family of systems

$$\begin{aligned}
\dot{x} &= a+cx+dy+gx^2+(h+1)xy,\\
\dot{y} &= b+ex+fy-x^2+gxy+hy^2.
\end{aligned} \tag{8}$$

For this systems we calculate

$$C_2(x,y)=yp_2(x,y)-xq_2(x,y)=x(x^2+y^2),\quad \theta=(h+1)\big[g^2+(h-1)^2\big]/2. \tag{9}$$

Considering Lemma 4 and the value of C_2, we deduce that, in the case of an ellipse, the quadratic homogeneous part of the conic (5) must be of the form $\lambda(x^2+y^2)$, with $\lambda\neq 0$. So, dividing all the coefficients of the conic by λ we may assume $\lambda=1$ and then we have $s=u=1$, $t=0$. So we have to detect the conditions under the coefficients of the above systems in order to possess an invariant ellipse of the form:

$$\Phi(x,y)=p+qx+ry+x^2+y^2=0. \tag{10}$$

Considering Remark 3, we have the next lemma (maybe it also follows from other well known results).

Lemma 5. *If a conic has the normal form* (10)*, then it is:* (i) *a real ellipse if* $\Delta<0$*;* (ii) *a complex ellipse if* $\Delta>0$*; and* (iii) *a reducible conic if* $\Delta=0$.

Proof: First of all we calculate for the above conic the corresponding determinant (see Remark 3):

$$\Delta = (4p - q^2 - r^2)/4.$$

On the other hand, applying the transformation $x = x_1 - q/2$ and $y = y_1 - r/2$, we obtain

$$\Phi(x_1, y_1) = \Delta + x_1^2 + y_1^2,$$

and the validity of the lemma is evident. ∎

We observe that systems (8) have generic linear and constant parts. Since for systems (8) we have

$$\theta = (h+1)\big[g^2 + (h-1)^2\big]/2, \qquad \widetilde{N} = 9\big[(2+g^2-2h)x^2 + 2g(h+1)xy + (h-1)(h+1)y^2\big],$$

then in order to simplify these canonical systems via a translation we examine the following three possibilities

$$(i)\ \theta \neq 0; \quad (ii)\ \theta = 0 \text{ and } \widetilde{N} \neq 0; \quad (iii)\ \theta = \widetilde{N} = 0.$$

3.1.1 The possibility $\theta \neq 0$

In this subsection we prove the next theorem which corresponds to the part of the Diagram 1 defined by the condition $\theta \neq 0$.

Theorem 1. *Assume that for a quadratic system* (6) *the conditions $\eta < 0$ and $\theta \neq 0$ hold. Then, this system could possess at most one invariant ellipse. And it possesses exactly one invariant ellipse (real or complex) if and only if $\widehat{\gamma}_1 = \widehat{\gamma}_2 = 0$ and one of the following sets of the conditions are satisfied:*

(i) If $\widehat{\beta}_1 \neq 0$, then either

(i.1) $\widehat{\beta}_2 \neq 0,\ \widehat{\beta}_3 \neq 0,\ \begin{cases} \widehat{\mathcal{R}}_1 < 0 \to \textit{real}; \\ \widehat{\mathcal{R}}_1 < 0 \to \textit{complex}, \end{cases}$ *or*

(i.2) $\widehat{\beta}_2 \neq 0,\ \widehat{\beta}_3 = 0,\ \widehat{\gamma}_3 = 0,\ \begin{cases} \widehat{\mathcal{R}}_1 < 0 \to \textit{real}; \\ \widehat{\mathcal{R}}_1 > 0 \to \textit{complex}, \end{cases}$ *or*

(i.3) $\widehat{\beta}_2 = 0,\ \widehat{\beta}_4 \neq 0,\ \widehat{\beta}_5 \neq 0,\ \begin{cases} \widehat{\mathcal{R}}_2 < 0 \to \textit{real}; \\ \widehat{\mathcal{R}}_2 > 0 \to \textit{complex}, \end{cases}$ *or*

(i.4) $\widehat{\beta}_2 = 0,\ \widehat{\beta}_4 \neq 0,\ \widehat{\beta}_5 = 0,\ \widehat{\gamma}_3 = 0,\ \begin{cases} \widehat{\mathcal{R}}_2 < 0 \to \textit{real}; \\ \widehat{\mathcal{R}}_2 > 0 \to \textit{complex}; \end{cases}$

(ii) If $\widehat{\beta}_1 = 0$, then either

(ii.1) $\widehat{\beta}_6 \neq 0,\ \widehat{\beta}_2 \neq 0,\ \widehat{\beta}_7^2 + \widehat{\beta}_8^2 \neq 0,\ \widehat{\gamma}_4 = 0,\ \begin{cases} \widehat{\mathcal{R}}_3 < 0 \to \textit{real}; \\ \widehat{\mathcal{R}}_3 > 0 \to \textit{complex}, \end{cases}$ *or*

(ii.2) $\widehat{\beta}_6 \neq 0,\ \widehat{\beta}_2 = 0,\ \widehat{\beta}_4 \neq 0,\ \widehat{\gamma}_5 = 0,\ \begin{cases} \widehat{\mathcal{R}}_2 < 0 \to \textit{real}; \\ \widehat{\mathcal{R}}_2 > 0 \to \textit{complex}, \end{cases}$ *or*

(ii.3) $\widehat{\beta}_6 = 0,\ \widehat{\beta}_2 \neq 0,\ \widehat{\gamma}_4 = 0,\ \widehat{\gamma}_8 = 0,\ \begin{cases} \widehat{\mathcal{R}}_3 < 0 \to \textit{real}; \\ \widehat{\mathcal{R}}_3 > 0 \to \textit{complex}, \end{cases}$ *or*

(ii.4) $\widehat{\beta}_6 = 0,\ \widehat{\beta}_2 = 0,\ \widehat{\gamma}_4 = 0,\ \widehat{\gamma}_9 = 0,\ \begin{cases} \widehat{\mathcal{R}}_4 < 0 \to \textit{real}; \\ \widehat{\mathcal{R}}_4 > 0 \to \textit{complex}. \end{cases}$

Proof: The condition $\theta \neq 0$ implies $h+1 \neq 0$ and due to a translation we may assume $c = d = 0$. So we arrive at the systems

$$\dot{x} = a + gx^2 + (h+1)xy, \quad \dot{y} = b + ex + fy - x^2 + gxy + hy^2, \tag{11}$$

for which we calculate

$$\widehat{\gamma}_1 = \frac{(h+1)^2}{32}\Psi_1\Psi_2,$$

where

$$\Psi_1 = \big[eg(1-3h) + f(1+2g^2-h^2)\big], \quad \Psi_2 = \big[(e - fg + 3eh)^2 + 4f^2(h+1)^2)\big].$$

So, since $h+1 \neq 0$, we obtain that the condition $\widehat{\gamma}_1 = 0$ is equivalent to $\Psi_1\Psi_2 = 0$.

On the other hand for systems (11) we calculate

$$\begin{aligned} \widehat{\beta}_1 =& -\frac{(h+1)^2}{16}\big[e^2(1+3h)^2 + f^2(8+9g^2+8h) - 2efg(9h-1)\big], \\ \widehat{\beta}_2 =& -g\big[g^2 + (3h+1)^2\big]/2, \\ \widehat{\beta}_3 =& (3h-1)\big[9g^2 + (3h+5)^2\big]/2. \end{aligned} \tag{12}$$

and we consider two cases: $\widehat{\beta}_1 \neq 0$ and $\widehat{\beta}_1 = 0$.

3.1.1.1 The case $\widehat{\beta}_1 \neq 0$.

We claim that in this case the condition $\widehat{\gamma}_1 = 0$ implies $\Psi_1 = 0$. Indeed, assuming $\Psi_2 = 0$ due to $h+1 \neq 0$ we necessarily get $f = 0$. Then we obtain

$$\Psi_2 = e^2(1+3h)^2, \ \ \widehat{\beta}_1 = -e^2(1+h)^2(1+3h)^2/16,$$

and clearly the condition $\widehat{\beta}_1 \neq 0$ implies $\Psi_2 \neq 0$. So the contradiction we obtained proves our claim.

Thus the condition $\widehat{\gamma}_1 = 0$ implies $\Psi_1 = 0$, i.e. we have the condition

$$eg(1-3h) + f(1+2g^2-h^2) = 0.$$

This equation is linear with respect to the parameter e with the coefficient $g(1-3h)$. We observe that the condition $g = 0$ is equivalent to $\widehat{\beta}_2 = 0$, and hence we consider two subcases: $\widehat{\beta}_2 \neq 0$ and $\widehat{\beta}_2 = 0$.

3.1.1.1.1 The subcase $\widehat{\beta}_2 \neq 0$.

Then $g \neq 0$ and we have that the second factor $1-3h$ could vanish if and only if $\widehat{\beta}_3 = 0$.

1) *The possibility* $\widehat{\beta}_3 \neq 0$. Then the condition $\Psi_1 = 0$ gives $e = \dfrac{f(1+2g^2-h^2)}{g(3h-1)}$, and then we calculate:

$$\widehat{\gamma}_2 = -\frac{1575f^2(1+h)^5}{g^2(3h-1)^3}\big[g^2 + (1-h)^2\big]^2\big[9g^2 + (1+3h)^2\big]\mathcal{B}_1,$$

where
$$\mathcal{B}_1 = (bg - ah)(3h-1)^2 - 2f^2g(h-1),$$
and clearly in the considered case the condition $\widehat{\gamma}_2 = 0$ is equivalent to $\mathcal{B}_1 = 0$. This last condition yields
$$b = \frac{2f^2g(h-1) + ah(3h-1)^2}{g(3h-1)^2}$$
and we arrive at the family of systems
$$\begin{aligned}
\dot{x} &= a + gx^2 + (h+1)xy,\\
\dot{y} &= \frac{ah}{g} + \frac{2f^2(h-1)}{(3h-1)^2} + \frac{f(1+2g^2-h^2)}{g(3h-1)}x + fy - x^2 + gxy + hy^2,
\end{aligned}$$
which possess the invariant conic
$$\Phi(x,y) = \frac{a}{g} + \frac{4f^2}{(3h-1)^2} - \frac{2f(1+h)}{g(3h-1)}x + \frac{4f}{3h-1}y + x^2 + y^2 = 0.$$

Since $g(1-3h) \neq 0$ we may apply to the above systems the following translation
$$x_1 = x - \frac{f(1+h)}{g(3h-1)}, \quad y_1 = y + \frac{2f}{3h-1}$$
and, after additional change of the parameters f and a by the new ones (d_1 and a_1) using the formulae
$$a = a_1 + \frac{d_1^2 g}{(1+h)^2}, \quad f = \frac{d_1 g(3h-1)}{(1+h)^2},$$
we arrive at the simpler canonical systems (we pass to the old notation: $x_1 \to x$, $y_1 \to y$, $a_1 \to a$ and $d_1 \to d$)
$$\begin{aligned}
\dot{x} &= a + dy + gx^2 + (h+1)xy,\\
\dot{y} &= \frac{ah}{g} - dx - x^2 + gxy + hy^2.
\end{aligned} \tag{13}$$
These systems possess the invariant conic
$$\Phi(x,y) = \frac{a}{g} + x^2 + y^2 = 0, \tag{14}$$
which clearly is a real ellipse for $ag < 0$, it is a complex ellipse for $ag > 0$ and it is reducible conic for $a = 0$.

On the other hand for systems (13) we calculate
$$\begin{aligned}
\widehat{\mathcal{R}}_1 &= 3agd^2(1+h)^2\big[g^2 + (h-1)^2\big]^4\big[9g^2 + (3h+1)^2\big]/128,\\
\widehat{\beta}_1 &= -d^2\big[g^2 + (h-1)^2\big]\big[9g^2 + (3h+1)^2\big]/16, \quad \widehat{\beta}_2 = -g\big[g^2 + (3h+1)^2\big]/2
\end{aligned} \tag{15}$$
and due to $\widehat{\beta}_1\widehat{\beta}_2 \neq 0$ we deduce that the condition $a = 0$ is equivalent to $\widehat{\mathcal{R}}_1 = 0$. Therefore, the conic (14) is irreducible if and only if $\widehat{\mathcal{R}}_1 \neq 0$. Moreover, we have $\operatorname{sign}(ag) = \operatorname{sign}(\widehat{\mathcal{R}}_1)$, and hence systems (13) possess a real (respectively, complex) ellipse (14) if and only if $\widehat{\mathcal{R}}_1 < 0$ (respectively $\widehat{\mathcal{R}}_1 > 0$).

2) *The possibility* $\widehat{\beta}_3 = 0$. Then $3h - 1 = 0$ (i.e. $h = 1/3$) and we obtain $\Psi_1 = 2f(4 + 9g^2)/9 = 0$, which implies $f = 0$. We observe that in this case we get $\widehat{\gamma}_1 = \widehat{\gamma}_2 = 0$.

Thus, we arrive at the family of systems

$$\dot{x} = a + gx^2 + 4xy/3, \quad \dot{y} = b + ex - x^2 + gxy + y^2/3, \quad g \neq 0, \tag{16}$$

for which, considering (7), we obtain

$$\begin{gathered}
s = u = 1, \ t = 0, \ U = 2g, \ V = 2/3, \ W = -gq - r, \\
Eq_6 = (6e + 2q - 3gr)/3, \quad Eq_7 = (3gq + 2r)/3, \quad Eq_8 = 2a - 2gp + gq^2 + er + qr, \\
Eq_9 = (6b - 2p + 3gqr + 3r^2)/3, \quad Eq_{10} = aq + gpq + br + pr, \\
Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_5 = 0.
\end{gathered}$$

So the conditions $Eq_6 = 0$ and $Eq_7 = 0$ give us

$$r = 18eg/(4 + 9g^2), \quad q = -12e/(4 + 9g^2),$$

and then we get

$$\begin{gathered}
Eq_8 = 2(a - gp) + \frac{162e^2g^3}{(4 + 9g^2)^2}, \quad Eq_9 = 2(3b - p)/3 + \frac{108e^2g^2}{(4 + 9g^2)^2}, \quad Eq_{10} = -\frac{6e(2a - 3bg - gp)}{4 + 9g^2}, \\
Res_p\,(Eq_8, Eq_9) = \frac{4\big[(a - 3bg)(4 + 9g^2)^2 - 81e^2g^3\big]}{3(4 + 9g^2)^2} \equiv \frac{4}{3(4 + 9g^2)^2}\mathcal{B}_2.
\end{gathered}$$

Evidently, for the existence of an invariant conic the condition $\mathcal{B}_2 = 0$ is necessary and this condition gives $b = \dfrac{a}{3g} - \dfrac{27e^2g^2}{(4 + 9g^2)^2}$, and then we have

$$Eq_8 = 2a - 2gp + \frac{162e^2g^3}{(4 + 9g^2)^2}, \quad Eq_9 = \frac{1}{3g}Eq_8, \quad Eq_{10} = \frac{-3e}{4 + 9g^2}Eq_8.$$

Clearly we obtain the unique condition $Eq_8 = 0$ which gives $p = \dfrac{a}{g} + \dfrac{81e^2g^2}{(4 + 9g^2)^2}$ and all the equations are satisfied. So we arrive at the canonical systems

$$\begin{aligned}
\dot{x} &= a + gx^2 + 4xy/3, \\
\dot{y} &= \frac{a}{3g} - \frac{27e^2g^2}{(4 + 9g^2)^2} + ex - x^2 + gxy + y^2/3,
\end{aligned}$$

which possess the invariant conic

$$\Phi(x, y) = \frac{a}{g} + \frac{81e^2g^2}{(4 + 9g^2)^2} - \frac{12e}{4 + 9g^2}x + \frac{18eg}{4 + 9g^2}y + x^2 + y^2 = 0.$$

It remains to observe that the condition $\mathcal{B}_2 = 0$ is governed by the invariant polynomial $\widehat{\gamma}_3$ as for systems (16) calculations yield:

$$\widehat{\gamma}_3 = -\frac{122512e}{27}\mathcal{B}_2, \quad \widehat{\beta}_1 = -4e^2/9,$$

and due to $\widehat{\beta}_1 \neq 0$ the condition $\mathcal{B}_2 = 0$ is equivalent to $\widehat{\gamma}_3 = 0$.

Next, we may apply to the above systems the following translation

$$x_1 = x - \frac{6e}{4+9g^2}, \quad y_1 = y + \frac{9eg}{4+9g^2},$$

and, after additional change of the parameters a and e by the new ones (a_1 and d_1) using the formulae

$$a = a_1 + \frac{9d_1^2 g}{16}, \quad e = \frac{d_1(4+9g^2)}{8},$$

we arrive at a simpler canonical form (we pass here to the old notation: $x_1 \to x$, $y_1 \to y$, $a_1 \to a$ and $d_1 \to d$)

$$\dot{x} = a + dy + gx^2 + 4xy/3, \quad \dot{y} = \frac{a}{3g} - dx - x^2 + gxy + y^2/3.$$

We observe that we get a subfamily of the family of systems (13) defined by $h = 1/3$. Therefore, the above systems possess the same invariant ellipse (14), because this ellipse does not depend on the parameter h. Considering (15) it is clear that in this particular case the invariant polynomial $\widehat{\mathcal{R}}_1$ is responsible for the type of the ellipse (14).

3.1.1.1.2 The subcase $\widehat{\beta}_2 = 0$. In this case, according to (12), we have $g = 0$ and then for systems (11) we have

$$\widehat{\gamma}_1 = f(h+1)^3(1-h)\big[4f^2(h+1)^2 + e^2(1+3h)^2\big]/32, \quad \theta = (h-1)^2(h+1)/2.$$

Therefore, due to $\theta \neq 0$, the condition $\widehat{\gamma}_1 = 0$ gives $f = 0$. So we get the following systems

$$\dot{x} = a + (h+1)xy, \quad \dot{y} = b + ex - x^2 + hy^2, \tag{17}$$

for which, solving the equations (7), we obtain

$$\begin{gathered} s = u = 1,\ t = 0,\ U = 0,\ V = 2h,\ W = -r, \\ Eq_6 = 2e + q(1-h), \quad Eq_7 = r(1-h), \quad Eq_8 = 2a + er + qr, \\ Eq_9 = 2b - 2hp + r^2, \quad Eq_{10} = aq + br + pr, \\ Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_5 = 0. \end{gathered}$$

Therefore, due to $h - 1 \neq 0$, equations $Eq_6 = 0$ and $Eq_7 = 0$ imply $r = 0$ and $q = 2e/(h-1)$, and then we get

$$Eq_8 = 2a, \quad Eq_9 = 2(b - hp), \quad Eq_{10} = 2ae/(h-1).$$

So equation $Eq_8 = 0$ gives the condition $a = 0$ (this implies $Eq_{10} = 0$). In this case we could not have $h = 0$, otherwise equation $Eq_9 = 0$ yields $b = 0$, and we get the degenerate system $\dot{x} = xy$, $\dot{y} = (e-x)x$. In the case $h \neq 0$ the condition $Eq_9 = 0$ gives $p = b/h$ and all the above equations are satisfied.

Thus, we conclude that for the existence of an invariant conic for systems (17) the conditions $a = 0$ and $h \neq 0$ must be satisfied. On the other hand for these systems we calculate

$$\begin{aligned} &\widehat{\gamma}_2 = -1575ae^2h(h-1)^2(h+1)^3(3h-1)(3h+1)^2, \quad \theta = (h-1)^2(h+1)/2, \\ &\widehat{\beta}_1 = -e^2(h+1)^2(3h+1)^2/16, \quad \widehat{\beta}_4 = 2h(1+h)^2, \quad \widehat{\beta}_5 = -2(h+1)(3h-1), \end{aligned} \tag{18}$$

and it is clear that the condition $h \neq 0$ is equivalent to $\widehat{\beta}_4 \neq 0$.

So in what follows we assume $\widehat{\beta}_4 \neq 0$ and then, due to $\theta\widehat{\beta}_1 \neq 0$, the necessary condition $\widehat{\gamma}_2 = 0$ implies $a(3h-1) = 0$ and therefore we examine two possibilities: $\widehat{\beta}_5 \neq 0$ and $\widehat{\beta}_5 = 0$.

1) *The possibility* $\widehat{\beta}_5 \neq 0$. In this case we obtain $a = 0$ and we arrive at systems

$$\dot{x} = (h+1)xy, \quad \dot{y} = b + ex - x^2 + hy^2, \tag{19}$$

which possess the invariant conic

$$\Phi(x,y) = \frac{b}{h} + \frac{2e}{h-1}x + x^2 + y^2 = 0, \tag{20}$$

Since $h(h-1)(h+1) \neq 0$, we may apply to the above systems the following translation

$$x_1 = x + \frac{e}{h-1}, \quad y_1 = y,$$

and, after an additional change of the parameters e and b by the new ones (d_1 and b_1) using the formulae

$$e = \frac{d_1(1-h)}{1+h}, \quad b = \frac{b_1(1+h)^2 + d_1^2 h}{(1+h)^2},$$

we arrive at the simpler canonical systems (we pass to the old notation: $x_1 \to x$, $y_1 \to y$, $b_1 \to b$ and $d_1 \to d$)

$$\dot{x} = dy + (h+1)xy, \quad \dot{y} = b - dx - x^2 + hy^2. \tag{21}$$

These systems possess the invariant conic

$$\Phi(x,y) = \frac{b}{h} + x^2 + y^2 = 0, \tag{22}$$

which clearly is a real ellipse for $bh < 0$, it is a complex ellipse for $bh > 0$, and it is reducible conic for $b = 0$.

On the other hand, for systems (21) we calculate

$$\begin{aligned} \widehat{\mathcal{R}}_2 &= bh(h-1)^2(1+h)^2(1+3h)^4/8, \\ \widehat{\beta}_1 = -d^2(h-1)^2(1+3h)^2/16, &\quad \widehat{\beta}_4 = 2h(1+h)^2, \end{aligned} \tag{23}$$

and due to $\widehat{\beta}_1\widehat{\beta}_4 \neq 0$ we deduce that the condition $b = 0$ is equivalent to $\widehat{\mathcal{R}}_2 = 0$. Therefore, the conic (22) is irreducible if and only if $\widehat{\mathcal{R}}_2 \neq 0$. Moreover, we have $\operatorname{sign}(bh) = \operatorname{sign}(\widehat{\mathcal{R}}_2)$ and hence systems (21) possess a real (respectively, complex) ellipse (22) if and only if $\widehat{\mathcal{R}}_2 < 0$ (respectively, $\widehat{\mathcal{R}}_2 > 0$).

2) *The possibility* $\widehat{\beta}_5 = 0$. In this case we obtain $h = 1/3$ and considering (18) this implies $\widehat{\gamma}_2 = 0$. However, it was proved above that for systems (17) the condition $a = 0$ is necessary for the existence of invariant ellipse. So for these systems with $h = 1/3$ we calculate

$$\widehat{\gamma}_3 = -1960192ae/27, \quad \widehat{\beta}_1 = -4e^2/9,$$

and clearly due to the condition $\widehat{\beta}_1 \neq 0$ we deduce that the condition $a = 0$ is equivalent to $\widehat{\gamma}_3 = 0$.

Thus, $a = 0$ and we arrive at the systems

$$\dot{x} = 4xy/3, \quad \dot{y} = b + ex - x^2 + y^2/3,$$

which possess the invariant conic

$$\Phi(x, y) = 3b - 3ex + x^2 + y^2 = 0.$$

We observe that the above systems belong to the family (19) with $h = 1/3$, as well as the above conic is a particular case of the conic (20) (when $h = 1/3$). So, following the same steps as before for the generic case of systems (19) (but considering $h = 1/3$), we arrive at the canonical systems (21) possessing the ellipse (22) in the particular case $h = 1/3$. Therefore, it is clear that the same invariant polynomial $\widehat{\mathcal{R}}_2 \neq 0$ is responsible for the type of this ellipse.

3.1.1.2 The case $\widehat{\beta}_1 = 0$.

We have the following result.

Lemma 6. *For systems* (11), *conditions* $\widehat{\beta}_1 = 0 = \widehat{\gamma}_1$ *and* $\theta \neq 0$ *imply* $f = 0$.

Proof: Suppose the contrary. If $f \neq 0$, according to (12), the conditions $\widehat{\beta}_1 = 0$ and $\theta \neq 0$ yield

$$\phi = 9f^2g^2 + 2efg(1 - 9h) + 8f^2(1 + h) + (e + 3eh)^2 = 0.$$

We observe that

$$\text{Discrim}[\phi, g] = -32f^2\big[e^2 + 9f^2 + 9(e^2 + f^2)h\big],$$

and in order to factorize the polynomial ϕ (with respect to the parameter g) into two linear factors, we set $e^2 + 9f^2 + 9(e^2 + f^2))h = -2u^2$. Then, we obtain

$$h = -\frac{e^2 + 9f^2 + 2u^2}{9(e^2 + f^2)}, \qquad \phi = \frac{1}{9(e^2 + f^2)^2}\phi_1\phi_2 = 0,$$
$$\phi_{1,2} = 2eu^2 \pm 4(e^2 + f^2)u + 2e(e^2 + 5f^2) + 9fg(e^2 + f^2),$$

and due to the change $u \to -u$, without loss of generality we can assume $\phi_1 = 0$. This equality gives us

$$g = -\frac{2(e^3 + 5ef^2 + 2e^2u + 2f^2u + eu^2)}{9f(e^2 + f^2)},$$

and then we calculate

$$\widehat{\beta}_1 = 0, \qquad \widehat{\gamma}_1 = -\frac{8(2e - u)^6(2e + u)^2\big[f^2 + (e + u)^2\big]^2}{3^{12}f(e^2 + f^2)^4},$$
$$\theta = \frac{4(2e - u)(2e + u)\big[e^4 + 34e^2f^2 + 81f^4 + 4e^3u + 20ef^2u + 6e^2u^2 + 22f^2u^2 + 4eu^3 + u^4\big]}{729f^2(e^2 + f^2)^2}.$$

Clearly, due to $\theta \neq 0$ and $f \neq 0$, we get $\widehat{\gamma}_1 \neq 0$ and this contradiction completes the proof of the lemma. ∎

Thus, by Lemma 6, we have $f = 0$, and for systems (11) we obtain

$$\begin{aligned}
&\widehat{\gamma}_1 = -e^3g(1 + h)^2(3h - 1)(1 + 3h)^2/32,\\
&\theta = (h + 1)\big[g^2 + (h - 1)^2\big]/2, \quad \widehat{\beta}_1 = -e^2(h + 1)^2(3h + 1)^2/16,\\
&\widehat{\beta}_2 = -g\big[g^2 + (3h + 1)^2\big]/2, \quad \widehat{\beta}_6 = (1 + 3h)\big[9g^2 + (3h + 1)^2\big]/8,
\end{aligned} \tag{24}$$

and we observe that due to $\theta \neq 0$ the condition $\widehat{\beta}_1 = 0$ gives $e(3h+1) = 0$ and this imply $\widehat{\gamma}_1 = \widehat{\gamma}_2 = 0$. So we examine two subcases: $\widehat{\beta}_6 \neq 0$ and $\widehat{\beta}_6 = 0$.

3.1.1.2.1 The subcase $\widehat{\beta}_6 \neq 0$. Then $3h+1 \neq 0$ and the condition $\widehat{\beta}_1 = 0$ implies $e = 0$. Thus, we arrive at the family of systems

$$\dot{x} = a + gx^2 + (1+h)xy, \quad \dot{y} = b - x^2 + gxy + hy^2, \quad 3h+1 \neq 0, \tag{25}$$

for which, considering (7), we obtain

$$\begin{gathered} s = u = 1,\ t = 0,\ U = 2g,\ V = 2h,\ W = -gq - r, \\ Eq_6 = q(1-h) - gr, \quad Eq_7 = gq + r(1-h), \quad Eq_8 = 2a - 2gp + gq^2 + qr, \\ Eq_9 = 2b - 2hp + gqr + r^2, \quad Eq_{10} = aq + gpq + br + pr, \\ Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_5 = 0. \end{gathered}$$

We observe that the linear system of equations $Eq_6 = 0$ and $Eq_7 = 0$ with respect to the parameters r and q has the determinant $g^2 + (h-1)^2 \neq 0$ due to $\theta \neq 0$. So the conditions $Eq_6 = 0$ and $Eq_7 = 0$ gives us $q = r = 0$ and then we obtain

$$Eq_8 = 2(a - gp), \quad Eq_9 = 2(b - hp), \quad Eq_{10} = 0. \tag{26}$$

Therefore we have to distinguish the possibilities $g \neq 0$ and $g = 0$. According to (24) these conditions are governed by the invariant polynomial $\widehat{\beta}_2$.

1) *The possibility* $\widehat{\beta}_2 \neq 0$. Then $g \neq 0$ and the condition $Eq_8 = 0$ yields $p = a/g$ and we get $Eq_9 = 2(bg - ah)/g = 0$. So we obtain the condition $bg - ah = 0$.

This condition gives us $b = ah/g$ and we get the family of systems

$$\dot{x} = a + gx^2 + (h+1)xy, \quad \dot{y} = \frac{ah}{g} - x^2 + gxy + hy^2, \tag{27}$$

which is a subfamily of systems (13) defined by the condition $d = 0$. These systems possess the invariant ellipse (14), which does not depend on the parameter d. However, considering (15) for the above systems we have $\widehat{\mathcal{R}}_1 = \widehat{\beta}_1 = 0$, i.e. we need another invariant polynomial for determining the type of the ellipse (14).

Thus, as it was shown above, systems (25) possess an invariant ellipse if and only if $bg - ah = 0$ and $ag \neq 0$. It remains to find out invariant polynomials which give us equivalent affine invariant conditions. We prove the next lemma.

Lemma 7. *Assume that for a system* (25) *the condition* $\theta\widehat{\beta}_2 \neq 0$ *holds. Then, this system possesses an invariant ellipse if and only if* $\widehat{\gamma}_4 = 0$, $\widehat{\beta}_7^2 + \widehat{\beta}_8^2 \neq 0$ *and* $\widehat{\mathcal{R}}_3 \neq 0$. *Moreover, this ellipse is real for* $\widehat{\mathcal{R}}_3 < 0$ *and it is complex if* $\widehat{\mathcal{R}}_3 > 0$.

Proof: Necessity. Assume that a system (25) with $\widehat{\beta}_2 \neq 0$ possesses an invariant ellipse, i.e. as it was mentioned above the conditions $bg - ah = 0$ and $ag \neq 0$ are satisfied. Since $g \neq 0$, we have $b = ah/g$ and then we calculate

$$\begin{gathered} \widehat{\gamma}_4 = 0, \quad \widehat{\mathcal{R}}_3 = 160ag(g^2+h^2)\big[g^2 + (3h+1)^2\big], \quad \widehat{\beta}_7 = 8(1+4h)\big[4g^2 + (2h+1)^2\big], \\ \widehat{\beta}_8 = \frac{8a}{g}\big[4g^2 + (2h+1)^2\big]\big[h + 3h^2(5+12h) + g^2(11+36h)\big]. \end{gathered} \tag{28}$$

We note that the condition $ag \neq 0$ implies $\widehat{\mathcal{R}}_3 \neq 0$ and, moreover $\mathrm{sign}\,(\widehat{\mathcal{R}}_3) = \mathrm{sign}\,(ag)$. It remains to observe that the condition $\widehat{\beta}_7 = \widehat{\beta}_8 = 0$ could not be satisfied. Indeed, assuming $\widehat{\beta}_7 = 0$, due to $g \neq 0$, we obtain $h = -1/4$ and then by (28) we obtain $\widehat{\beta}_8 = a(1+16g^2)^2/(4g) \neq 0$ due to $a \neq 0$. This completes the proof of the necessity.

Sufficiency. Assume now that for a system (25) the conditions $\theta\widehat{\beta}_2\widehat{\mathcal{R}}_3(\widehat{\beta}_7^2 + \widehat{\beta}_8^2) \neq 0$ and $\widehat{\gamma}_4 = 0$ are satisfied. For this system calculations yield

$$\begin{aligned}\widehat{\gamma}_4 =& -4608(bg-ah)(1+h)\big[g^2+(h-1)^2\big]\widetilde{\Psi},\\ \widetilde{\Psi} =& a^2(1+4h)^2+(b-2ag+2bh)^2.\end{aligned}$$

We observe that, in the case $\widetilde{\Psi} \neq 0$, due to $\theta\widehat{\beta}_2 \neq 0$ (i.e. $g(h+1) \neq 0$), the condition $\widehat{\gamma}_4 = 0$ implies $bg - ah = 0$ and it was proved above that in this case, due to $\widehat{\mathcal{R}}_3 \neq 0$, the system under examination possesses an invariant conic. So it remains to prove the following claim: the conditions $\widehat{\beta}_7^2 + \widehat{\beta}_8^2 \neq 0$ and $\widehat{\mathcal{R}}_3 \neq 0$ imply $\widetilde{\Psi} \neq 0$.

Indeed, assuming $\widetilde{\Psi} = 0$, we get $a(1+4h) = 0 = b - 2ag + 2bh$. Therefore, we could have the following three possibilities:

$$\begin{aligned} a=0,\ b=0 \quad &\Rightarrow \quad \widehat{\mathcal{R}}_3 = 0,\\ a=0,\ h=-1/2 \quad &\Rightarrow \quad \widehat{\mathcal{R}}_3 = 0,\\ h=-1/4,\ b=4ag \quad &\Rightarrow \quad \widehat{\beta}_7 = 0 = \widehat{\beta}_8,\end{aligned}$$

and evidently our claim is proved. This completes the proof of Lemma 7. ∎

2) *The possibility* $\widehat{\beta}_2 = 0$. In this case we get $g = 0$ and this leads to the systems

$$\dot{x} = a + (1+h)xy, \quad \dot{y} = b - x^2 + hy^2, \quad 3h+1 \neq 0. \tag{29}$$

Considering (26) the condition $Eq_8 = 0$ yields $a = 0$ for this family. Moreover, the condition $h \neq 0$ must be fulfilled for the above systems in order to have an invariant conic, otherwise the equation $Eq_9 = 0$ from (26) implies $b = 0$, and we arrive degenerate systems (29).

On the other hand, for systems (29) calculations yield:

$$\begin{gathered}\widehat{\gamma}_5 = -ah(1+3h)^4/4, \quad \theta = (h+1)(h-1)^2/2,\\ \widehat{\beta}_4 = 2h(1+h)^2, \quad \widehat{\beta}_6 = (3h+1)^3/8.\end{gathered}$$

Therefore, due to $\theta\widehat{\beta}_6 \neq 0$, the condition $h \neq 0$ is equivalent to $\widehat{\beta}_4 \neq 0$, whereas the condition $a = 0$ is equivalent to $\widehat{\gamma}_5 = 0$. So we arrive at the family of systems

$$\dot{x} = (1+h)xy, \quad \dot{y} = b - x^2 + hy^2,$$

which is a subfamily of (21) defined by the condition $d = 0$. This means that the above systems possess the invariant ellipse (22), which does not depend on the parameter d. Moreover, the type of this ellipse is governed by the invariant polynomial $\widehat{\mathcal{R}}_2$, the value of which is given in (23) and also does not depend on d.

3.1.1.2.2 The subcase $\widehat{\beta}_6 = 0$. Then $3h+1=0$ (i.e. $h=-1/3$) and we obtain the following family of systems:

$$\dot{x} = a + gx^2 + 2xy/3, \quad \dot{y} = b + ex - x^2 + gxy - y^2/3, \tag{30}$$

for which, considering (7), we obtain

$$\begin{gathered} s = u = 1,\ t = 0,\ U = 2g,\ V = -2/3,\ W = -gq - r, \\ Eq_6 = (6e + 4q - 3gr)/3, \quad Eq_7 = (3gq + 4r)/3, \quad Eq_8 = 2a - 2gp + gq^2 + er + qr, \\ Eq_9 = (6b + 2p + 3gqr + 3r^2)/3, \quad Eq_{10} = aq + gpq + br + pr, \\ Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_5 = 0. \end{gathered}$$

Therefore, conditions $Eq_6 = Eq_7 = 0$ give

$$r = 18eg/(16 + 9g^2), \quad q = -24e/(16 + 9g^2),$$

and then we have

$$Eq_8 = 2a - 2gp + \frac{54e^2g(8 + 3g^2)}{(16 + 9g^2)^2}, \quad Eq_9 = 2b + 2p/3 - \frac{108e^2g^2}{(16 + 9g^2)^2}.$$

So, the condition $Eq_9 = 0$ gives $p = -3b + \dfrac{162e^2g^2}{(16 + 9g^2)^2}$ and then we obtain

$$\begin{aligned} Eq_8 &= \frac{2\big[(a + 3bg)(16 + 9g^2)^2 - 27e^2g(3g^2 - 8)\big]}{(16 + 9g^2)^2} \equiv \frac{2}{(16 + 9g^2)^2}\mathcal{B}_3, \\ Eq_{10} &= \frac{12e\big[-81e^2g^3 - (3bg - 2a)(16 + 9g^2)^2\big]}{(16 + 9g^2)^3} \equiv \frac{12e}{(16 + 9g^2)^3}\mathcal{B}_4, \end{aligned}$$

and hence the conditions $\mathcal{B}_3 = 0$ and $e\mathcal{B}_4 = 0$ are necessary for systems (30) to possess an invariant ellipse.

We claim that the condition $e = 0$ must hold in order to get an irreducible invariant conic. Indeed, assuming $e \neq 0$ and solving the system of equations $\mathcal{B}_3 = 0$ and $\mathcal{B}_4 = 0$ (which are linear with respect to the parameters a and b), we obtain

$$a = -\frac{72e^2g}{(16 + 9g^2)^2}, \quad b = \frac{3e^2(9g^2 - 16)}{(16 + 9g^2)^2}.$$

It could be checked directly that systems (30) with these values of the parameters a and b possess the following invariant conic:

$$9e^2 + 6e(-4x + 3gy) + (16 + 9g^2)(x^2 + y^2) = 0.$$

However, this invariant conic is not irreducible, because the corresponding determinant (see Remark 3) vanishes and this completes the proof of our claim. Thus, for systems (30), the conditions $\mathcal{B}_3 = 0$ and $e = 0$ must hold.

On the other hand, for systems (30) we calculate

$$\widehat{\gamma}_8 = -eg/9, \quad \widehat{\beta}_2 = -g^3/2,$$

and we consider two possibilities: $\widehat{\beta}_2 \neq 0$ and $\widehat{\beta}_2 = 0$.

1) *The possibility* $\widehat{\beta}_2 \neq 0$*.* Then $g \neq 0$ and the condition $e = 0$ is equivalent to $\widehat{\gamma}_8 = 0$. In the case $e = 0$ we obtain $\mathcal{B}_3 = (a + 3bg)(16 + 9g^2)^2$. On the other hand, for systems (30) with $e = 0$ we calculate

$$\widehat{\gamma}_4 = -1024(a + 3bg)(16 + 9g^2)\big[a^2 + (b - 6ag)^2\big]/81,$$

and clearly the condition $\widehat{\gamma}_4 = 0$ is equivalent to $a + 3bg = 0$, otherwise supposing $a + 3bg \neq 0$ we get $a = b = 0$ which leads to a contradiction. As $g \neq 0$ (due to $\widehat{\beta}_2 \neq 0$), we get $b = -a/(3g)$ and we arrive at the family of systems

$$\dot{x} = a + gx^2 + 2xy/3, \quad \dot{y} = -\frac{a}{3g} - x^2 + gxy - y^2/3,$$

which is a subfamily of (27) defined by the condition $h = -1/3$. Therefore, the above systems possess the same invariant ellipse (14), because this ellipse does not depend on the parameters h and d. Moreover, the type of this ellipse is determined by the invariant polynomial $\widehat{\mathcal{R}}_3$ given in (28), because in the case $h = -1/3$ it gives the sign of the product ag.

2) *The possibility* $\widehat{\beta}_2 = 0$*.* In this case we have $g = 0$ (which implies $\widehat{\gamma}_8 = 0$) and therefore we need another invariant polynomial which governs the condition $e = 0$. So for systems (30) with $g = 0$ we obtain

$$\begin{aligned} \mathcal{B}_3 = 256a = 0 \quad &\Leftrightarrow \quad \widehat{\gamma}_4 = -64a\big[256a^2 + (16b + 3e^2)^2\big]/81 = 0,\\ e = 0 \quad &\Leftrightarrow \quad \widehat{\gamma}_9 = -2e^2/3 = 0. \end{aligned}$$

Therefore, we arrive at the family of systems

$$\dot{x} = 2xy/3, \quad \dot{y} = b - x^2 - y^2/3,$$

which is a subfamily of (21) defined by the condition $d = 0$ and $h = -1/3$. So we conclude that the above system possesses the invariant conic (22), which in this case becomes

$$\Phi(x, y) = -3b + x^2 + y^2 = 0.$$

However, considering (23) we observe that the invariant polynomial $\widehat{\mathcal{R}}_2$ vanishes, and hence we need another one which is responsible for the sign of the parameter b. So for the above systems we calculate

$$\widehat{\mathcal{R}}_4 = -32b(3x^2 + y^2)/9 \quad \Rightarrow \quad \operatorname{sign}(\widehat{\mathcal{R}}_4) = -\operatorname{sign}(b),$$

and we conclude that the above ellipse is real if $\widehat{\mathcal{R}}_4 < 0$ and it is complex if $\widehat{\mathcal{R}}_4 > 0$.

Since all the cases are examined we conclude that Theorem 1 is proved. ∎

3.1.2 The possibility $\theta = 0$ and $\widetilde{N} \neq 0$

In this subsection we prove the next theorem which corresponds to the part of the Diagram 1 defined by the conditions $\theta = 0$ and $\widetilde{N} \neq 0$.

Theorem 2. *Assume that for a quadratic system* (6) *the conditions* $\eta < 0$*,* $\theta = 0$ *and* $\widetilde{N} \neq 0$ *hold. Then, this system could possess at most one invariant ellipse. And it possesses exactly one invariant ellipse (real or complex) if and only if* $\widehat{\gamma}_1 = \widehat{\gamma}_2 = 0$ *and one of the following sets of the conditions are satisfied:*

(i) $\widehat{\beta}_1 \neq 0,\ \widehat{\beta}_2 \neq 0,\ \widehat{\gamma}_6 = 0,\ \begin{cases}\widehat{\mathcal{R}}_5 < 0 \to \textit{real};\\ \widehat{\mathcal{R}}_5 > 0 \to \textit{complex},\end{cases}$ *or*

(ii) $\widehat{\beta}_1 \neq 0,\ \widehat{\beta}_2 = 0,\ \widehat{\gamma}_6 = 0,\ \begin{cases}\widehat{\mathcal{R}}_6 < 0 \to \textit{real};\\ \widehat{\mathcal{R}}_6 > 0 \to \textit{complex},\end{cases}$ *or*

(iii) $\widehat{\beta}_1 = 0,\ \widehat{\gamma}_6 = 0,\ \widehat{\gamma}_7 = 0,\ \begin{cases}\widehat{\mathcal{R}}_3 < 0 \to \textit{real};\\ \widehat{\mathcal{R}}_3 > 0 \to \textit{complex}.\end{cases}$

Proof: According to (9), the condition $\theta = 0$ implies

$$(h+1)\big[g^2 + (h-1)^2\big] = 0.$$

On the other hand, for systems (8) we have

$$\widetilde{N} = 9\big[(2+g^2-2h)x^2 + 2g(h+1)xy + (h-1)(h+1)y^2\big], \tag{31}$$

and, as we can observe, the condition $g = 0 = h-1$ is equivalent to $\widetilde{N} = 0$. Since $\widetilde{N} \neq 0$ we deduce that the condition $\theta = 0$ implies $h = -1$ and we may assume $f = 0$ due to a translation in systems (8). This leads to the family of systems

$$\dot{x} = a + cx + dy + gx^2, \quad \dot{y} = b + ex - x^2 + gxy - y^2, \tag{32}$$

for which we calculate

$$\begin{gathered}\widehat{\gamma}_1 = \frac{d^2}{32}(4+g^2)(4+9g^2)\big[4(d+e)g - c(g^2-4)\big] \equiv \frac{d^2}{32}(4+g^2)(4+9g^2)\widetilde{\Psi},\\ \widehat{\beta}_1 = -d\big[d(3g^2-4)^2 + 16g(cg^2-4c-4eg)\big], \quad \widehat{\beta}_2 = -g(g^2+4)/2.\end{gathered} \tag{33}$$

Therefore, the condition $\widehat{\gamma}_1 = 0$ yields $d\widetilde{\Psi} = 0$ (this implies $\widehat{\gamma}_2 = 0$ because it contains the factors $d\widetilde{\Psi}$), and we consider two subcases: $\widehat{\beta}_1 \neq 0$ and $\widehat{\beta}_1 = 0$.

3.1.2.1 The subcase $\widehat{\beta}_1 \neq 0$. Then $d \neq 0$ and the condition $\widetilde{\Psi} = 0$ is equivalent to $\widehat{\gamma}_1 = 0$. So, we have $4(d+e)g - c(g^2-4) = 0$, and we consider two possibilities: $\widehat{\beta}_2 \neq 0$ and $\widehat{\beta}_2 = 0$.

3.1.2.1.1 The possibility $\widehat{\beta}_2 \neq 0$. According to (33), we have $g \neq 0$ and we obtain $e = (cg^2 - 4c - 4dg)/(4g)$. Then considering the equations (7), for the systems (32) with this value of the parameter e we get the following values of the parameters of a conic as well as of its corresponding cofactor:

$$s = u = 1,\ t = 0,\ U = 2g,\ V = -2,\ W = c/2,\ r = c/2,\ q = c/g.$$

For these values of the parameters, we obtain that $Eq_1 = 0, \ldots, Eq_7 = 0$, and

$$\begin{gathered}Eq_8 = (16a - 4cd + c^2g - 16gp)/8, \quad Eq_9 = (4cd + 8bg - c^2g + 8gp)/(4g),\\ Eq_{10} = c(2a + bg - gp)/(2g).\end{gathered}$$

Since $g \neq 0$ the equation $Eq_8 = 0$ yields $p = \dfrac{16a - 4cd + c^2g}{16g}$ and we obtain

$$Eq_9 = (16a + 4cd + 16bg - c^2g)/(8g), \qquad Eq_{10} = \frac{c}{4}Eq_9.$$

Therefore we deduce that for the existence of an invariant conic the condition $16a+4cd+16bg-c^2g=0$ is necessary and sufficient. On the other hand for systems (32) with $e=(cg^2-4c-4dg)/(4g)$ we calculate

$$\widehat{\gamma}_6 = 4(16a+4cd+16bg-c^2g)(4+g^2),$$

and clearly the condition $\widehat{\gamma}_6=0$ yields $a=(c^2g-4cd-16bg)/16$.

Thus we arrive at the family of systems

$$\dot{x}=\frac{c^2g-4cd-16bg}{16}+cx+dy+gx^2, \quad \dot{y}=b+\frac{cg^2-4c-4dg}{4g}x-x^2+gxy-y^2,$$

which possess the invariant conic

$$\Phi(x,y)=\frac{c^2g-4cd-8bg}{8g}+\frac{c}{g}x+\frac{c}{2}y+x^2+y^2=0.$$

Since $g\neq 0$, we may apply to the above systems the following translation

$$x_1=x+\frac{c}{2g}, \quad y_1=y+\frac{c}{4},$$

and after additional change of the parameter b by a_1 and using the formula

$$b=\frac{c^2(g^2-4)-16a_1g-8cdg}{16g^2},$$

we arrive at a simpler canonical form (we pass here to the old notation: $x_1\to x,\ y_1\to y$ and $a_1\to a$)

$$\dot{x}=a+dy+gx^2, \quad \dot{y}=-\frac{a}{g}-dx-x^2+gxy-y^2.$$

We observe that the family of systems we obtained is a subfamily of (13) defined by the condition $h=-1$ and clearly possesses the same ellipse (14) which does not depend on the parameter h. However by (15) the condition $h=-1$ cancels the invariant polynomial $\widehat{\mathcal{R}}_1$ and hence, other polynomial is needed. We calculate

$$\widehat{\mathcal{R}}_5=12ag(4+g^2),$$

and therefore the ellipse (14) is real for $\widehat{\mathcal{R}}_5<0$ and it is complex for $\widehat{\mathcal{R}}_5>0$.

3.1.2.1.2 The possibility $\widehat{\beta}_2=0$. According to (33), we have $g=0$ and we calculate

$$\widehat{\gamma}_1=2cd^2, \quad \widehat{\beta}_1=-d^2.$$

So due to $\widehat{\beta}_1\neq 0$ the condition $\widehat{\gamma}_1=0$ gives $c=0$ and we arrive at the family of systems

$$\dot{x}=a+dy, \quad \dot{y}=b+ex-x^2-y^2. \tag{34}$$

Then, from the first seven equations (7), we determine

$$s=u=1,\ t=0,\ U=0,\ V=-2,\ W=0,\ r=0,\ q=-(d+e),$$

and the last three equations have the form

$$Eq_8 = 2a, \quad Eq_9 = 2b - d^2 - de + 2p, \quad Eq_{10} = -a(d+e).$$

So, the equation $Eq_9 = 0$ yields $p = (d^2 - 2b + de)/2$ and we deduce that the condition $a = 0$ is necessary and sufficient for a system (34) to possess an invariant conic.

On the other hand for these systems we calculate $\widehat{\gamma}_6 = 512a$, i.e. this invariant polynomial is responsible for the existence of an invariant conic. Thus we arrive at the family of systems

$$\dot{x} = dy, \quad \dot{y} = b + ex - x^2 - y^2,$$

which possess the invariant conic

$$\Phi(x,y) = \frac{d^2 - 2b + de}{2} - (d+e)x + x^2 + y^2 = 0.$$

However, even though the canonical form of the above systems is simple, we would like to present them in the same form as the previous ones.

So, we apply the following translation $x_1 = x + (d+e)/2$ and, after additional change of the parameter b by b_1 using the formula $b = (4b_1 + d^2 - e^2)/4$, we arrive at the canonical form (we pass here to the old notation: $x_1 \to x$, $y_1 \to y$ and $b_1 \to b$)

$$\dot{x} = dy, \quad \dot{y} = b - dx - x^2 - y^2. \tag{35}$$

We observe that the above family of systems is a subfamily of (21) defined by the condition $h = -1$ and clearly possess the same ellipse (22) which takes the form

$$\Phi(x,y) = -b + x^2 + y^2 = 0.$$

However from (23) we detect that the condition $h = -1$ cancels the invariant polynomial $\widehat{\mathcal{R}}_2$. So for systems (35) we calculate

$$\widehat{\mathcal{R}}_6 = -4b,$$

and therefore the ellipse (22) with $h = -1$ is real for $\widehat{\mathcal{R}}_6 < 0$ and it is complex for $\widehat{\mathcal{R}}_6 > 0$.

3.1.2.2 The subcase $\widehat{\beta}_1 = 0$. We claim that the conditions $\widehat{\beta}_1 = \widehat{\gamma}_1 = 0$ imply $d = 0$. Indeed suppose the contrary, that $d \neq 0$. Considering (33) we obtain that the condition $\widehat{\gamma}_1 = 0$ implies $\widetilde{\Psi} = 4(d+e)g - c(g^2 - 4) = 0$. If $g \neq 0$ then we get $e = (cg^2 - 4c - 4dg)/(4g)$ and then we obtain

$$\widehat{\beta}_1 = -d^2(g^2+4)(9g^2+4)/16 \neq 0.$$

In the case $g = 0$ the condition $\widetilde{\Psi} = 0$ gives $c = 0$ and we get $\widehat{\beta}_1 = -d^2 \neq 0$, i.e. in both cases we arrive at a contradiction which proves our claim.

Thus $d = 0$ and considering (7) for systems (32) with $d = 0$, we obtain

$$\begin{gathered} s = u = 1,\ t = 0,\ U = 2g,\ V = -2,\ W = r, \\ Eq_6 = 2e + 2q - gr, \quad Eq_8 = 2a - 2gp + cq + er - qr, \\ Eq_5 = 2c - gq - 2r, \quad Eq_9 = 2b + 2p - r^2, \quad Eq_{10} = aq + br - pr, \\ Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_7 = 0. \end{gathered}$$

We detect that equations $Eq_5 = 0$ and $Eq_6 = 0$ yield

$$r = 2(2c + eg)/(4 + g^2), \quad q = -2(2e - cg)/(4 + g^2),$$

and then the equation $Eq_9 = 0$ gives

$$p = \frac{2(2c + eg)^2 - b(4 + g^2)^2}{(4 + g^2)^2}.$$

For these values of the parameters q, r and p for the equation $Eq_8 = 0$ we obtain

$$Eq_8 = \frac{2\big[(a + bg)(4 + g^2)^2 + g(g^2 - 8)(c^2 - e^2) + 2ce(4 - 5g^2)\big]}{(4 + g^2)^2} \equiv \frac{2\mathcal{B}_5}{(4 + g^2)^2}.$$

So, the condition $\mathcal{B}_5 = 0$ is necessary for the existence of invariant conic and this condition yields

$$a = -bg - \frac{g(g^2 - 8)(c^2 - e^2) + 2ce(4 - 5g^2)}{(4 + g^2)^2}. \tag{36}$$

In this case we obtain that all the $Eq_i = 0$ $(i = 1, \dots, 9)$ vanish identically, except $Eq_{10} = 0$, which becomes

$$Eq_{10} = \frac{2}{(4 + g^2)^3}(4c + 4eg - cg^2)\big[b(4 + g^2)^2 + (c^2 - e^2)(g^2 - 4) - 8ceg\big] \equiv \frac{2\mathcal{B}_6\mathcal{B}_7}{(4 + g^2)^3} = 0.$$

So, the condition $Eq_{10} = 0$ implies $\mathcal{B}_6\mathcal{B}_7 = 0$ and we consider two possibilities: $\mathcal{B}_6 \neq 0$ and $\mathcal{B}_6 = 0$.

a) *The possibility* $\mathcal{B}_6 \neq 0$. This implies $\mathcal{B}_7 = 0$ and we obtain

$$b = \frac{8ceg - (c^2 - e^2)(g^2 - 4)}{(4 + g^2)^2}.$$

Thus, we arrive at the family of systems

$$\begin{aligned} \dot{x} &= \frac{2(cg - 2e)(2c + eg)}{(4 + g^2)^2} + cx + gx^2 = \frac{(cg - 2e + 4x + g^2x)(4c + 2eg + 4gx + g^3x)}{(4 + g^2)^2}, \\ \dot{y} &= -\frac{(cg - 2c - 2e - eg)(2c - 2e + cg + eg)}{(4 + g^2)^2} + ex - x^2 + gxy - y^2, \end{aligned}$$

which possess the invariant conic

$$\Phi(x, y) = \frac{c^2 + e^2}{4 + g^2} - \frac{2(2e - cg)}{4 + g^2}x + \frac{2(2c + eg)}{4 + g^2}y + x^2 + y^2 = 0.$$

However, this invariant conic is not irreducible, because the corresponding determinant (see Remark 3) vanishes. In other words, we get a couple of two complex invariant lines. Moreover, we observe that this family of systems also possesses a couple of parallel real invariant lines in the direction $x = 0$.

b) *The possibility* $\mathcal{B}_6 = 0$. Then we have $4c + 4eg - cg^2 = 0$, and we observe that the condition $g \neq 0$ must hold. Indeed, setting $g = 0$, we get $c = 0$, and this implies $a = 0$ which leads to degenerate systems (32).

So, $g \neq 0$ and $\mathcal{B}_6 = 0$ yields $e = c(g^2-4)/(4g)$, and we obtain the systems

$$\dot{x} = \frac{g(c^2-16b)}{16} + cx + gx^2, \quad \dot{y} = b + \frac{c(g^2-4)}{4g}x - x^2 + gxy - y^2, \tag{37}$$

which possess the invariant conic

$$\Phi(x,y) = \frac{c^2-8b}{8} + \frac{c}{g}x + \frac{c}{2}y + x^2 + y^2 = 0.$$

The corresponding determinant of this conic is

$$\Delta = \frac{c^2g^2 - 4c^2 - 16bg^2}{16g^2} \equiv \frac{\mathcal{D}_4}{16g^2},$$

and according to Remark 3 we have an irreducible conic (which is an ellipse) if and only if $\mathcal{D}_4 \neq 0$.

Thus we deduce that a system (32) with $d = 0$ possess an irreducible invariant conic if and only if $\mathcal{B}_5 = \mathcal{B}_6 = 0$ and $\mathcal{D}_4 \neq 0$. We have the next lemma.

Lemma 8. *Assume that for a system* (32) *with $d = 0$ and $g \neq 0$ the conditions $\mathcal{B}_5 = \mathcal{B}_6 = 0$ and $\mathcal{D}_4 \neq 0$ are satisfied. Then, for this system we must have $c^2 - 4ag \neq 0$.*

Proof: Suppose the contrary, i.e. we have $c^2 - 4ag = 0$. Then $a = c^2/(4g)$ and the condition $\mathcal{B}_6 = 4c + 4eg - cg^2 = 0$ gives $e = c(g^2-4)/(4g)$. For these values of the parameters a and e, we obtain

$$\mathcal{B}_5 = -\frac{(g^2+4)^2}{16g}(c^2g^2 - 4c^2 - 16bg^2), \qquad \mathcal{D}_4 = c^2g^2 - 4c^2 - 16bg^2,$$

and clearly the condition $\mathcal{D}_4 \neq 0$ implies $\mathcal{B}_5 \neq 0$. So, we arrive at a contradiction, which completes the proof of the lemma. ∎

On the other hand for systems (32) with $d = 0$ we calculate

$$\widehat{\gamma}_7 = -12(c^2 - 4ag)\mathcal{B}_5.$$

So the condition $\widehat{\gamma}_7 = 0$ implies $\mathcal{B}_5 = 0$, i.e. we have the condition (36). For this value of the parameter a we obtain

$$\widehat{\gamma}_6 = \frac{4g(10+g^2)}{4+g^2}\mathcal{B}_6^2.$$

We claim that, in the case under consideration for non-degenerate systems, the condition $g \neq 0$ must hold. Indeed, assuming $g = 0$ and considering the values of the polynomials $\mathcal{B}_5$ and $\mathcal{B}_6$ given above, we obtain

$$\mathcal{B}_5 = 8(2a + ce) = 0, \quad \mathcal{B}_6 = 4c = 0,$$

and we get $c = a = 0$. However, this leads to the degenerate systems $\dot{x} = 0$, $\dot{y} = b + ex - x^2 - y^2$, and our claim is proved.

Thus, $g \neq 0$ and the condition $\mathcal{B}_6 = 0$ is equivalent to $\widehat{\gamma}_6 = 0$, and we obtain $e = c(g^2-4)/(4g)$. This leads to the systems (37) which could be simplified. Indeed, since $g \neq 0$, we may apply to the above systems the following translation

$$x_1 = x + \frac{c}{2g}, \quad y_1 = y + \frac{c}{4}$$

(which replaces the center of the circle to the origin) and after additional change of the parameter b by a_1 and using the formula

$$b = \frac{c^2(g^2-4) - 16a_1 g}{16},$$

we arrive at a simpler canonical form (we pass here to the old notation: $x_1 \to x,\ y_1 \to y$ and $a_1 \to a$)

$$\dot{x} = a + gx^2, \quad \dot{y} = -\frac{a}{g} - x^2 + gxy - y^2.$$

We observe that the family of systems we obtained is a subfamily of (13) defined by the condition $h = -1$ and $d = 0$. It is clear that the above systems possess the same ellipse (14) which does not depend on the parameters h and d. Moreover, considering the value of the invariant polynomial $\widehat{\mathcal{R}}_3$ from (28), for $h = -1$ we obtain $\widehat{\mathcal{R}}_3 = 160ag(1+g^2)(4+g^2)$. So we deduce that the ellipse (14) is real for $\widehat{\mathcal{R}}_3 < 0$ and it is complex for $\widehat{\mathcal{R}}_3 > 0$.

Since all the cases are examined we conclude that Theorem 2 is proved. ∎

3.1.3 The possibility $\theta = \widetilde{N} = 0$

In this subsection we prove the next theorem which corresponds to the part of the Diagram 1 defined by the condition $\theta = \widetilde{N} = 0$.

Theorem 3. *Assume that for a quadratic system (6) the conditions $\eta < 0$ and $\theta = 0 = \widetilde{N}$ hold. Then, this system either has no invariant ellipse or it has an infinite family of invariant ellipses. Moreover it possesses an 1-parameter family of invariant ellipses (real and complex) if and only if $\widehat{\beta}_1 = \widehat{\gamma}_5 = 0$. In addition, the system possesses an invariant line and the position of these invariant ellipses with respect to the invariant line is determined by the following conditions, correspondingly:*

(i) if $\widehat{\mathcal{R}}_7 < 0$, then all the ellipses are real and they have two (real) common points of intersection as it is shown in the picture $\mathcal{F}_1$ of FIGURE *1;*

(ii) if $\widehat{\mathcal{R}}_7 = 0$, then all the ellipses are real and they have a unique (real) point of intersection as it is shown in the picture $\mathcal{F}_2$ of FIGURE *1;*

(iii) if $\widehat{\mathcal{R}}_7 > 0$, then the invariant ellipses of this family could either be real or complex, depending on the parameter of the family of ellipses. The subfamily of real invariant ellipses has no real intersection points and it is presented in FIGURE *1 by the family $\mathcal{F}_3$.*

Proof: Considering (31) the condition $\widetilde{N} = 0$ yields, for systems (8), $g = 0$ and $h = 1$ (this implies $\theta = 0$), and applying an additional translation we may assume $c = d = 0$, arriving at the family of systems

$$\dot{x} = a + 2xy, \quad \dot{y} = b + ex + fy - x^2 + y^2. \tag{38}$$

Then, from the first five equations (7) for these systems, we determine

$$s = u = 1,\ t = 0,\ U = 0,\ V = 2,\ W = -r,$$

and

$$Eq_6 = 2e, \quad Eq_7 = 2f, \quad Eq_8 = 2a + er + qr,$$
$$Eq_9 = 2b - 2p + fr + r^2, \quad Eq_{10} = aq + br + pr.$$

So, the equations $Eq_6 = Eq_7 = 0$ and $Eq_9 = 0$ give us $e = f = 0$ and $p = (2b + r^2)/2$, and then we obtain

$$Eq_8 = 2a + qr, \quad Eq_{10} = (2aq + 4br + r^3)/2. \tag{39}$$

We shall examine two subcases: $a = 0$ and $a \neq 0$.

1) *The subcase* $a = 0$. So we get the family of systems

$$\dot{x} = 2xy, \quad \dot{y} = b - x^2 + y^2. \tag{40}$$

On the other hand, the conditions $Eq_8 = Eq_{10} = 0$ imply $qr = r(r^2 + 4b) = 0$.

a) If $r = 0$, we get the family of conics depending on the parameter q:

$$\Phi(x, y) = b + qx + x^2 + y^2 = 0, \tag{41}$$

and having the corresponding determinant

$$\Delta = (4b - q^2)/4.$$

So, by Lemma 5, for any fixed value of the parameter b, the ellipses from the family (41) are real if and only if $4b - q^2 < 0$.

We observe that systems (40) possess the invariant line $x = 0$ and two real and two complex finite singularities if $b \neq 0$, and one real invariant line of multiplicity four if $b = 0$ (since we get the homogeneous system).

Then, it is clear that the real singularities are located on the invariant line if $b \leq 0$ and outside of the axis $x = 0$ if $b > 0$.

As a result, we obtain the family of ellipses $\mathcal{F}_1$ if $b < 0$, the family $\mathcal{F}_2$ if $b = 0$, and the family $\mathcal{F}_3$ if $b > 0$ (see FIGURE 1).

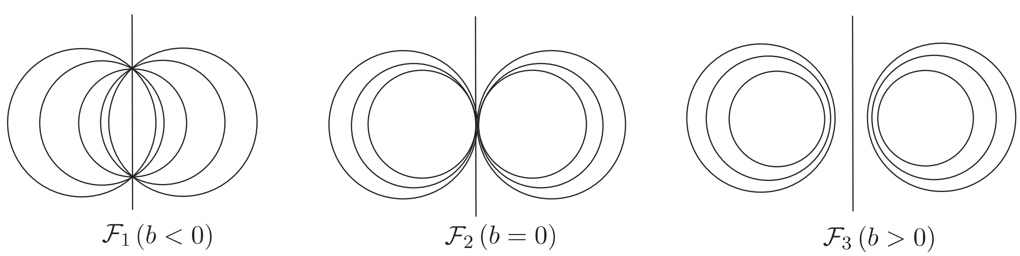

FIGURE 1: **The family** (41) **of invariant ellipses of systems** (40)**.**

b) Assume now $r \neq 0$. Then, $Eq_{10} = r(4b + r^2)/2 = 0$ implies $4b + r^2 = 0$. On the other hand, considering the conditions $s = u = 1,\ t = 0,\ q = 0,\ p = (2b + r^2)/2$, we obtain the conics

$$\Phi(x, y) = b + r^2/2 + x^2 + ry + y^2 = 0,$$

whose determinant is

$$\Delta = (4b + r^2)/4.$$

Evidently, we have that the condition $4b + r^2 = 0$ implies $\Delta = 0$, for any value of the parameter b, and hence we could not have irreducible invariant conics in the considered case.

2) *The subcase* $a \neq 0$. Then, considering (39), from $Eq_8 = 2a + qr = 0$ we get $qr \neq 0$ and $q = -2a/r$. In this case we obtain

$$Eq_{10} = (r^4 + 4br^2 - 4a^2)/(2r).$$

We claim that the condition $Eq_{10} = 0$ leads to degenerate conics. Indeed, taking into consideration the equalities $s = u = 1,\ t = 0,\ q = -2a/r,\ p = (2b + r^2)/2$, we arrive at the family of conics

$$\Phi(x, y) = b + r^2/2 - 2ax/r + ry + x^2 + y^2 = 0,$$

whose determinant is

$$\Delta = (r^4 + 4br^2 - 4a^2)/(4r^2).$$

By Lemma 5, the above conics are irreducible if and only if $\Delta \neq 0$. However, the condition $Eq_{10} = 0$ implies $\Delta = 0$, and this completes the proof of our claim.

Thus, we conclude that a system (38) either has no invariant ellipse or it has an infinite family of invariant ellipses. As it was proved, this system possesses an infinite family of invariant ellipses if and only if the condition $a = f = e = 0$ holds. Moreover the position of the real invariant ellipses with respect to the invariant line $x = 0$ of systems (38) (in the case $a = 0$) is governed by the parameter b.

On the other hand for (38) we calculate

$$\widehat{\beta}_1 = -4(e^2 + f^2), \quad \widehat{\gamma}_5 = -32(2a - ef), \quad \widehat{\mathcal{R}}_7 = 32b,$$

and it is clear that the conditions $\widehat{\beta}_1 = \widehat{\gamma}_5 = 0$ is equivalent to $a = f = e = 0$. Moreover we have $\operatorname{sign}(\widehat{\mathcal{R}}_7) = \operatorname{sign}(b)$ and this completes the proof of Theorem 3. ∎

3.2 Systems with $C_2 = 0$

We remark that if $C_2 = 0$ then a non-degenerate real planar quadratic differential system has all points at infinity (in the Poincaré compactification) as singularities.

In [22] the full study of the whole family of such systems was done. In this subsection we would like to determine the necessary and sufficient affine invariant conditions for a quadratic system (6) with $C_2 = 0$ to possess an invariant ellipse. Therefore for the conic (5), we impose the condition

$$t^2 - su < 0 \quad \Rightarrow \quad su > 0 \tag{42}$$

to be satisfied. It is clear that, if a system possesses an invariant conic $\Phi(x, y) = 0$, then the conic $\alpha\Phi(x, y) = 0$, with $\alpha \in \mathbb{R}$, is also invariant for this system. So, we may assume $u = 1$, and in what follows we will consider the following conic:

$$\Phi(x, y) \equiv p + qx + ry + sx^2 + 2txy + y^2 = 0.$$

We prove the next theorem.

Theorem 4. *Assume that for a quadratic system (6) the condition $C_2 = 0$ is fulfilled. Then this system either has no invariant ellipse or it has an infinite family of invariant ellipses. Moreover it possesses an 1-parameter family of invariant ellipses (real and complex) if and only if one of the sets of conditions indicated below are satisfied. In addition, the system possesses an invariant line and the position of these invariant ellipses with respect to the invariant line is determined by the following conditions, correspondingly:*

(i) $H_{10} \neq 0$ *and* $N_7 = 0$. *Furthermore,*

- *(i_1) if $H_9 < 0$, then all the invariant ellipses are real and they have two (real) common points of intersection located on the invariant line. At these points the ellipses are all tangent to each other, as it is shown in the picture $\mathcal{F}_4$ of* FIGURE *2;*
- *(i_2) if $H_9 = 0$, then all the invariant ellipses are real and they have a unique (real) point of intersection located on the invariant line. The conics are all tangent to the line at this point, as it is shown in the picture $\mathcal{F}_5$ of* FIGURE *2;*
- *(i_3) if $H_9 > 0$, then the invariant ellipses of this family could either be real or complex, depending on the parameter of the family of ellipses. The subfamily of real invariant ellipses has no real intersection points, and it is presented in* FIGURE *2 by the family $\mathcal{F}_6$.*

(ii) $H_{10} = 0$, $H_{12} \neq 0$, *and* $H_2 = 0$. *The invariant ellipses of this family are complex for $H_{11} < 0$ and they are real if $H_{11} > 0$, and in this case their position with respect to the corresponding invariant line is shown in the picture $\mathcal{F}_4$ of* FIGURE *2.*

Proof: Assume that for a quadratic system (6) the condition $C_2(x, y) = 0$ is satisfied. Then, the line at infinity is filled up with singularities and, according to Lemma 2, via a linear transformation and time rescaling, quadratic systems could be brought to the systems $(\mathbf{S}_V)$. Applying the additional translation $(x, y) \mapsto (x - f, y - e)$, we can assume $e = f = 0$ and this leads to the following systems

$$\dot{x} = \hat{a} + \hat{c}x + \hat{d}y + x^2, \quad \dot{y} = \hat{b} + xy. \tag{43}$$

Following [22], for the above systems we calculate $H_{10} = 36\hat{d}^2$. We observe that for $\hat{d} = 0$ these systems possess two parallel invariant lines, and we consider two subcases: $H_{10} \neq 0$ and $H_{10} = 0$.

3.2.1 The case $H_{10} \neq 0$

Then, $\hat{d} \neq 0$. As it was shown in [22, page 749], in this case, via some parametrization and using an additional affine transformation and time rescaling, we arrive at the following 2-parameter family of systems

$$\dot{x} = a + y + (x + c)^2, \quad \dot{y} = xy. \tag{44}$$

Considering (7) for these systems, we obtain

$$Eq_1 = s(2 - U), \quad Eq_2 = 2t(2 - U) - sV, \quad Eq_3 = 2 - U - 2tV, \quad Eq_4 = -V,$$

and evidently the equations $Eq_3 = Eq_4 = 0$ imply $U = 2$ and $V = 0$. Then, we have

$$Eq_5 = -q + 4cs - sW = 0, \quad Eq_6 = -r + 2s + 4ct - 2tW = 0,$$
$$Eq_7 = 2t - W = 0, \quad Eq_8 = -2p + 2cq + 2as + 2c^2s - qW = 0,$$

and this gives

$$q = s(4c - W), \quad r = 2s + 2cW - W^2, \quad t = W/2,$$
$$p = s(2a + 10c^2 - 6cW + W^2)/2.$$

Considering these values of the parameters q, r, t, and p, we finally calculate

$$Eq_i = 0, \quad i = 1, 2, \ldots, 8, \quad Eq_{10} = s(2c - W)\big[4a + (W - 2c)^2\big]/2 = 0,$$
$$Eq_9 = 4cs + (a - 3s + c^2)W - 2cW^2 + W^3 = 0.$$

Since $s \neq 0$, we consider the two subcases defined by the equality $(2c - W)\big[4a + (W - 2c)^2\big] = 0$.

3.2.1.1 The subcase $W = 2c$. This implies $Eq_{10} = 0$ and $Eq_9 = 2c(a - s + c^2) = 0$.

3.2.1.1.1 The possibility $c = 0$. In this case we obtain the 1-parameter family of systems

$$\dot{x} = a + y + x^2, \quad \dot{y} = xy, \tag{45}$$

which possesses the 2-parameter family of invariant conics

$$\Phi(x, y) = as + 2sy + sx^2 + y^2 = 0.$$

Since $s > 0$ (see condition (42)), we may set a new parameter m as follows: $s = 1/m^2$, and this leads to the 1-parameter family of ellipses

$$\widetilde{\Phi}(s, x, y) = a + 2y + x^2 + m^2y^2 = 0. \tag{46}$$

We determine that, for this family, the corresponding determinant $\Delta = am^2 - 1$. So, by Lemma 5, for any fixed value of the parameter a, the ellipses from the family (46) are real if and only if $am^2 - 1 < 0$.

We observe that systems (45) possess the invariant line $y = 0$ and two real or two complex finite singularities located on this line if $a \neq 0$. In the case $a = 0$, we have on this invariant line one double real singularity.

As a result, we obtain the family of ellipses $\mathcal{F}_4$ if $a < 0$, the family $\mathcal{F}_5$ if $a = 0$, and the family $\mathcal{F}_6$ if $a > 0$ (see FIGURE 2). It remains to note that, for systems (45), we have $H_9 = 2304a^3$ and, therefore, the invariant polynomial H_9 distinguishes these cases.

3.2.1.1.2 The possibility $a - s + c^2 = 0$. Then, $s = a + c^2$ and systems (44) possess the following invariant conic

$$\Phi(x, y) = (a + c^2)(a + c^2 + 2cx + 2y) + (a + c^2)x^2 + 2cxy + y^2 = 0,$$

for which we calculate $\Delta = 0$. Then, by Remark 3, this conic is reducible.

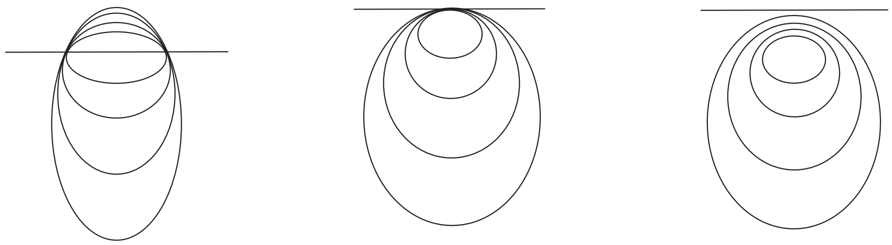

FIGURE 2: **The family** (46) **of invariant ellipses of systems** (45)**.**

3.2.1.2 The subcase $4a + (W-2c)^2 = 0$**.** This implies $Eq_{10} = 0$. If $a = 0$, then $W = 2c$ and, as it was shown above, for the existence of an invariant conic it is necessary $c = 0$. So, we arrive at the particular case of the family of ellipses (46), defined by the condition $a = 0$. Therefore, we consider two possibilities: $a < 0$ and $a > 0$.

On the other hand, for systems (44), we have $H_9 = 2304a(a+c^2)^2$, and clearly the above conditions are governed by this invariant polynomial.

3.2.1.2.1 The possibility $H_9 < 0$**.** Then, $a < 0$ and we may assume $a = -k^2$. After the rescaling $(x, y, t) \mapsto (kx, k^2y, t/k)$, we obtain the systems

$$\dot{x} = y - 1 + (x+c)^2, \quad \dot{y} = xy, \tag{47}$$

for which we have $W = 2(c \pm 1)$, and we obtain $Eq_{10} = 0$, $Eq_9 = 2(c \pm 3)\big[(c \pm 1)^2 - s\big] = 0$. We consider the two subcases given by two factors of the polynomial Eq_9.

1) *The subcase* $c \pm 3 = 0$. We may assume $c > 0$ because of the rescaling $(x, y, t) \mapsto (-x, y, -t)$ in the above systems. Therefore, we set $c = 3$ and then systems (47) could be brought to system (44) with $c = 0$ and $a = -1$ via the transformation

$$(x, y, t) \mapsto \big(2(x-1),\ 4(y-x-1),\ t/2\big).$$

So, we arrive at the system (45) with $a = -1$ and, as it was shown above, this system possesses the family of ellipses (46) with $a = -1$.

2) *The subcase* $(c \pm 1)^2 - s = 0$. Then, $s = (c \pm 1)^2$ and this leads to the reducible conics $\Phi(x, y) = (c^2 - 1 \pm x + cx + y)^2 = 0$.

3.2.1.2.2 The possibility $H_9 > 0$**.** Then, $a > 0$ and we may assume $a = k^2$. Applying the same rescaling as earlier, we arrive at the family systems

$$\dot{x} = 1 + y + (x+c)^2, \quad \dot{y} = xy.$$

In this case, the condition $4a + (W-2c)^2 = 0$ with $a = 1$ gives $W = 2(c \pm i)$, and we obtain $Eq_{10} = 0$ and $Eq_9 = 2(c \pm 3i)\big[(c \pm i)^2 - s\big] = 0$. Since $c \in \mathbb{R}$, we obtain $s = (c \pm i)^2$ and this again leads to the reducible conics $\Phi(x, y) = (c^2 + 1 \pm ix + cx + y)^2 = 0$.

Thus, we detect that, in the case $H_{10} \neq 0$, a system (44) could possess an invariant ellipse if and only if either the conditions $c = 0$ or $c^2 + 9a = 0$ hold. On the other hand, for systems (44), we calculate $N_7 = 16c(c^2 + 9a)$ and, therefore, we conclude that an arbitrary quadratic system (6), with $C_2 = 0$ and $H_{10} \neq 0$, possesses at least one invariant ellipse if and only if $N_7 = 0$.

3.2.2 The case $H_{10} = 0$.

This implies $\hat{d} = 0$. In this case, systems (43) become as systems

$$\dot{x} = \hat{a} + \hat{c}x + x^2, \quad \dot{y} = \hat{b} + xy, \tag{48}$$

for which, following [22], we calculate the value of invariant polynomial $H_{12} = -8\hat{a}^2x^2$ and we consider two possibilities: $H_{12} \neq 0$ and $H_{12} = 0$.

3.2.2.1 The possibility $H_{12} \neq 0$. Then, $\hat{a} \neq 0$ and as it was shown in [22, page 750], in this case, via an affine transformation and time rescaling after some additional parametrization, we arrive at the following 2-parameter family of systems

$$\dot{x} = a + (x + c)^2, \quad \dot{y} = xy, \tag{49}$$

for which the condition $H_{12} = -8(a + c^2)^2x^2 \neq 0$ must hold.

Next, in order to determine the conditions for the existence of an invariant conic, we apply as earlier the equations (7). Since the quadratic parts of the above systems coincide with quadratic parts of systems (44), by the same reasons from the first four equations (7), we determine that $U = 2$ and $V = 0$, and then calculations yield

$$Eq_5 = -q + 4cs - sW = 0, \quad Eq_6 = -r + 4ct - 2tW = 0,$$
$$Eq_7 = -W = 0, \quad Eq_8 = -2p + 2cq + 2as + 2c^2s - qW = 0.$$

So, equation $Eq_7 = 0$ gives $W = 0$, and then we obtain $q = 4cs$, $r = 4ct$, and $p = s(a + 5c^2)$. In this case, we calculate

$$Eq_i = 0, \quad i = 1, 2, \ldots, 8, \quad Eq_9 = 2(a + c^2)t = 0, \quad Eq_{10} = 4c(a + c^2)s = 0.$$

By the conditions (42) and $H_{12} \neq 0$, we have $s(a + c^2) \neq 0$ and, therefore, we obtain $t = c = 0$. So, we arrive at the 1-parameter family of systems

$$\dot{x} = a + x^2, \quad \dot{y} = xy, \tag{50}$$

which possesses the family of conics

$$\Phi(x, y) = as + sx^2 + y^2 = 0.$$

Since $s > 0$, we may set a new parameter m as follows: $s = 1/m^2$, and then we arrive at the 1-parameter family of ellipses:

$$\widetilde{\Phi}(x, y) = a + x^2 + m^2y^2 = 0.$$

It is clear that the above ellipses are real if $a < 0$ and complex if $a > 0$. On the other hand, for the systems (50) we have $H_{11} = -192ax^4$, and hence the above conditions are governed by this invariant polynomial.

3.2.2.2 The possibility $H_{12} = 0$. Then, for systems (48), we have $\hat{a} = 0$, and this condition implies $\hat{b} \neq 0$ (otherwise we obtain degenerate systems). So, we may assume $\hat{b} = 1$, due to the rescaling $y \to \hat{b}y$, and this leads to the 1-parameter family of systems (we set $\hat{c} = c$)

$$\dot{x} = cx + x^2, \quad \dot{y} = 1 + xy.$$

And again, since the quadratic parts of the above systems coincide with quadratic parts of systems (44), by the same reasons from the first four equations (7), we determine that $U = 2$ and $V = 0$, and then calculations yield

$$Eq_5 = -q + 2cs - sW = 0, \quad Eq_6 = -r + 2ct - 2tW = 0, \quad Eq_7 = -W = 0,$$
$$Eq_8 = -2p + cq + 2t - qW = 0, \quad Eq_9 = 2 - rW = 0, \quad Eq_{10} = r - pW.$$

So, it is evident that the conditions $Eq_7 = Eq_9 = 0$ lead to a contradiction.

Then, it was shown that in the case $H_{10} = 0$ (i.e $\hat{d} = 0$) systems (43) for $H_{12} \neq 0$ (i.e $\hat{a} \neq 0$) could be brought via an affine transformation to the systems (49) with $a + c^2 \neq 0$. Moreover, it was proved that these systems possess at least one invariant ellipse if and only if $c = 0$.

On the other hand, for systems (49) we calculate $H_2 = 4cx^2$, and hence this invariant polynomial is responsible for the condition $c = 0$.

Since all the cases are investigated we deduce that Theorem 4 is proved and hence the conditions from the Diagram 2 are valid. ∎

4 The proof of the Main Theorem: statement (*B*)

Consider the family of quadratic systems (6) and assume that these systems possess an invariant ellipse, i.e. one of the set of the conditions provided by the statement **(*A*)** of the Main Theorem (see Diagrams 1 and 2) is fulfilled. Then, according to [14], via an affine change and time rescaling, they could be brought to the canonical systems of Qin Yuan-Xun (see also [13, page 412])

$$\dot{x} = 1 - cy - x^2 - axy - (b+1)y^2, \quad \dot{y} = x(c + ax + by). \tag{51}$$

These systems possess the invariant conic

$$\Phi(x, y) = x^2 + y^2 - 1 = 0.$$

According to [13] (see page 412), this conic is a limit cycle if and only if

$$a^2 + b^2 < c^2, \quad a \neq 0. \tag{52}$$

We note that these conditions are different from the ones which appear in [7], where the conditions contain a minor mistake.

For systems (51), we calculate

$$\mathcal{T}_3\mathcal{F} = a^2c^2(a^2 + b^2 - c^2)\big[a^2 + (b-2)^2\big]^2/8,$$

and, since from the conditions (52) we have $ac \neq 0$, the following lemma is valid.

Lemma 9. *If a quadratic system* (6) *possesses an invariant ellipse, then this ellipse is a limit cycle of the system if and only if* $\mathcal{T}_3\mathcal{F} < 0$.

So, it remains only to detect which of the conditions, provided by Theorems 1, 2, 3, and 4, in the case of systems (51), are compatible with the conditions (52).

Lemma 10. *For systems* (51) *satisfying the conditions* (52)*, the following conditions hold:*

$$\widehat{\gamma}_1 = \widehat{\gamma}_2 = 0, \quad \eta < 0, \quad C_2\widetilde{N} \neq 0 \quad \textit{and} \quad \widehat{\beta}_1\widehat{\beta}_2 \neq 0.$$

Proof: For systems (51) we calculate:

$$\begin{aligned}
\widehat{\gamma}_1 =&0, \quad \widehat{\gamma}_2 = 0, \quad \eta = -4\big[a^2 + (b+1)^2\big],\\
C_2 =& -\big[ax + (b+1)y\big](x^2 + y^2),\\
\widetilde{N} =& \big[18a^2 - 9b(b+2)\big]x^2 + 18a(3b+4)xy + 9\big[a^2 - 2(b+1)(b+2)\big]y^2,\\
\widehat{\beta}_1 =& -c^2\big[a^2 + (b-2)^2\big]\big[a^2 + (b+2)^2\big]/16,\\
\widehat{\beta}_2 =& -a\big[(a^2 + b^2)^2 - (2b+3)(a^2 + b^2) + 4(b+1)\big]/2.
\end{aligned}$$

Considering the conditions (52) and the formulae above, the conditions $C_2 \neq 0$, $\eta < 0$, $\widehat{\beta}_1 \neq 0$, and $\widehat{\beta}_4 \neq 0$ follow immediately. We examine the remaining conditions: $\widetilde{N} \neq 0$ and $\widehat{\beta}_2 \neq 0$.

Suppose first that $\widetilde{N} = 0$. Then, due to $a \neq 0$, we obtain $b = -4/3$, and then we get

$$\widetilde{N} = (9a^2 + 4)(2x^2 - y^2) \neq 0,$$

leading to a contradiction.

Next we examine the condition $\widehat{\beta}_2 \neq 0$. We write $\widehat{\beta}_2 = -\dfrac{a\tilde{\beta}_2}{2}$, where

$$\tilde{\beta}_2 = (a^2 + b^2)^2 - (2b+3)(a^2 + b^2) + 4(b+1)$$

and assume the contrary, that there exists $(a_0, b_0) \in \mathbb{R}^2$ such that $\tilde{\beta}_2(a_0, b_0) = 0$. Denoting by $u = a^2 + b^2$ we obtain

$$\tilde{\beta}_2 = u^2 - (2b+3)u + 4(b+1) \equiv \varphi_1(b, u)$$

and clearly we have $\varphi_1(b_0, u_0) = 0$ where $u_0 = a_0^2 + b_0^2$. We consider the plane (b, u) and examine the graphics of the two conics: hyperbola $\varphi_1(b, u) = 0$ and the parabola $\varphi_2(b, u) = u - b^2 = 0$ (see Figure 3). We observe that these conics have two common points at which the conics are tangent. Let b_0 be a point on the axis Ob for which the equation $\varphi_1(b_0, u) = 0$ has at least one real solution. Clearly we have exactly one solution if and only b_0 is the abscissa of one of the common points of the conics. In other cases, due to the choice of b_0, the equation $\varphi_1(b_0, u) = 0$ possesses two solutions $u_0^{(i)}$, $i = 1, 2$, where $u_0^{(i)} = \left(a_0^{(i)}\right)^2 + b_0^2$.

On the other hand from Figure 3 it follows that for each $i = 1, 2$ we have $u_0^{(i)} \leq b_0^2$. However in this case we get $\left(a_0^{(i)}\right)^2 + b_0^2 \leq b_0^2$, which is impossible due to $a \neq 0$ and this completes the proof of this lemma. ∎

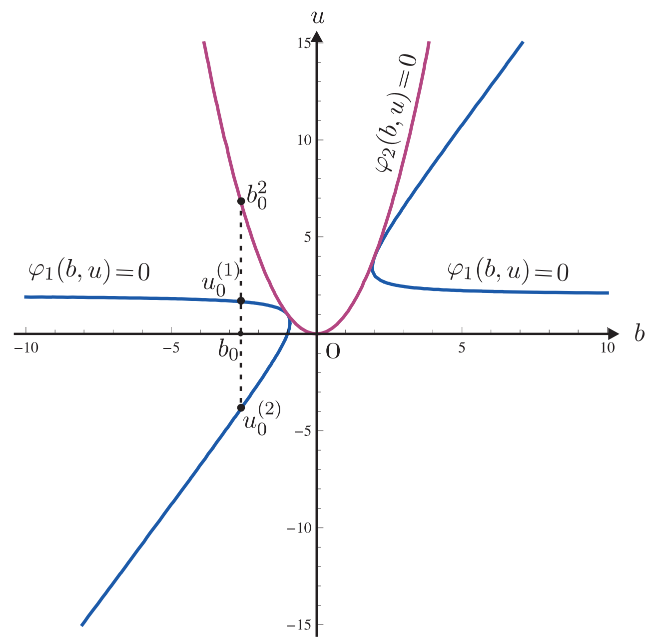

FIGURE 3: **Grafics of the conics $\phi_1(b,u)=0$ and $\phi_2(b,u)=0$.**

We claim that for any non-degenerate quadratic system with $\eta<0$ the condition $\widehat{\beta}_2\neq 0$ implies $\widetilde{N}\neq 0$. Indeed, assume the contrary, that for a quadratic system the conditions $\eta<0$, $\widehat{\beta}_2\neq 0$ and $\widetilde{N}=0$ are satisfied. Then via an affine transformation these systems could be brought to the systems (38). However, for these systems we have $\widehat{\beta}_2=0$, which leads to a contradiction.

Therefore considering Lemmas 9 and 10, and the conditions provided by Theorem 1 (in the case $\theta\neq 0$) and by Theorem 2 (in the case $\theta=0$ and $\widetilde{N}\neq 0$) for the existence of a real invariant ellipse, we arrive at the next assertion.

Theorem 5. *A quadratic system* (6) *possesses an invariant ellipse, which in addition is an algebraic limit cycle, if and only if* $\eta<0$, $\mathcal{T}_3\mathcal{F}<0$, $\widehat{\beta}_1\widehat{\beta}_2\neq 0$, $\widehat{\gamma}_1=\widehat{\gamma}_2=0$, *and one of the following sets of conditions is satisfied:*

- *(i)* $\theta\neq 0$, $\widehat{\beta}_3\neq 0$, $\widehat{\mathcal{R}}_1<0$; $\quad$ *(*$a=-1,\ b=-5,\ c=-6$*);*
- *(ii)* $\theta\neq 0$, $\widehat{\beta}_3=0$, $\widehat{\gamma}_3=0$, $\widehat{\mathcal{R}}_1<0$; $\quad$ *(*$a=-3/4,\ b=-(10+3\sqrt{3})/4,\ c=-8$*);*
- *(iii)* $\theta=0$, $\widehat{\gamma}_6=0$, $\widehat{\mathcal{R}}_5<0$; $\quad$ *(*$a=-1/4,\ b=-(2+\sqrt{3})/4,\ c=-2$*).*

We remark that, in the second column in parentheses in Theorem 5, we present examples which prove that the corresponding sets of the invariant conditions are not empty.

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References

[1] J. C. Artés, J. Llibre, D. Schlomiuk and N. Vulpe, *Geometric configurations of singularities of planar polynomial differential systems [A global classification in the quadratic case]*, Accepted for publication by Springer Nature Switzerland AG (January, 2019).

[2] V. Baltag, *Algebraic equations with invariant coefficients in qualitative study of the polynomial homogeneous differential systems*. Bull. of Acad. of Sci. of Moldova. Mathematics **2**(2003), 31–46.

[3] D. Bularas, Iu. Calin, L. Timochouk and N. Vulpe, T–comitants of quadratic systems: a study via the translation invariants, Delft University of Technology, Faculty of Technical Mathematics and Informatics, Report no. 96-90, 1996; (URL: `ftp://ftp.its.tudelft.nl/publications/techreports/1996/DUT-TWI-96- 90.ps.gz`).

[4] L. Cairó and J. Llibre, *Darbouxian first integrals and invariants for real quadratic systems having an invariant conic*, J. Phys. A: Math. Gen. **35** (2002), 589 - 608.

[5] L. Cairó, M.R. Feix and J. Llibre, *Integrability and algebraic solutions for planar polynomial differential systems with emphasis on the quadratic systems*, Resenhas **4** (1999), no. 2, 127–161.

[6] Iu. Calin, *On rational bases of $GL(2,\mathbb{R})$-comitants of planar polynomial systems of differential equations*, Bull. of Acad. of Sci. of Moldova. Mathematics **2** (2003), 69–86.

[7] J. Chavarriga, H. Giacomini and J. Llibre, *Uniqueness of algebraic limit cycles for quadratic systems*, J. Math. Anal. Appl. **261** (2001), 85–99.

[8] C. Christopher, *Quadratic systems having a parabola as an integral curve*. Proceedings of the Royal Society of Edinburgh **112A** (1989), 113–134.

[9] C. Christopher, *Invariant algebraic curves and conditions for a center*, Proc. Roy. Soc. Edinburgh **124A** (1994), 1209–1229.

[10] T.A. Druzhkova *The algebraic integrals of a certain differential equation.* Differ. Equ. **4** 736, 1968

[11] J.H. Grace and A. Young, The algebra of invariants, New York: Stechert, 1941.

[12] J.D. Lawrence, *A catalog of special planar curves.* Dover Publication, 1972.

[13] J. Llibre and G. Ṡwirszcz, *Classification of quadratic systems admitting the existence of an algebraic limit cycle*. Bull. Sci. Math. **131** (2007), 405–421.

[14] Qin Yuan-xun, *On the algebraic limit cycles of second degree of the differential equation $dy/dx = \Sigma_{0\le i+j\le 2} a_{ij}x^i y^j / \Sigma_{0\le i+j\le 2} b_{ij}x^i y^j$*, Chin. Math. Acta **8** 608 (1996).

[15] R.D.S. Oliveira, A.C. Rezende and N. Vulpe, *Family of quadratic differential systems with invariant hyperbolas: a complete classification in the space $\mathbb{R}^{12}$*, Electron. J. Differential Equations **162** (2016), 1–50.

[16] P.J. Olver, Classical Invariant Theory, London Mathematical Society, Student Texts: **44**, Cambridge University Press, 1999.

[17] M.N. Popa, *Applications of algebraic methods to differential systems*, Romania, Piteşti Univers., The Flower Power Edit., 2004.

[18] D. Schlomiuk and N. Vulpe, *Geometry of quadratic differential systems in the neighbourhood of the line at infinity.* J. Differential Equations, **215** (2005), 357–400.

[19] D. Schlomiuk and N. Vulpe, *Global classification of the planar Lotka–Volterra differential systems according to their configurations of invariant straight lines*, Journal of Fixed Point Theory and Applications, **8** (2010), no. 1, 177–245.

[20] D. Schlomiuk and N. Vulpe, *Integrals and phase portraits of planar quadratic differential systems with invariant lines of at least five total multiplicity*, Rocky Mountain Journal of Mathematics, **38** (2008), No. 6, 1–60.

[21] D. Schlomiuk and N. Vulpe, *Planar quadratic differential systems with invariant straight lines of at least five total multiplicity*, Qualitative Theory of Dynamical Systems, **5** (2004), 135–194.

[22] D. Schlomiuk and N. Vulpe, *The full study of planar quadratic differential systems possessing a line of singularities at infinity*, J. Dynam. Differential Equations, **20** (2008), No. 4, 737–775.

[23] K.S. Sibirskii, *Introduction to the algebraic theory of invariants of differential equations*, Translated from the Russian. Nonlinear Science: Theory and Applications. Manchester University Press, Manchester, 1988.

[24] N. Vulpe, *Characterization of the finite weak singularities of quadratic systems via invariant theory.* Nonlinear Analysis, **74** (2011), No. 4, 6553–6582.

[25] N.I. Vulpe, Polynomial bases of comitants of differential systems and their applications in qualitative theory. (Russian) “Shtiintsa”, Kishinev, 1986.