AN ESTIMATIVE FOR THE NUMBER OF LIMIT CYCLES IN A LIÉNARD-LIKE PERTURBATION OF A QUADRATIC NON-LINEAR CENTER

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ABSTRACT. We study the maximum number of limit cycles that can bifurcate from an integrable non-linear quadratic isochronous center, when perturbed with a Liénard-like polynomial perturbation of arbitrary degree.

1. INTRODUCTION

The number of limit cycles in systems that are perturbation of a linear non-denegerate center has been extensively studied in [2, 5, 4] . The main tool used to study such problems are the averaging methods [2].

In this paper we consider systems of the form

$$\left\{\begin{array}{rcl}\dot{x} & = & p(x,y)+\varepsilon P_n(x,y),\\ \dot{y} & = & q(x,y)+\varepsilon Q_n(x,y),\end{array}\right. \tag{1.1}$$

where p, q are quadratic polynomials and P_n, Q_n are n-degree polynomials to be choosen inside a particular family to be specified later. It is natural to consider that, for $\varepsilon = 0$, system (1.1) has a center at the origin. In addition, we shall consider that for $\varepsilon = 0$, (1.1) has a isochronous system at origin.

Our aim is to study the maximum number of limit cycles that can appear on system (1.1) when $\varepsilon \neq 0$. The main tool we employ is the averaging method described in [2].

In [2] is proved that if we consider a 2-degree polynomial perturbation of system (1.1), then at most 2 limit cycles bifurcate from this center. The same result was proved in [9]. A computational-numeric approach to this problem (using general perturbations up to some degree) can be found in [12, 6].

A correlated problem is the study of the number of limit cycles that bifurcates from the linear center of system (1.2)

$$\left\{\begin{array}{rcl}\dot{x} & = & -y+\varepsilon P_n(x,y),\\ \dot{y} & = & x+\varepsilon Q_n(x,y),\end{array}\right. \tag{1.2}$$

with $\varepsilon = 0$, when $\varepsilon \neq 0$. This problem is considered in a huge number of papers (see [2] and references), including its analog in higher dimension. The averaging theory has been the main technique to attack this problem, especially in the last years. We shall compare our result, for system (1.1), with the existing results for system (1.2). In particular, in Section 3, we emphasize the main differences between the

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application of averaging theory to linear centers (system (1.2)) and the general case (system (1.1)).

The main result we prove in this paper is the following. Consider system

$$\left\{\begin{array}{rcl} \dot{x} & = & -y+x^2, \\ \dot{y} & = & x+xy+\varepsilon\left(yf_n(x)+g_n(x)\right), \end{array}\right. \tag{1.3}$$

where f_n, g_n are polynomials of degree n.

Theorem 1.1. *Suppose that in system* (1.3)*, f_n, g_n are polynomials of degree n. Then the maximum number of limit cycles that bifurcates from the center ($\varepsilon = 0$ in* (1.3)*) is:*
i) 1*, if $n = 2, 3$,*
ii) 2*, if $n = 4, 5$,*
iii) lower of equal than $n-2$ (if $n > 4$ is even) or $n-3$ (if $n > 5$ is odd).

One of the goals of this paper is consider perturbations of system (1.3) of arbitrary degree (inside a specific family), while the existing results are for perturbations of fixed degree.

This paper is organized as follows. In Section 2 we provide a classification of quadratic systems with an isochronous center at origin. In Section 3 we describe the results of the averaging theory that we shall need to our purposes, and briefly compare the applications of the averaging theorems for systems (1.1) and (1.2). Moreover, we define the special family of perturbation we shall consider. The proof of Theorem 1.1 is contained in Section 5.

2. Quadratic centers

In [7] is proved that the origin is an isochronous center of the quadratic system

$$\left\{\begin{array}{rcl} \dot{x} & = & -y+a_{2,0}x^2+a_{1,1}xy+a_{0,2}y^2 \\ \dot{y} & = & x+b_{2,0}x^2+b_{1,1}xy+b_{0,2}y^2 \end{array}\right.$$

if and only if the system can be brought to one of the systems (2.1), (2.2), (2.3) or (2.4) through a linear change of coordinates and a rescaling of time:

$$\left\{\begin{array}{rcl} \dot{x} & = & -y+x^2-y^2 \\ \dot{y} & = & x(1+2y) \end{array}\right. \tag{2.1}$$

$$\left\{\begin{array}{rcl} \dot{x} & = & -y+x^2 \\ \dot{y} & = & x(1+y) \end{array}\right. \tag{2.2}$$

$$\left\{\begin{array}{rcl} \dot{x} & = & -y-\frac{4}{3}x^2 \\ \dot{y} & = & x(1-\frac{16}{3}y) \end{array}\right. \tag{2.3}$$

$$\left\{\begin{array}{rcl} \dot{x} & = & -y+\frac{16}{3}x^2-\frac{4}{3}y^2 \\ \dot{y} & = & x(1+\frac{8}{3}y) \end{array}\right. \tag{2.4}$$

All these systems are integrable. Regarding the expression of the first integral, system (2.2) is the simplest of all. In this paper, we just consider perturbations of system (2.2).

3. Averaging theory

In this section we briefly describe some results on periodic averaging of first order. This is the simplest form of averaging, and is concerned with approximating solutions of a non-autonomous differential equation by solutions of a autonomous one. In particular, the first order averaging method is equivalent to the study of the first order Melnikov function (both are equivalent to the study of the displacement function). For more references on this, see [13].

The averaging theory was formalized by 1930, but some naive results were conjectured even in the 18th century. For a historical description, we suggest [10].

The next theorem is the classical averaging theorem for periodic differential system.

Theorem 3.1. *We consider the following differential system*

$$x' = \varepsilon f(t,x) + \varepsilon^2 g(t,x,\epsilon) \tag{3.1}$$

where $x \in D$ (D is an open subset of $\mathbb{R}$), $t \in [0,\infty)$, $\varepsilon \in (0,\varepsilon_0]$, f,g are T-periodic in the variable t. Suppose that f and g are maps of class C^2. Consider the average function of $f(t,x)$ with respect to t

$$f^0(y) = \int_0^T f(t,y)\,dt. \tag{3.2}$$

If $p \in D$ is such that $f^0(p) = 0$ and $Df^0(p) \neq 0$, then for every $|\varepsilon| > 0$ small, there existes a T-periodic solution $\varphi_\varepsilon(t)$ of system (3.1) such that $\phi(t,\varepsilon) \to p$ as $\varepsilon \to 0$.

Systems in the form of (3.1) are said to be in the standard form. We remark that the regularity condition that f,g are of class C^2 is not really necessary, but simplify the statement. For a proof and more comments, see chapter 6 of [10].

Theorem 3.1 is about the existence of periodic solutions for non-autonomous systems. The following construction allows us to use this theorem for proving the existence of limit cycles.

Consider the planar system

$$\left\{ \begin{array}{rcl} \dot{x} &=& p(x,y) + \varepsilon P(x,y) \\ \dot{y} &=& q(x,y) + \varepsilon Q(x,y) \end{array} \right. \tag{3.3}$$

where $p,q,P,Q : \mathbb{R}^2 \to \mathbb{R}$ are continuous functions.

Let us first consider the case $p(x,y) = -y$ and $q(x,y) = x$. Then, for $\varepsilon = 0$, system (3.3) is the linear center.

In this case, using a polar change of coordinates $x = r\cos(\theta)$, $y = r\sin(\theta)$, we obtain

$$\left\{ \begin{array}{rcl} \dot{r} &=& \varepsilon\left[\cos(\theta)P(r\cos(\theta), r\sin(\theta)) + \sin(\theta)Q(r\cos(\theta), r\sin(\theta))\right] \\ \dot{\theta} &=& 1 + (\varepsilon/r)\left[\cos(\theta)Q(r\cos(\theta)) - \sin(\theta)P(r\cos(\theta), r\sin(\theta))\right] \end{array} \right. \tag{3.4}$$

For small $\varepsilon > 0$, the second equation of (3.4) is not zero. Then we can reparametrize (3.4) to obtain

$$\left\{ \begin{array}{rcl} \dot{r} &=& \varepsilon\left[\cos(\theta)P(r\cos(\theta), r\sin(\theta)) + \sin(\theta)Q(r\cos(\theta), r\sin(\theta))\right] + o(\varepsilon^2) \\ \dot{\theta} &=& 1 \end{array} \right. \tag{3.5}$$

Taking θ as the new time, (3.5) turns into a non-autonomous differential equation

(3.6)
$$\left\{ \; r' \;\; = \;\; \varepsilon\left[\cos(\theta)P(r\cos(\theta), r\sin(\theta)) + \sin(\theta)Q(r\cos(\theta), r\sin(\theta))\right] + o(\varepsilon^2) \right.$$

where the prime is derivative with respect to θ.

Note that we can apply Theorem 3.1 to system (3.6).

Let

$$G(r) = \int_0^{2\pi} \left[\cos(\theta)P(r\cos(\theta), r\sin(\theta)) + \sin(\theta)Q(r\cos(\theta), r\sin(\theta))\right]\, d\theta.$$

By Theorem 3.1, each simple zero of G is associated with a periodic solution of (3.6) and then with a limit cycle of (3.3).

If P, Q are polynomials with no constant terms, P with degree M and Q with degree N, for instance,

$$P(x,y) = \sum_{k=1}^{M}\sum_{i+j=k} a_{i,j}x^i y^j, \;\; Q(x,y) = \sum_{k=1}^{N}\sum_{i+j=k} b_{i,j}x^i y^j, \tag{3.7}$$

then $G(r)$ writes as

$$\begin{aligned} G(r) \;\; &= \;\; \sum_{k=1}^{M} r^k \sum_{i+j=k} a_{i,j} \int_0^{2\pi} \cos^{i+1}(\theta)\sin^j(\theta)\, d\theta. \\ &+ \;\; \sum_{k=1}^{N} r^k \sum_{i+j=k} b_{i,j} \int_0^{2\pi} \cos^{i}(\theta)\sin^{j+1}(\theta)\, d\theta. \end{aligned}$$

Note that $G(r)$ is a polynomial with degree $\max\{M, N\}$, that is, the maximum number of limit cycles of system (3.3) with $p(x,y) = -y$, $q(x,y) = x$ and P, Q polynomials given by (3.7) is $\max\{M, N\}$. In particular, the number of limit cycles, and even the function G, depends directly on the coefficients of P, Q: the coefficient of r^k depends just on $a_{i,j}, b_{i,j}$ with $i + j = k$.

Now we turn to the general case. Our aim is to put (3.3) in the standard form to apply the Averaging Theorem 3.1.

The polar coordinates do not help anymore - in fact, they just work when the system is a linear center for $\varepsilon = 0$.

From now, we assume that (3.3), for $\varepsilon = 0$, has a first integral H, and a continuous family of closed orbits

$$\{\Gamma_h\} \subset \{(x,y) : H(x,y) = h, h_1 < h < h_2\}.$$

This allow us to find a change of coordinates that put (3.3) in the standard form.

Assume that $xp(x,y) - yp(x,y) \neq 0$ for all (x,y) in $\bigcup \Gamma_h$. Let

$$\rho : (\sqrt{h_1}, \sqrt{h_2}) \times [0, 2\pi) \to [0, \infty)$$

be a continuous function such that

$$H(\rho(R,\phi)\cos(\phi), \rho(R,\phi)\sin(\phi)) = R^2$$

for all $R \in (\sqrt{h_1}, \sqrt{h_2})$ and all $\phi \in [0, 2\pi)$.

The change of coordinates $x = \rho(R,\phi)\cos(\phi)$, $y = \rho(R,\phi)\sin(\phi)$ applied to system (3.3) give us

$$\left\{ \begin{array}{rcl} \dot{R} & = & \varepsilon L(R,\phi), \\ \dot{\phi} & = & 1 + \varepsilon S(R,\phi), \end{array} \right.$$

for some smooth functions L, S. Now, using the same argument as in the linear center, we can obtain a non-autonomous system in the form

$$R' = \varepsilon L(R,\phi) + o(\varepsilon^2), \tag{3.8}$$

where the prime is the derivative with respect to ϕ. Note that (3.8) is in the standard form.

Applying the Averaging Theorem 3.1 to (3.8) and writing the expression of L, we obtain the following result:

Theorem 3.2 ([2], Theorem 5.2)**.** *Consider that system* (3.3) *has a first integral H for $\varepsilon = 0$, and a continuous family of closed orbits*

$$\{\Gamma_h\} \subset \{(x,y) : H(x,y) = h, h_1 < h < h_2\}.$$

Let $\mu(x,y)$ be an integrating factor for system (3.3)*. If*

$$F(R) = \int_0^{2\pi} \frac{\mu \cdot (x^2+y^2) \cdot (Pq - Qp)}{2R \cdot (qx - py)}\, d\phi, \tag{3.9}$$

where μ, P, p, Q, q depends on $x = \rho(R,\phi)\cos(\phi)$ and $y = \rho(R,\phi)\sin(\phi)$, then each simple zero of $F(R)$ give us a limit cycle of (3.3)*.*

Remark 3.3. The integrand of $F(R)$ is exactly the function $L(R,\phi)$ we defined above.

Now we workout an example that shows that the dependence of F on P, Q is not so directly as in the linear case.

Consider the system of differential equations

$$\left\{ \begin{array}{rcl} \dot{x} & = & -y + x^2 + \varepsilon c x y^2, \\ \dot{y} & = & x + xy + \varepsilon(a x^2 + b y^3), \end{array} \right. \tag{3.10}$$

where $\varepsilon > 0$ is a small parameter and $a, b, c \in \mathbb{R}$. For $\varepsilon = 0$, this system has a center at the origin.

The function F given by (3.9) is

$$F(Z) = \rho(Z)(2aZ^2 + (a + c - 3b)Z - 2c),$$

where $R = \sqrt{1 - Z^2}$ and ρ is a C^1 function without zeroes for $Z \in (0,1)$.

Note that the leader coefficient in the polynomial $d(Z) = \dfrac{F(Z)}{\rho(Z)}$ is $2a$, but a is not associated with the higher degree terms in (3.10); furthermore, the coefficient associated to the third order term y^3 in the second line of (3.10) appears just on the coefficient of the linear term of d.

This indirect dependence of F on the coefficients of the perturbation make difficult to consider general perturbations of non-linear centers.

As we want to study systems with perturbations of arbitrary degree, we fix a system, given by

$$\left\{ \begin{array}{rcl} \dot{x} & = & -y + x^2, \\ \dot{y} & = & x + xy + \varepsilon\left(y f_n(x) + g_n(x)\right), \end{array} \right. \tag{3.11}$$

where f_n, g_n are polynomials of degree n.

We call this perturbation Lienard like due to its similarity with the classical Lienard system.

4. Proof of Theorem 1.1

Recall system (1.3)

$$\left\{ \begin{array}{rcl} \dot{x} & = & -y + x^2, \\ \dot{y} & = & x + xy + \varepsilon\left(y f_n(x) + g_n(x)\right), \end{array} \right. \tag{4.1}$$

with f_n, g_n polynomials of degree n. If we take $H(x,y) = \dfrac{x^2+y^2}{(1+y)^2}$, then H is a first integral for (4.1) with $\varepsilon = 0$, and all the level curves $H = c$ are closed for $c \in (0,1)$.

The solution of $H(\rho\cos(\psi), \rho\sin(\psi)) = R^2$ is $\rho = \dfrac{R}{1 - R\sin(\psi)}$. Note also that the integrand factor is $\mu = \dfrac{2}{(1+y)^3}$.

The integrand in expression (3.9), for this case, is given by

$$\begin{array}{rcl} L(R,\psi) & = & \zeta(R,\psi) f_n\left(\dfrac{R\cos(\psi)}{1-R\sin(\psi)}\right) \\ & + & \eta(R,\psi) g_n\left(\dfrac{R\cos(\psi)}{1-R\sin(\psi)}\right), \end{array} \tag{4.2}$$

where

$$\begin{array}{rcl} \zeta(R,\psi) & = & -R^2\sin(\psi) - R\cos^2(\psi) + R, \\ \eta(R,\psi) & = & R^2\sin(\psi) - 2R + R\cos^2(\psi) + \sin(\psi). \end{array}$$

Finally, the first Melnikov function (given by (3.9)) is

$$F(R) = \int_0^{2\pi} \left(\zeta(R,\psi) f_n\left(\frac{R\cos(\psi)}{1-R\sin(\psi)}\right) + \eta(R,\psi) g_n\left(\frac{R\cos(\psi)}{1-R\sin(\psi)}\right)\right) d\psi.$$

Note that we are interested in the isolated zeros of $F(R)$ with $R \in (0,1)$ (as the center is contained in $H^{-1}((0,1))$.

Put $f_n(x) = \sum_{j=1}^n a_j x^j$ and $g_n(x) = \sum_{j=1}^n b_j x^j$. Then $F(R)$ is given by

(4.3)

$$\begin{array}{rcl} F(R) & = & \displaystyle\int_0^{2\pi} \left(\zeta(R,\psi)\sum_{j=1}^n a_j \left(\frac{R\cos(\psi)}{1-R\sin(\psi)}\right)^j + \eta(R,\psi)\sum_{j=1}^n b_j \left(\frac{R\cos(\psi)}{1-R\sin(\psi)}\right)^j\right) d\psi \\ & = & \displaystyle\int_0^{2\pi} \left(\zeta(R,\psi)\sum_{j=1}^n a_j \frac{R^j\cos^j(\psi)}{(1-R\sin(\psi))^j} + \eta(R,\psi)\sum_{j=1}^n b_j \frac{R^j\cos^j(\psi)}{(1-R\sin(\psi))^j}\right) d\psi \\ & = & \displaystyle -\sum_{j=1}^n a_j R^{j+2} \int_0^{2\pi} \frac{\sin(\psi)\cos^j(\psi)}{(1-R\sin(\psi))^j}\, d\psi - \sum_{j=1}^n a_j R^{j+1} \int_0^{2\pi} \frac{\cos^{j+2}(\psi)}{(1-R\sin(\psi))^j}\, d\psi \\ & + & \displaystyle \sum_{j=1}^n a_j R^{j+1} \int_0^{2\pi} \frac{\cos^j(\psi)}{(1-R\sin(\psi))^j}\, d\psi + \sum_{j=1}^n b_j R^{j+2} \int_0^{2\pi} \frac{\sin(\psi)\cos^j(\psi)}{(1-R\sin(\psi))^j}\, d\psi \\ & + & \displaystyle \sum_{j=1}^n b_j R^{j+1} \int_0^{2\pi} \frac{\cos^{j+2}(\psi)}{(1-R\sin(\psi))^j}\, d\psi - 2\sum_{j=1}^n b_j R^{j+1} \int_0^{2\pi} \frac{\cos^j(\psi)}{(1-R\sin(\psi))^j}\, d\psi \\ & + & \displaystyle \sum_{j=1}^n b_j R^j \int_0^{2\pi} \frac{\sin(\psi)\cos^j(\psi)}{(1-R\sin(\psi))^j}\, d\psi. \end{array}$$

Denote

$$J_{\alpha,\beta,\gamma}(R) = \int_0^{2\pi} \frac{\sin^\alpha(\psi)\cos^\beta(\psi)}{(1-R\sin(\psi))^\gamma}\,d\psi. \tag{4.4}$$

The next lemmas make explicit expressions for $J_{1,j,j}(R)$, $J_{0,j+2,j}(R)$ and $J_{0,j,j}(R)$. We just present the proofs for two of them, the others are similar.

Lemma 4.1. *$J_{0,2l+1,2l+1}(R) \equiv 0$, for all l.*

Proof. Note that

$$\begin{aligned}
J_{0,2l+1,2l+1}(R) &= \int_0^{2\pi} \frac{\cos^{2l+1}(\psi)}{(1-R\sin(\psi))^{2l+1}}\,d\psi \\
&= \int_0^{2\pi} \frac{\cos^{2l}(\psi)\cos(\psi)}{(1-R\sin(\psi))^{2l+1}}\,d\psi \\
&= \int_0^{1} \frac{(1-u^2)^l\,du}{(1-Ru))^{2l+1}} - \int_{-1}^{1} \frac{(1-u^2)^l\,du}{(1-Ru))^{2l+1}} + \int_{-1}^{0} \frac{(1-u^2)^l\,du}{(1-Ru))^{2l+1}} \\
&= 0
\end{aligned}$$

□

Lemma 4.2. *$J_{0,2l,2l}(R) = \lambda \dfrac{u_{l-1}(Z)}{Z^{2l-1}(1+Z)^l}$, for all l, where $Z = \sqrt{1-R^2}$, λ is some constant and u_j is a j-degree polynomial.*

Proof. If we write $z = \tan(\psi/2)$ then $\cos(\psi) = \dfrac{1-z^2}{1+z^2}$, $\sin(\psi) = \dfrac{2z}{1+z^2}$, $d\psi = \dfrac{2\,dz}{1+z^2}$; then we have to solve

$$\int_{-\infty}^{\infty} \frac{2(1-z^2)^{2l}}{(1+z^2)(z^2-2Rz+1)^{2l}}\,dz.$$

We proceed using an well-know application of the Residue Theorem [1].

Lemma 4.3 ([1], section 5.3)**.** *If G is a rational function, an integral of the form $\int_{-\infty}^{\infty} G(x)\,dx$ converges if and only if the degree of the denominator of G is at least two units higher than the degree of the numerator, and if no poles lies on the real axis. In this case,*

$$\int_{-\infty}^{\infty} G(x)\,dx = 2\pi i \sum_j \mathrm{Res}_{w_j} G, \tag{4.5}$$

where $\mathrm{Res}_{w_j} G$ is the residue of G on the pole w_j, and the summation is done over all poles in the upper half plane.

Remark 4.4. Obviously, the complex product in (4.5) is a real number.

Let

$$G(w) = \frac{2(1-w^2)^{2l}}{(1+w^2)(w^2-2Rw+1)^{2l}}$$

where w is a complex variable.

The poles on the upper half plane are $w_1 = i$ (simple pole) and $w_2 = R + i\sqrt{1-R^2}$ (pole of order $2l$). We have to compute the residues of G over these poles.

Lemma 4.5 ([1]). *The point z_0 is a pole of order $m \geq 1$ for G if and only if $G(w) = \dfrac{\phi(w)}{(w-w_0)^m}$ for some analytic function ϕ. In this case, the residue of G over w_0 is given by*

$$Res_{w_0} G = \frac{\phi^{(m-1)}(w_0)}{(m-1)!},$$

where $\phi^{(m-1)}$ is the $(m-1)$-derivative of ϕ.

The residue over w_1 is easy to compute: we write

$$G_1(w) = (w-i)G(w) = \frac{2(1-w^2)^{2l}}{(w+i)(w^2-2Rw+1)^{2l}}$$

and then $Res_{w_1} G = G_1(w_1) = \dfrac{i(-1)^{l+1}}{R^{2l}}$.

Now let $\rho_1 = R + i\sqrt{1-R^2}$ and $\rho_2 = R - i\sqrt{1-R^2}$. Then

$$G(w) = \frac{1}{(w-\rho_1)^{2l}} \frac{2(1-w^2)^{2l}}{(1+w^2)(w-\rho_2)^{2l}}.$$

If we define $\phi(w) = \dfrac{2(1-w^2)^{2l}}{(1+w^2)(w-\rho_2)^{2l}}$, then to obtain the residue of G over w_1 we need to compute $\phi^{(2l-1)}(\rho_1)$.

For $l = 1$, $\phi(w) = \dfrac{2(1-w^2)^2}{(1+w^2)(w-\rho_2)^2}$ and

$$\phi^{(1)}(\rho_1) = \frac{i\left(-i + iR^2 - R\sqrt{1-R^2}\right)^2}{\left(iR - \sqrt{1-R^2}\right)^2 R^2 \left(1-R^2\right)^{3/2}}$$

Using the change $Z = \sqrt{1-R^2}$ in both residues we obtain

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{2(1-z^2)^2}{(1+z^2)(z^2-2Rz+1)^2}\,dz &= 2\pi i \left(\frac{i}{1-Z^2} + \frac{-i\left(iZ + \sqrt{1-Z^2}\right)^2}{Z\left(i\sqrt{1-Z^2} - Z\right)^2(-1+Z^2)} \right) \\
&= \frac{2\pi\left(-2\,Z^2 + 2\,i\sqrt{1-Z^2}Z + 1\right)}{\left(i\sqrt{1-Z^2} - Z\right)^2 (Z+1)\,Z} \\
&= \frac{2\pi}{Z(Z+1)}
\end{aligned}$$

where $Z = \sqrt{1-R^2}$. So the case $l = 1$ is done. The case $l > 1$ is similar. The specific degree in the statement is obtained when we simplify the sum of residues. □

Lemma 4.6. *$J_{0,2l+3,2l+1}(R) \equiv 0$, for all l.*

Lemma 4.7. *$J_{0,2,0}(R) = \pi$, $J_{0,4,2}(R) = \dfrac{3\pi}{(1+Z)^2}$, $J_{0,2l+2,2l}(R) = \dfrac{2l+1}{2^{l-1}} \dfrac{v_{l-2}(Z)}{(Z+1)^{l+1}Z^{2l-3}}$ for all $l \geq 2$, where $Z = \sqrt{1-R^2}$ and v_j is a j-degree polynomial.*

Lemma 4.8. *$J_{1,2l+1,2l+1}(R) \equiv 0$, for all l.*

Lemma 4.9. $J_{1,2,2}(R) = \dfrac{2\pi(1-Z)}{RZ(1+Z)}$, $J_{1,2l,2l}(R) = \dfrac{\lambda_l \pi w_l(Z)}{R(1+Z)^l Z^{2l-1}}$, *for all* $l \geq 2$, *where* $Z = \sqrt{1-R^2}$ *and* w_j *is a* j*-degree polynomial.*

Now we apply Lemmas 4.1-4.9 to simplify (4.3).

Lemma 4.10. *The coefficients* a_j, b_j *of* f_n, g_n *with* j *odd don't contribute to* (4.3).

Proof. Just note that these coefficients are associated to the integrals $J_{0,2l+1,2l+1}$ or $J_{0,2l+3,2l+1}$ or $J_{1,2l+1,2l+1}$, and according to Lemmas (4.1), (4.6), (4.8), these integrals vanishes. □

By Lemma (4.10) we may consider f_n, g_n even degree polynomials; moreover, we may take these polynomials even functions, that is, without odd degree terms. So from now on we consider just this case.

Lemma 4.11. *Consider* $n = 2$ *in* (4.3). *Then*

$$F(Z) = \pi \frac{(Z-1)^2((2a_2 - 2b_2)Z - a_2 - b_2)}{\sqrt{1-Z^2}},$$

where $Z = \sqrt{1-R^2}$. *The equation* $F(Z) = 0$ *has exactly one solution for* $Z \in (0,1)$ *when* $a_2 \neq b_2$ *and* $0 < \dfrac{a_2 + b_2}{2a_2 - 2b_2} < 1$; *otherwise there is no solution.*

Proof. Straightforward calculations. □

Lemma 4.12. *Consider* $n = 2m$ $(m > 1)$ *in* (4.3). *Then the numerator of* $R \cdot F(Z)$ *is given by*

$$\begin{aligned} \text{numer}(R \cdot F(Z)) &= \sum_{j=0}^{m} \alpha_{2j}(a,b) Z^{2j} + \beta_{2m-3}(a,b) Z^{2m-3} + \beta_{2m-1}(a,b) Z^{2m-1} \\ &= (Z-1)^2 \sum_{j=0}^{2m-2} \mu_j(a,b) Z^j, \end{aligned}$$

where $Z = \sqrt{1-R^2}$ *and* $\alpha_l(a,b)$, $\beta_s(a,b)$, $\mu_j(a,b)$, *depend on* a_j, b_j *(recall that* j *is always even), while the denominator is a function without zeroes in the interval* $(0,1)$. *In particular, the equation* $F(Z) = 0$ *have at most* $2m-2$ *solutions for* $Z \in (0,1)$.

Example 4.13. Consider the system

$$\left\{ \begin{array}{rcl} \dot{x} &=& -y + x^2, \\ \dot{y} &=& x + xy + \varepsilon\left(y(a_2x^2 + a_4x^4) + (b_2x^2 + b_4x^4)\right), \end{array} \right. \tag{4.6}$$

where $\varepsilon > 0$ is a small parameter and a_2, a_4, b_2, b_4 are real constants. Then

$$F(Z) = \pi(Z-1)^2 \frac{(-3a_4 + 2a_2 + 3b_4 - 2b_2)Z^2 + (5a_4 - a_2 - 3b_3 - b_2)Z - 2a_4}{Z\sqrt{1-Z^2}}$$

The equation $F(Z) = 0$ has at most two solutions for $Z \in (0,1)$. In particular, if
$\Delta = {a_4}^2 - 6\,a_4b_4 + 6\,a_4a_2 - 26\,a_4b_2 + 9\,{b_4}^2 + 6\,b_4a_2 + 6\,b_4b_2 + {a_2}^2 + 2\,a_2b_2 + {b_2}^2 > 0$,
$\Gamma = -5\,a_4 + 3\,b_4 + a_2 + b_2$ and
$\zeta = 2(3\,b_4 - 2\,b_2 + 2\,a_2 - 3\,a_4) \neq 0$,

the solutions are $Z_{\pm} = \dfrac{\Gamma \pm \sqrt{\Delta}}{\zeta}$. For instance, choosing $a_2 = -\frac{2}{3}$, $a_4 = -\frac{1}{12}$, $b_2 = 1$ and $b_4 = \frac{1}{36}$ we obtain two solutions.

The estimative in the last lemma that the equation $F(Z) = 0$ have at most $2m - 2$ solutions is just based on the degree of $\dfrac{R \cdot F(Z)}{\pi(Z-1)^2}$ and is not sharp.

Example 4.14. Consider the system

$$
\left\{ \begin{array}{rcl} \dot{x} & = & -y + x^2, \\ \dot{y} & = & x + xy + \varepsilon\left(y(a_2x^2 + a_4x^4 + a_6x^6) + (b_2x^2 + b_4x^4 + b_6x^6)\right), \end{array} \right. \tag{4.7}
$$

where $\varepsilon > 0$ is a small parameter and $a_2, a_4, a_6, b_2, b_4, b_6$ are real constants. Then

$$
\begin{array}{rcl} G_6(Z) = \dfrac{R \cdot F(Z)}{\pi(Z-1)^2} & = & (12b_4 - 15b_6 + 15a_6 - 12a_4 - 8b_2 + 8a_2)\, Z^4 \\ & + & (-12b_4 + 30b_6 - 4b_2 + 20a_4 - 4a_2 - 38a_6)\, Z^3 \\ & + & (-15b_6 - 8a_4 + 29a_6)\, Z^2 - 4a_6 Z - 2a_6 \end{array}
$$

The degree of polynomial G_6 is 4, but we cannot choose coefficients such that this polynomial has 4 roots in the interval $(0, 1)$. In this case we have just 3 roots.

Remark 4.15 (Conjecture). Consider system (4.1) with f_n, g_n polynomials of degree n. We conjecture that the maximum number of limit cycles (solutions of $F(Z) = 0$ for $Z \in (0, 1)$) is $n - 3$ when n is odd and $n - 4$ when n is even.

Remark 4.16. We note that the difficult in proving general quotas is common in papers dealing with perturbation of non-linear centers. Note that in the proof of Theorem 1.1 (in special in the proof of Lemma 4.2) the degree of u_{l-1} was easy obtained, but its exact dependence on the coefficients is hard to determine. Similar problems are found in [3].

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