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GEOMETRIC ANALYSIS OF QUADRATIC DIFFERENTIAL SYSTEMS WITH INVARIANT ELLIPSES

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Abstract

In this article we study the whole class **QSE** of non-degenerate planar quadratic differential systems possessing at least one invariant ellipse. We classify this family of systems according to their geometric properties encoded in the configurations of invariant ellipses and invariant straight lines which these systems could possess. The classification, which is taken modulo the action of the group of real affine transformations and time rescaling, is given in terms of algebraic geometric invariants and also in terms of invariant polynomials and it yields a total of 35 distinct such configurations. This classification is also an algorithm which makes it possible to verify for any given real quadratic differential system if it has invariant ellipses or not and to specify its configuration of invariant ellipses and straight lines.

Key-words: quadratic differential systems, configuration, invariant ellipses and lines, affine invariant polynomials, group action

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1 Introduction and statement of the main results

We consider planar polynomial differential systems which are systems of the form

$$dx/dt = p(x, y), \quad dy/dt = q(x, y) \tag{1}$$

where $p(x,y), q(x,y)$ are polynomials in x, y with real coefficients ($p, q \in \mathbb{R}[x,y]$) and their associated vector fields

$$X = p(x,y)\frac{\partial}{\partial x} + q(x,y)\frac{\partial}{\partial y}. \tag{2}$$

We call *degree* of such a system the number $\max\{\deg(p), \deg(q)\}$. In the case where the polynomials p and q are coprime we say that (1) is *non-degenerate*.

A real quadratic differential system is a polynomial differential system of degree 2, i.e.

$$\begin{aligned} dx/dt &= p_0 + p_1(\tilde{a},x,y) + p_2(\tilde{a},x,y) \equiv p(\tilde{a},x,y), \\ dy/dt &= q_0 + q_1(\tilde{a},x,y) + q_2(\tilde{a},x,y) \equiv q(\tilde{a},x,y), \end{aligned} \tag{3}$$

with $\max\{\deg(p), \deg(q)\} = 2$ and

$$\begin{aligned} p_0 &= a, \quad p_1(\tilde{a},x,y) = cx + dy, \quad p_2(\tilde{a},x,y) = gx^2 + 2hxy + ky^2, \\ q_0 &= b, \quad q_1(\tilde{a},x,y) = ex + fy, \quad q_2(\tilde{a},x,y) = lx^2 + 2mxy + ny^2. \end{aligned}$$

Here we denote by $\tilde{a} = (a,b,c,d,e,f,g,h,k,l,m,n)$ the 12-tuple of the coefficients of system (3). Thus a quadratic system can be identified with a point $\tilde{a}$ in $\mathbb{R}^{12}$.

We denote the class of all quadratic differential systems with **QS**.

Planar polynomial differential systems occur very often in various branches of applied mathematics, in modeling natural phenomena, for example, modeling the time evolution of interacting species in biology and in chemical reactions and economics [15, 31], in astrophysics [7], in the equations of continuity describing the interactions of ions, electrons and neutral species in plasma physics [21]. Polynomial systems appear also in shock waves, in neural networks, etc. Such differential systems have also theoretical importance. Several problems on polynomial differential systems, which were stated more than one hundred years ago, are still open: the second part of Hilbert's 16th problem stated by Hilbert in 1900 [11], the problem of algebraic integrability stated by Poincaré in 1891 [19, 20], the problem of the center stated by Poincaré in 1885 [18], and problems on integrability resulting from the work of Darboux [9] published in 1878. With the exception of the problem of the center, which was solved only for quadratic differential systems, all the other problems mentioned above, are still unsolved even in the quadratic case.

Definition 1 (Darboux)**.** *An algebraic curve $f(x,y) = 0$, where $f \in \mathbb{C}[x,y]$, is an invariant curve of the planar polynomial system* (1) *if and only if there exists a polynomial $k(x,y) \in \mathbb{C}[x,y]$ such that*

$$p(x,y)\frac{\partial f}{\partial x} + q(x,y)\frac{\partial f}{\partial y} = k(x,y)f(x,y).$$

Definition 2 (Darboux)**.** *We call algebraic solution of a planar polynomial system an invariant algebraic curve over $\mathbb{C}$ which is irreducible.*

One of our motivations in this article comes from integrability problems related to the work of Darboux [9].

Theorem 1 (Darboux)**.** *Suppose that a polynomial system* (1) *has m invariant algebraic curves $f_i(x,y) = 0$, $i \le m$, with $f_i \in \mathbb{C}[x,y]$ and with $m > n(n+1)/2$, where n is the degree of the system. Then there exist complex numbers $\lambda_1, \ldots, \lambda_m$ such that $f_1^{\lambda_1} \ldots f_m^{\lambda_m}$ is a first integral of the system.*

The condition in Darboux's theorem is only sufficient for Darboux integrability (integrability in terms of invariant algebraic curves) and it is not also necessary. Thus the lower bound on the number of invariant curves sufficient for Darboux integrability stated in the theorem of Darboux is larger than necessary. Darboux's theory has been improved by including for example the multiplicities of the curves ([14]). Also, the number of invariant algebraic curves needed was reduced but by adding some conditions, in particular the condition that any two of the curves be transversal. But a deeper understanding about Darboux integrability is still lacking. Algebraic integrability, which intervenes in the open problem stated by Poincaré in 1891 ([19] and [20]), and which means the existence of a rational first integral for the system, is a particular case of Darboux integrability.

Theorem 2 (Jouanolou [12])**.** *Suppose that a polynomial system* (1)*, defined by polynomials* $p(x, y)$*,* $q(x, y) \in \mathbb{C}[x, y]$*, has* m *invariant algebraic curves* $f_i(x, y) = 0$*,* $i \le m$*, with* $f_i \in \mathbb{C}[x, y]$ *and with* $m \ge n(n+1)/2+2$*, where* n *is the degree of the system. Then the system has a rational first integral* $h(x, y)/g(x, y)$ *where* $h(x, y), g(x, y) \in \mathbb{C}[x, y]$*.*

To advance knowledge on algebraic or more generally Darboux integrability it is necessary to have a large number of examples to analyze. In the literature, scattered isolated examples were analyzed but a more systematic approach was still needed. Schlomiuk and Vulpe initiated a systematic program to construct such a data base for quadratic differential systems. Since the simplest case is of systems with invariant straight lines, their first works involved only invariant lines for quadratic systems (see [23, 25, 26, 28, 29]). In this work we study a class of quadratic systems with invariant conics, namely the class **QSE** of non-degenerate (i.e. p and q are relatively prime) quadratic differential systems having an invariant ellipse. Such systems could also have some invariant lines and in many cases the presence of these invariant curves turns them into Darboux integrable systems. We always assume here that systems (3) are non-degenerate because otherwise doing a time rescaling, they can be reduced to linear or constant systems. Under this assumption all the systems in **QSE** have a finite number of finite singularities.

The irreducible affine conics over the field $\mathbb{R}$ are the hyperbolas, ellipses and parabolas. One way to distinguish them is to consider their points at infinity (see [1]). The term hyperbola is used for a real irreducible affine conic which has two real points at infinity. This distinguishes it from the other two irreducible real conics: the parabola has just one real point at infinity at which the multiplicity of intersection of the conic with the line at infinity is two, and the ellipse which has two complex points at infinity.

In the theory of Darboux the invariant algebraic curves are considered (and rightly so) over the complex field $\mathbb{C}$. We may extend the notion of hyperbola (parabola or ellipse) for conics over $\mathbb{C}$. A hyperbola (respectively parabola or ellipse) is an algebraic curve C in $\mathbb{C}^2$, $C : f(x, y) = 0$ with $f \in \mathbb{C}[x, y]$, $\deg(f) = 2$ which is irreducible and which has two real points at infinity (respectively one real point at infinity with intersection multiplicity two, or two complex (non-real) points at infinity).

Remark 1. *We draw attention to the fact that if we have a curve* $C : f(x, y) = 0$ *over* $\mathbb{C}$ *it could happen that multiplying the equation by a number* $\lambda \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$*, the coefficients of the new equation become real. In this case, to the equation* $f(x, y) = 0$ *we can associate two curves: one real* $\{(x, y) \in \mathbb{R}^2 | \lambda f(x, y) = 0\}$ *and one complex* $\{(x, y) \in \mathbb{C}^2 | f(x, y) = 0\}$*. In particular, if*

$f(x, y) \in \mathbb{R}[x, y]$ then we could talk about two curves, one in the real and the other in the complex plane. If the coefficients of an algebraic curve $C : f(x, y) = 0$ cannot be made real by multiplication with a constant, then clearly to the equation $f(x, y) = 0$ we can associate just one curve, namely the complex curve $\{(x, y) \in \mathbb{C}^2 | f(x, y) = 0\}$.

In this paper we consider real polynomial differential equations. To each such system of equations there corresponds the complex system with the same coefficients to which we can apply the theory of Darboux using complex invariant algebraic curves. Some of these curves may turn out to be with real coefficients in which case they also yield, as in the remark above, invariant algebraic curves in $\mathbb{R}^2$ of the real differential system. It is one way, but not the only way, in which the theory of Darboux yields applications to real systems. It is by juggling both with complex and real systems and their invariant complex or real algebraic curves that we get a full understanding of the classification problem we consider here.

Let us suppose that a polynomial differential system has an algebraic solution $f(x, y) = 0$ where $f(x, y) \in \mathbb{C}[x, y]$ is of degree n, $f(x, y) = a_{00} + a_{10}x + a_{01}y + \cdots + a_{n0}x^n + a_{n-1,1}x^{n-1}y + \cdots + a_{0n}y^n$ with $\hat{a} = (a_{00}, \ldots, a_{0n}) \in \mathbb{C}^N$ where $N = (n + 1)(n + 2)/2$. We note that the equation $\lambda f(x, y) = 0$ where $\lambda \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$ yields the same locus of complex points in the plane as the locus induced by $f(x, y) = 0$. So, a curve of degree n defined by $\hat{a}$ can be identified with a point $[\hat{a}] = [a_{00} : a_{10} : \cdots : a_{0n}]$ in $P_{N-1}(\mathbb{C})$. We say that a sequence of curves $f_i(x, y) = 0$ of degree n converges to a curve $f(x, y) = 0$ if and only if the sequence of points $[a_i] = [a_{i00} : a_{i10} : \cdots : a_{i0n}]$ converges to $[\hat{a}] = [a_{00} : a_{10} : \cdots : a_{0n}]$ in the topology of $P_{N-1}(\mathbb{C})$.

On the class **QS** acts the group of real affine transformations and time rescaling and because of this, modulo this group action quadratic systems ultimately depend on five parameters. In particular, restricting this group action on **QSE**, modulo this action the **QSE** is a union of 1-dimensional, 2-dimensional and 3-dimensional families of systems as it can be seen from the normal forms obtained in [16] for this family.

We observe that if we rescale the time $t' = \lambda t$ by a positive constant λ the geometry of the systems (1) does not change. So, for our purposes we can identify a system (1) of degree n with a point $[a_{00}, a_{10}, \ldots, a_{0n}, b_{00}, \ldots, b_{0n}]$ in $\mathbb{S}^{N-1}(\mathbb{R})$, with $N = (n + 1)(n + 2)$.

We compactify the space of all the polynomial differential systems of degree n on $\mathbb{S}^{N-1}$ with $N = (n + 1)(n + 2)$ by multiplying the coefficients of each systems with $1/(\sum(a_{ij}^2 + b_{ij}^2))^{1/2}$.

Definition 3. *(1) We say that an invariant curve $\mathcal{L} : f(x, y) = 0$, $f \in \mathbb{C}[x, y]$, for a polynomial system (S) of degree n has* multiplicity m *if there exists a sequence of real polynomial systems (S_k) of degree n converging to (S) in the topology of $\mathbb{S}^{N-1}$, $N = (n+1)(n+2)$, such that each (S_k) has m distinct invariant curves $\mathcal{L}_{1,k} : f_{1,k}(x, y) = 0, \ldots, \mathcal{L}_{m,k} : f_{m,k}(x, y) = 0$ over $\mathbb{C}$, $deg(f) = deg(f_{i,k}) = r$, converging to $\mathcal{L}$ as $k \to \infty$, in the topology of $\mathbb{P}_{R-1}(\mathbb{C})$, with $R = (r + 1)(r + 2)/2$ and this does not occur for $m + 1$.*

(2) We say that the line at infinity $\mathcal{L}_\infty : Z = 0$ of a polynomial system (S) of degree n has multiplicity m *if there exists a sequence of real polynomial systems (S_k) of degree n converging to (S) in the topology of $\mathbb{S}^{N-1}$, $N = (n + 1)(n + 2)$, such that each (S_k) has $m - 1$ distinct invariant lines $\mathcal{L}_{1,k} : f_{1,k}(x, y) = 0, \ldots, \mathcal{L}_{m-1,k} : f_{m-1,k}(x, y) = 0$ over $\mathbb{C}$, converging to the line at infinity $\mathcal{L}_\infty$ as $k \to \infty$, in the topology of $\mathbb{P}_2(\mathbb{C})$ and this does not occur for m.*

Remark 2. *(a) In order to describe the various kinds of multiplicities for infinite singularities we use the concepts and notations introduced in [23]. Thus we denote by "(a, b)" the maximum number a (respectively b) of infinite (respectively finite) singularities which can be obtained by perturbation of a multiple infinite singularity.*

(b) In DIAGRAM *3 we draw the multiple curves with bold lines and we place a number without parentheses next to the curve which corresponds to its multiplicity (see for example Config. E.24). However, there exist two configurations in which we draw the invariant ellipse with thicker line (without a number next to it) in order to indicate that it is a limit cycle (see Config. E.5 and Config. E.9).*

An important ingredient in this work is the notion of *configuration of algebraic solutions* of a polynomial differential system. This notion appeared for the first time in [23].

Definition 4. *Consider a planar polynomial system which has a finite number of algebraic solutions and a finite number of singularities, finite or infinite. By* configuration of algebraic solutions *of this system we mean the set of algebraic solutions over $\mathbb{C}$ of the system, each one of these curves endowed with its own multiplicity and together with all the real singularities of this system located on these curves, each one of these singularities endowed with its own multiplicity.*

We may have two distinct systems which may be non-equivalent modulo the action of the group but which may have "the same configuration" of invariant ellipses and straight lines. We need to say when two configurations are "the same" or equivalent.

Definition 5. *Suppose we have two systems (S_1) and (S_2) in **QSE** with a finite number of singularities, finite or infinite, a finite set of invariant ellipses $\mathcal{E}_i : e_i(x, y) = 0$, $i = 1, \ldots, k$, of (S_1) (respectively $\mathcal{E}'_i : e'_i(x, y) = 0$, $i = 1, \ldots, k$, of (S_2)) and a finite set (which could also be empty) of invariant straight lines $\mathcal{L}_j : f_j(x, y) = 0$, $j = 1, 2, \ldots, k'$, of (S_1) (respectively $\mathcal{L}'_j : f'_j(x, y) = 0$, $j = 1, 2, \ldots, k'$, of (S_2)). We say that the two configurations C_1 and C_2 of ellipses and lines of these systems are equivalent if there is a one-to-one correspondence ϕ_e between the ellipses of C_1 and C_2 and a one-to-one correspondence ϕ_l between the lines of C_1 and C_2 such that:*

(i) the correspondences conserve the multiplicities of the ellipses and lines and also send a real invariant curve to a real invariant curve and a complex invariant curve to a complex invariant curve;

(ii) for each ellipse $\mathcal{E} : e(x, y) = 0$ of C_1 (respectively each line $\mathcal{L} : f(x, y) = 0$) we have a one-to-one correspondence between the real singularities on $\mathcal{E}$ (respectively on $\mathcal{L}$) and the real singularities on $\phi_e(\mathcal{E})$ (respectively $\phi_l(\mathcal{L})$) conserving their multiplicities and their location;

(iii) furthermore, consider the total curves $\mathcal{F} : \prod E_i(X, Y, X) \prod F_j(X, Y, Z)Z = 0$ (respectively $\mathcal{F}' : \prod E'_i(X, Y, X) \prod F'_j(X, Y, Z)Z = 0$ where $E_i(X, Y, X) = 0$ and $F_j(X, Y, X) = 0$ (respectively $E'_i(X, Y, X) = 0$ and $F'_j(X, Y, X) = 0$) are the projective completions of $\mathcal{E}_i$ and $\mathcal{L}_j$ (respectively $\mathcal{E}'_i$ and $\mathcal{L}'_j$). Then, there is a correspondence ψ between the singularities of the curves $\mathcal{F}$ and $\mathcal{F}'$ conserving their multiplicities as singularities of the total curves.

In the family **QSE** we also have cases where we have an infinite number of ellipses. Thus, according to the theorem of Jouanolou (Theorem 2), we have a rational first integral. In this case the multiplicity of an ellipse in the family is either considered to be undefined or we may say that this multiplicity is infinite. Such situations occur either when we have (*i*) a finite number of singularities, finite or infinite, or (*ii*) an infinite number of singularities which could only be at infinity (recall that the systems in **QSE** are non-degenerate). In both cases however we show that we have a finite number of invariant affine straight lines with finite multiplicities. In fact it was proved in [27] that all quadratic systems which have the line at infinity filled up with singularities have invariant affine straight lines of total multiplicity three. Furthermore, the multiplicities of singularities of the systems are finite in the case (*i*) and this is also true in the case (*ii*) if we only take into consideration the affine lines. We therefore can talk about the *configuration of invariant affine lines associated to the system*. Two such configurations of invariant affine lines C_{1L} and C_{2L} associated to systems (S_1) and (S_2) are said to be equivalent if and only if there is a one-to-one correspondence ϕ_l between the lines of C_{1L} and C_{2L} such that:

(i) the correspondence conserves the multiplicities of lines and also sends a real invariant line to a real invariant line and a complex invariant line to a complex invariant line;

(ii) for each line $\mathcal{L} : f(x, y) = 0$ we have a one-to-one correspondence between the real singularities on $\mathcal{L}$ and the real singularities on $\phi_l(\mathcal{L})$ conserving their multiplicities and their order on the lines.

We use this to extend our previous definition further above to cover these cases.

Definition 6. *Suppose we have two systems (S_1) and (S_2) in **QSE**, each one with a finite number of finite singularities and an infinite number of invariant ellipses. Suppose we have a non-empty finite set of invariant affine straight lines $\mathcal{L}_j : f_j(x, y) = 0$, $j = 1, 2, \ldots, k$, of (S_1) (respectively $\mathcal{L}'_j : f'_j(x, y) = 0$, $j = 1, 2, \ldots, k$, of (S_2)). We now consider only the two configurations C_{1L} and C_{2L} of invariant affine lines of (S_1) and (S_2) associated to the systems, respectively. We say that the two configurations C_{1L} and C_{2L} are* equivalent with respect to the ellipses of the systems *if and only if (i) they are equivalent as configurations of invariant lines and in addition the following property (ii) is satisfied: we take any ellipse $\mathcal{E} : e(x, y) = 0$ of (S_1) and any ellipse $\mathcal{E}' : e'(x, y) = 0$ of (S_2). Then, we must have a one-to-one correspondence between the real singularities of the system (S_1) located on $\mathcal{E}$ and of real singularities of the system (S_2) located on $\mathcal{E}'$, conserving their multiplicities and their location. Furthermore, consider the curves $\mathcal{F} : \prod e(x, y) \prod f_j(x, y) = 0$ and $\mathcal{F}' : \prod e'(x, y) \prod f'_j(x, y) = 0$. Then we have a one-to-one correspondence between the singularities of the curve $\mathcal{F}$ with those of the curve $\mathcal{F}'$ conserving their multiplicities as singularities of these curves.*

It can be easily shown that the definition above is independent of the choice of the two ellipses $\mathcal{E} : e(x, y) = 0$ of (S_1) and $\mathcal{E}' : e'(x, y) = 0$ of (S_2).

In this work we are interested in systems possessing an invariant ellipse. The conics $f(x, y) = 0$ with $f(x, y) \in \mathbb{R}[x, y]$ are classified via the group action of real affine transformation. The conics for which $f(x, y)$ is an irreducible polynomial over $\mathbb{C}$ can be brought by a real affine transformation to one of the following four forms: 1) $x^2 + y^2 - 1 = 0$ (ellipses); 2) $x^2 - y^2 - 1 = 0$ (hyperbolas); 3) $y - x^2 = 0$ (parabolas); 4) $x^2 + y^2 + 1 = 0$, these are empty in $\mathbb{R}^2$ with points only in $\mathbb{C}^2$. Some authors call these conics *complex ellipses* (see [6], for instance). These complex ellipses will play a helpful role in our classification problem.

Definition 7. *By an ellipse we will mean a conic $f(x,y)=0$ with real coefficients which can be brought by a real affine transformation to an equation $x^2+y^2+a=0$ with $a=-1$ (a real ellipse) or $a=1$ (a complex ellipse).*

Remark 3. *In the family **QSE** we can have cases of systems which possess simultaneously an infinite number of real ellipses as well as an infinite number of complex ellipses. For such systems we present the respective configurations containing only real ellipses (besides, of course, the corresponding invariant lines, if there are any).*

In [16] the authors provide necessary and sufficient affine invariant conditions for a non-degenerate quadratic differential system to have at least one invariant ellipse and these conditions are expressed in terms of the coefficients of the systems. In this paper we denote by $\mathbf{QSE}_{(\eta<0)}$ the family of non-degenerate quadratic systems in **QSE** possessing two complex singularities at infinity and by $\mathbf{QSE}_{(C_2=0)}$ the systems in **QSE** possessing the line at infinity filled up with singularities. We classify these families of systems, modulo the action of the group of real affine transformations and time rescaling, according to their geometric properties encoded in the configurations of invariant ellipses and/or invariant straight lines which these systems possess.

As we want this classification to be intrinsic, independent of the normal form given to the systems, we use here geometric invariants and invariant polynomials for the classification. For example, it is clear that the configuration of algebraic solutions of a system in **QSE** is an affine invariant. The classification is done according to the configurations of invariant ellipses and straight lines encountered in systems belonging to **QSE**. We put in the same equivalence class systems which have equivalent configurations of invariant ellipses and lines (in the sense of Definitions 5 and 6). In particular the notion of multiplicity in Definition 3 is invariant under the group action, i.e. if a quadratic system S has an invariant curve $\mathcal{L}=0$ of multiplicity m, then each system S' in the orbit of S under the group action has a corresponding invariant line $\mathcal{L}'=0$ of the same multiplicity m. To distinguish configurations of algebraic solutions we need some geometric invariants, and we also use invariant polynomials both of which are introduced in our Section 2.

Main Theorem. *Consider the class* **QSE** *of all non-degenerate quadratic differential systems* (3) *possessing an invariant ellipse.*

- **(*A*)** *This family is classified according to the configurations of invariant ellipses and of invariant straight lines of the systems, yielding 35 distinct such configurations, 30 of which belong to the class* $\mathbf{QSE}_{(\eta<0)}$ *and 5 to* $\mathbf{QSE}_{(C_2=0)}$. *This geometric classification is described in Theorems 5.*
- **(*B*)** *Using invariant polynomials, we obtain the bifurcation diagram in the space $\mathbb{R}^{12}$ of the coefficients of systems in* **QS** *according to their configurations of invariant ellipses and invariant straight lines (this diagram is presented in part (B) of Theorem 5). Moreover, this diagram gives an algorithm to compute the configuration of a system with an invariant ellipse for any quadratic differential system, presented in any normal form.*

This paper is organized as follows: In Section 2 we define all the geometric and algebraic invariants used in the paper and we introduce the basic auxiliary results we need for the proof of our theorems. In Section 3 we consider the class $\mathbf{QSE}_{(\eta<0)}$ (respectively $\mathbf{QSE}_{(C_2=0)}$) of all non-degenerate quadratic differential systems (3) possessing exactly one real singularity at infinity (respectively all non-degenerate quadratic differential systems (3) possessing an invariant ellipse and the line at infinity filled up with singularities) and we classify this family according to the geometric configurations of invariant ellipses and invariant straight lines which they possess. We also give their bifurcation diagram in the 12-dimensional space $\mathbb{R}^{12}$ of the coefficients of quadratic systems, in terms of invariant polynomials. In section 4 we give some concluding comments, stressing the fact that the bifurcation diagrams in $\mathbb{R}^{12}$ give us an algorithm to compute the configuration of a system with an invariant ellipse for any system presented in any normal form.

2 Basic concepts and auxiliary results

In this section we define all the geometric invariants we use in the Main Theorem and we state some auxiliary results. A quadratic system possessing an invariant ellipse could also possesses invariant lines. We classify the systems possessing an invariant ellipse in terms of their configurations of invariant ellipses and invariant lines. Each one of these invariant curves has a multiplicity in the sense of Definition 3 (see also [8]). We encode this picture in the multiplicity divisor of invariant ellipses and lines. We first recall the algebraic-geometric definition of an r-cycle on an irreducible algebraic variety of dimension n.

Definition 8. *Let V be an irreducible algebraic variety of dimension n over a field K. A cycle of dimension r or r-cycle on V is a formal sum $\Sigma_W n_W W$, where W is a subvariety of V of dimension r which is not contained in the singular locus of V, $n_W \in \mathbb{Z}$, and only a finite number of n_W's are non-zero. We call* degree of an r-cycle *the sum Σn_W. An $(n-1)$-cycle is called a* divisor. *We denote by $Max(C)$ the maximum value of the coefficients n_W in C. For every $m \leq Max(C)$ let $s(m)$ be the number of the coefficients n_W in C which are equal to m. We call* type of the cycle C *the set of ordered couples $(s(m), m)$, where $1 \leq m \leq Max(C)$.*

Now we define the geometrical invariants needed for distinguishing the configurations given by the Main Theorem.

Definition 9. *We denote the number of invariant ellipses by N_ε, which assumes the value 1 if the systems possess only one invariant ellipse or ∞ if they possess a family of invariant ellipses (real or complex ones).*

Definition 10. *1. Suppose that a real quadratic system has a finite number of invariant ellipses $\mathcal{E}_i : f_i(x,y) = 0$ and a finite number of invariant affine lines $\mathcal{L}_j$. We denote the line at infinity $\mathcal{L}_\infty : Z = 0$. Let us assume that on the line at infinity we have a finite number of singularities. The divisor of invariant ellipses and invariant lines on the complex projective plane of the system is the following:*

$$ICD = n_1\mathcal{E}_1 + \cdots + n_k\mathcal{E}_k + m_1\mathcal{L}_1 + \cdots + m_k\mathcal{L}_k + m_\infty\mathcal{L}_\infty,$$

where n_j (respectively m_i) is the multiplicity of the ellipse $\mathcal{E}_j$ (respectively of the line $\mathcal{L}_i$), and m_∞ is the multiplicity of $\mathcal{L}_\infty$. We also mark the complex (non-real) invariant ellipses (respectively lines) denoting them by $\mathcal{E}_i^C$ (respectively $\mathcal{L}_i^C$). We denote by ILD the invariant lines divisor, i.e.

$$ILD = m_\infty\mathcal{L}_\infty + m_1\mathcal{L}_1 + \cdots + m_k\mathcal{L}_k.$$

2. The zero-cycle on the real projective plane, of real singularities of a system (3) *located on the configuration of invariant lines and invariant ellipses, is given by:*

$$MS_{0C} = l_1 U_1 + \cdots + l_k U_k + m_1 s_1 + \cdots + m_n s_n,$$

where U_i (respectively s_j) are all the real infinite (respectively finite) such singularities of the system and l_i (respectively m_j) are their corresponding multiplicities.

The zero-cycle on the real affine plane, of real singularities of a quadratic system located on the configuration of invariant lines and invariant ellipses, is given by:

$$MS_{0C}^{Af} = m_1 s_1 + \cdots + m_n s_n,$$

where s_j are all the real finite such singularities of the system and m_j are their corresponding multiplicities.

In case we have a real finite singularity located on invariant curves we denote it by $\overset{j}{s}_r$, where $j \in \{e, l, el, ll, \ldots\}$. Here e (respectively $l, el, ll, \ldots$) means that the singular point s_r is located on an ellipse (respectively located on a line, on the intersection of an ellipse and a line, on the intersection of two lines, etc.).

A few more definitions and results which play an important role in the proof of the part (B) of the Main Theorem are needed. We do not prove these results here but we indicate where they can be found.

Consider the differential operator $\mathcal{L} = x \cdot L_2 - y \cdot L_1$ constructed in [4] and acting on the ring $\mathbb{R}[\tilde{a}, x, y]$, where

$$\begin{aligned} L_1 =& 2a_{00}\frac{\partial}{\partial a_{10}} + a_{10}\frac{\partial}{\partial a_{20}} + \frac{1}{2}a_{01}\frac{\partial}{\partial a_{11}} + 2b_{00}\frac{\partial}{\partial b_{10}} + b_{10}\frac{\partial}{\partial b_{20}} + \frac{1}{2}b_{01}\frac{\partial}{\partial b_{11}},\\ L_2 =& 2a_{00}\frac{\partial}{\partial a_{01}} + a_{01}\frac{\partial}{\partial a_{02}} + \frac{1}{2}a_{10}\frac{\partial}{\partial a_{11}} + 2b_{00}\frac{\partial}{\partial b_{01}} + b_{01}\frac{\partial}{\partial b_{02}} + \frac{1}{2}b_{10}\frac{\partial}{\partial b_{11}}. \end{aligned}$$

Using this operator and the affine invariant $\mu_0 = \mathrm{Res}_x\big(p_2(\tilde{a}, x, y), q_2(\tilde{a}, x, y)\big)/y^4$ we construct the following polynomials

$$\mu_i(\tilde{a}, x, y) = \frac{1}{i!}\mathcal{L}^{(i)}(\mu_0),\ \ i = 1, \ldots, 4,$$

where $\mathcal{L}^{(i)}(\mu_0) = \mathcal{L}(\mathcal{L}^{(i-1)}(\mu_0))$ and $\mathcal{L}^{(0)}(\mu_0) = \mu_0$.

These polynomials are in fact comitants of systems (3), invariant with respect to the group $GL(2, \mathbb{R})$ (see [4]). Their geometrical meaning is revealed in the next lemma.

Lemma 1 ([3, 4])**.** *Assume that a quadratic system* (S) *with coefficients $\tilde{a}$ belongs to the family* (3)*. Then:*

(i) Let λ be an integer such that $\lambda \le 4$. The total multiplicity of all finite singularities of this system equals $4 - \lambda$ if and only if for every $i \in \{0, 1, \ldots, \lambda - 1\}$ we have $\mu_i(\tilde{a}, x, y) = 0$ in the ring $\mathbb{R}[x, y]$ and $\mu_\lambda(\tilde{a}, x, y) \neq 0$. In this case, the factorization $\mu_\lambda(\tilde{a}, x, y) = \prod_{i=1}^{\lambda}(u_i x - v_i y) \neq 0$ over $\mathbb{C}$ indicates the coordinates $[v_i : u_i : 0]$ of points at infinity which coalesced with finite singularities of the system (S)*. Moreover, the number of distinct factors in this factorization is less than or equal to three (the maximum number of infinite singularities of a quadratic system in the projective plane)*

and the multiplicity of each one of the factors $u_i x - v_i y$ gives us the number of the finite singularities of the system (S) *which have coalesced with the infinite singularity* $[v_i : u_i : 0]$.

(ii) System (S) *is degenerate (i.e.* $\gcd(P,Q) \neq \text{const}$*) if and only if* $\mu_i(\tilde{a}, x, y) = 0$ *in* $\mathbb{R}[x,y]$ *for every* $i = 0, 1, 2, 3, 4$.

The following zero-cycle on the complex plane was introduced in [13] based on previous work in [22].

Definition 11. *For a polynomial system* (S) *we define* $\mathcal{D}_{\mathbb{C}^2}(S) = \sum_{s\in\mathbb{C}^2} n_s s$ *where* n_s *is the intersection multiplicity at* s *of the curves* $p(x,y) = 0$ *and* $q(x,y) = 0$, *with* p *and* q *being the polynomials defining the equations* (1).

According to [32] (see also [2]) we have the following proposition.

Proposition 1. *The form of the zero-cycle* $\mathcal{D}_{\mathbb{C}^2}(S)$ *for non-degenerate quadratic systems* (3) *is determined by the corresponding conditions indicated in Table 1, where we write* $p + q + r^c + s^c$ *if two of the finite points, i.e.* r^c, s^c, *are complex but not real, and*

$$
\begin{aligned}
\mathbf{D} &= \Big[3\big((\mu_3,\mu_3)^{(2)},\ \mu_2\big)^{(2)} - \big(6\mu_0\mu_4 - 3\mu_1\mu_3 + \mu_2^2,\ \mu_4\big)^{(4)}\Big]/48,\\
\mathbf{P} &= 12\mu_0\mu_4 - 3\mu_1\mu_3 + \mu_2^2,\\
\mathbf{R} &= 3\mu_1^2 - 8\mu_0\mu_2,\\
\mathbf{S} &= \mathbf{R}^2 - 16\mu_0^2\mathbf{P},\\
\mathbf{T} &= 18\mu_0^2(3\mu_3^2 - 8\mu_2\mu_4) + 2\mu_0(2\mu_2^3 - 9\mu_1\mu_2\mu_3 + 27\mu_1^2\mu_4) - \mathbf{PR},\\
\mathbf{U} &= \mu_3^2 - 4\mu_2\mu_4,\\
\mathbf{V} &= \mu_4.
\end{aligned}
\tag{4}
$$

Table 1: Number and multiplicity of the finite singularities of **QS**

No.	Zero–cycle $\mathcal{D}_{\mathbb{C}^2}(S)$	Invariant criteria	No.	Zero–cycle $\mathcal{D}_{\mathbb{C}^2}(S)$	Invariant criteria
1	$p+q+r+s$	$\mu_0 \neq 0, \mathbf{D} < 0, \mathbf{R} > 0, \mathbf{S} > 0$	10	$p+q+r$	$\mu_0 = 0, \mathbf{D} < 0, \mathbf{R} \neq 0$
2	$p+q+r^c+s^c$	$\mu_0 \neq 0, \mathbf{D} > 0$	11	$p+q^c+r^c$	$\mu_0 = 0, \mathbf{D} > 0, \mathbf{R} \neq 0$
3	$p^c+q^c+r^c+s^c$	$\mu_0 \neq 0, \mathbf{D} < 0, \mathbf{R} \leq 0$ $\mu_0 \neq 0, \mathbf{D} < 0, \mathbf{S} \leq 0$	12	$2p+q$	$\mu_0 = \mathbf{D} = 0, \mathbf{PR} \neq 0$
4	$2p+q+r$	$\mu_0 \neq 0, \mathbf{D} = 0, \mathbf{T} < 0$	13	$3p$	$\mu_0 = \mathbf{D} = \mathbf{P} = 0, \mathbf{R} \neq 0$
5	$2p+q^c+r^c$	$\mu_0 \neq 0, \mathbf{D} = 0, \mathbf{T} > 0$	14	$p+q$	$\mu_0 = \mathbf{R} = 0, \mathbf{P} \neq 0, \mathbf{U} > 0$
6	$2p+2q$	$\mu_0 \neq 0, \mathbf{D} = \mathbf{T} = 0, \mathbf{PR} > 0$	15	p^c+q^c	$\mu_0 = \mathbf{R} = 0, \mathbf{P} \neq 0, \mathbf{U} < 0$
7	$2p^c+2q^c$	$\mu_0 \neq 0, \mathbf{D} = \mathbf{T} = 0, \mathbf{PR} < 0$	16	$2p$	$\mu_0 = \mathbf{R} = 0, \mathbf{P} \neq 0, \mathbf{U} = 0$
8	$3p+q$	$\mu_0 \neq 0, \mathbf{D} = \mathbf{T} = 0, \mathbf{P} = 0, \mathbf{R} \neq 0$	17	p	$\mu_0 = \mathbf{R} = \mathbf{P} = 0, \mathbf{U} \neq 0$
9	$4p$	$\mu_0 \neq 0, \mathbf{D} = \mathbf{T} = 0, \mathbf{P} = \mathbf{R} = 0$	18	0	$\mu_0 = \mathbf{R} = \mathbf{P} = 0, \mathbf{U} = 0, \mathbf{V} \neq 0$

The next result, stated in [16], gives us for non-degenerate quadratic systems (3) the necessary and sufficient conditions for the existence of at least one invariant ellipse. The invariant polynomials which appear in the statement of the next theorem and in the corresponding diagrams are constructed in [16] and we present them further below.

Theorem 3 ([16])**.** *Consider a non-degenerate quadratic system.*

- **(*A*)** *The conditions $\widehat{\gamma}_1 = \widehat{\gamma}_2 = 0$ and either $\eta < 0$ or $C_2 = 0$ are necessary for this system to possess at least one invariant ellipse. Assume that the condition $\widehat{\gamma}_1 = \widehat{\gamma}_2 = 0$ is satisfied for this system.*
  - **(*A*$_1$)** *If $\eta < 0$ and $\widetilde{N} \neq 0$, then the system could possesses at most one invariant ellipse. Moreover, the necessary and sufficient conditions for the existence of such an ellipse are given in* Diagram *1.*
  - **(*A*$_2$)** *If $\eta < 0$ and $\widetilde{N} = 0$, then the system either has no invariant ellipse or it has an infinite family of invariant ellipses. Moreover, the necessary and sufficient conditions for the existence of a family of invariant ellipses are given in* Diagram *1.*
  - **(*A*$_3$)** *If $C_2 = 0$, then the system either has no invariant ellipse or it has an infinite family of invariant ellipses. Moreover, the necessary and sufficient conditions for the existence of a family of invariant ellipses are given in* Diagram *2.*

- **(*B*)** *A non-degenerate quadratic system possesses an algebraic limit cycle, which is an ellipse, if and only if $\widehat{\gamma}_1 = \widehat{\gamma}_2 = 0$, $\eta < 0$, $\mathcal{T}_3\mathcal{F} < 0$, $\widehat{\beta}_1\widehat{\beta}_2 \neq 0$, and one of the following sets of conditions is satisfied:*
  - **(*B*$_1$)** $\theta \neq 0$, $\widehat{\beta}_3 \neq 0$, $\widehat{\mathcal{R}}_1 < 0$;
  - **(*B*$_2$)** $\theta \neq 0$, $\widehat{\beta}_3 = 0$, $\widehat{\gamma}_3 = 0$, $\widehat{\mathcal{R}}_1 < 0$;
  - **(*B*$_3$)** $\theta = 0$, $\widehat{\gamma}_6 = 0$, $\widehat{\mathcal{R}}_5 < 0$.

- **(*C*)** *The* Diagrams *1 and 2 actually contain the global "bifurcation" diagram in the 12-dimensional space of parameters of non-degenerate systems which possess at least one invariant ellipse. The corresponding conditions are given in terms of 37 invariant polynomials with respect to the group of affine transformations and time rescaling.*

Remark 4. *An invariant ellipse is denoted by $\mathcal{E}^r$ if it is real and by $\mathcal{E}^c$ if it is complex. In the case of an $\mathcal{E}^r$ when the drawing is done with thicker line it means that this ellipse is a limit cycle (see Remark 2 (b)).*

The following result is included in [16] as a corollary of Theorem 3.

Corollary 1 ([16])**.** *Consider a non-degenerate quadratic system with the coefficients corresponding to a point $\tilde{a} \in \mathbb{R}^{12}$. According to [16] this system could possess an invariant ellipse only if the conditions $\widehat{\gamma}_1(\tilde{a}) = \widehat{\gamma}_2(\tilde{a}) = 0$ and either $\eta(\tilde{a}) < 0$ or $C_2(\tilde{a}, x, y) = 0$ in the ring $\mathbb{R}[x, y]$ are satisfied.*

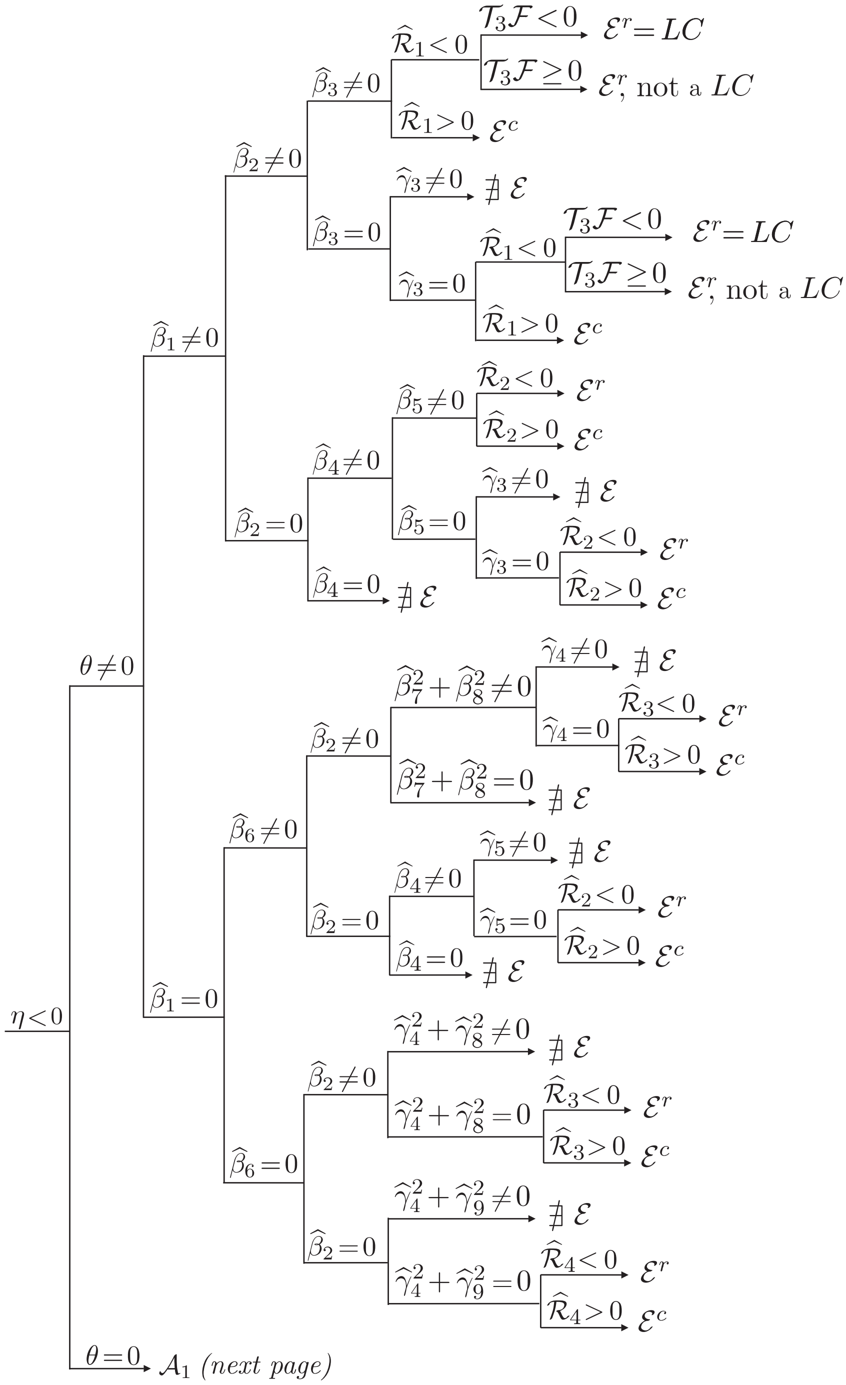

DIAGRAM 1: **The existence of invariant ellipse: the case $\eta<0$**

So we define the following two affine invariant subsets in $\mathbb{R}^{12}$, which must contain all quadratic systems (1) possessing an invariant ellipse:

$$\mathfrak{B}=\{\tilde a\in\mathbb{R}^{12}\big|\ \widehat{\gamma}_1(\tilde a)=\widehat{\gamma}_2(\tilde a)=0,\eta(\tilde a)<0\};$$
$$\mathfrak{C}=\{\tilde a\in\mathbb{R}^{12}\big|\ \widehat{\gamma}_1(\tilde a)=\widehat{\gamma}_2(\tilde a)=0,C_2(\tilde a,x,y)=0\}.$$

Next, following DIAGRAMS *1 and 2 we split the invariant sets $\mathfrak{B}$ and $\mathfrak{C}$ into affine invariant subsets*

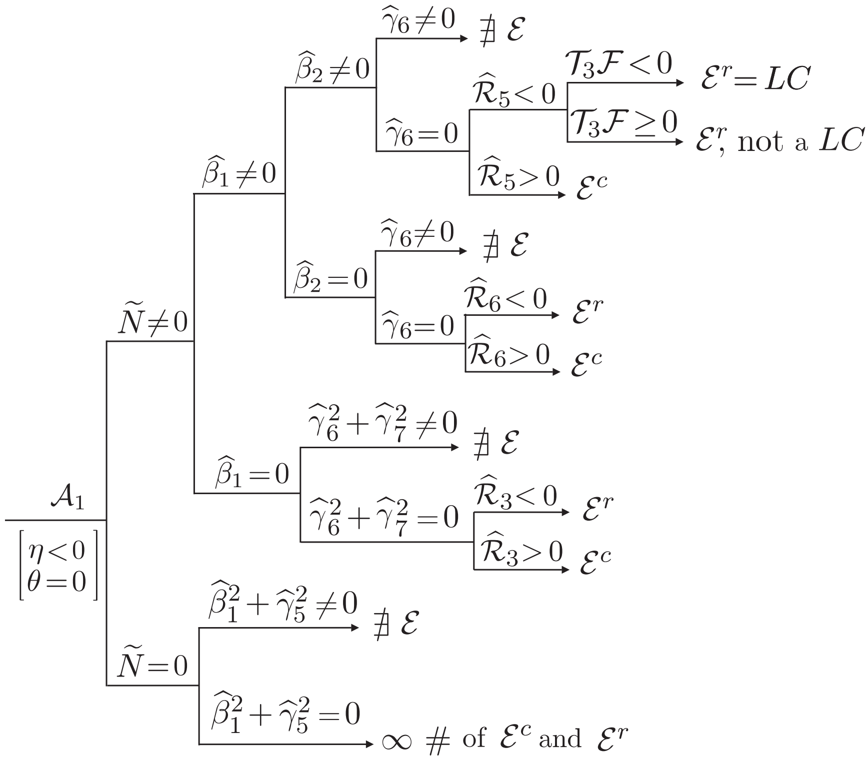

Diagram 1: (*Cont.*) **The existence of invariant ellipse: the case $\eta < 0$**

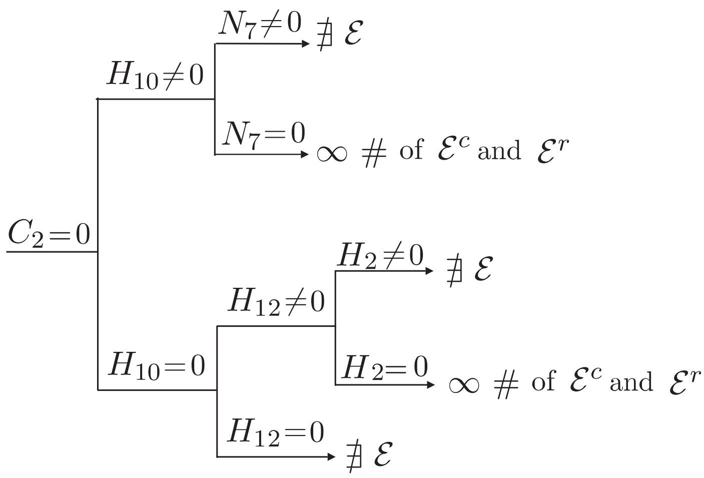

Diagram 2: **The existence of invariant ellipse: the case $C_2 = 0$**

$\mathfrak{B}_i$ *and* $\mathfrak{C}_j$ *(defined below):*

$$\mathfrak{B} = \bigcup_{i=1}^{12} \mathfrak{B}_i,\ \ \mathfrak{B}_i \bigcap_{i\neq j} \mathfrak{B}_j = \emptyset; \quad \mathfrak{C} = \bigcup_{j=1}^{2} \mathfrak{C}_j,\ \ \mathfrak{C}_1 \bigcap \mathfrak{C}_2 = \emptyset,$$

and we define the corresponding invariant subsets $\widetilde{\mathfrak{B}}_i$ $(i = 1, 2, \ldots, 12)$ *and* $\widetilde{\mathfrak{C}}_j$ $(j = 1, 2)$*:*

$(\mathfrak{B}_1):\ \theta\neq 0, \widehat{\beta}_1\neq 0, \widehat{\beta}_2\neq 0, \widehat{\beta}_3\neq 0;$
$(\mathfrak{B}_2):\ \theta\neq 0, \widehat{\beta}_1\neq 0, \widehat{\beta}_2\neq 0, \widehat{\beta}_3= 0,$
$(\mathfrak{B}_3):\ \theta\neq 0, \widehat{\beta}_1\neq 0, \widehat{\beta}_2= 0, \widehat{\beta}_5\neq 0,$
$(\mathfrak{B}_4):\ \theta\neq 0, \widehat{\beta}_1\neq 0, \widehat{\beta}_2= 0, \widehat{\beta}_5= 0,$
$(\mathfrak{B}_5):\ \theta\neq 0, \widehat{\beta}_1= 0, \widehat{\beta}_6\neq 0, \widehat{\beta}_2\neq 0,$
$(\mathfrak{B}_6):\ \theta\neq 0, \widehat{\beta}_1= 0, \widehat{\beta}_6\neq 0, \widehat{\beta}_2= 0,$
$(\mathfrak{B}_7):\ \theta\neq 0, \widehat{\beta}_1= 0, \widehat{\beta}_6= 0, \widehat{\beta}_2\neq 0,$
$(\mathfrak{B}_8):\ \theta\neq 0, \widehat{\beta}_1= 0, \widehat{\beta}_6= 0, \widehat{\beta}_2= 0,$
$(\mathfrak{B}_9):\ \theta= 0, \widetilde{N}\neq 0, \widehat{\beta}_1\neq 0, \widehat{\beta}_2\neq 0,$
$(\mathfrak{B}_{10}):\ \theta= 0, \widetilde{N}\neq 0, \widehat{\beta}_1\neq 0, \widehat{\beta}_2= 0,$
$(\mathfrak{B}_{11}):\ \theta= 0, \widetilde{N}\neq 0, \widehat{\beta}_1= 0,$
$(\mathfrak{B}_{12}):\ \theta= 0, \widetilde{N}= 0,$
$(\mathfrak{C}_1):\ C_2=0, H_{10}\neq 0,$
$(\mathfrak{C}_2):\ C_2=0, H_{10}= 0,$

$(\widetilde{\mathfrak{B}}_1):\ \widehat{\mathcal{R}}_1\neq 0;$
$(\widetilde{\mathfrak{B}}_2):\ \widehat{\gamma}_3=0, \widehat{\mathcal{R}}_1\neq 0;$
$(\widetilde{\mathfrak{B}}_3):\ \widehat{\mathcal{R}}_2\neq 0;$
$(\widetilde{\mathfrak{B}}_4):\ \widehat{\gamma}_3=0, \widehat{\mathcal{R}}_2\neq 0;$
$(\widetilde{\mathfrak{B}}_5):\ \widehat{\beta}_7^2+\widehat{\beta}_8^2\neq 0, \widehat{\gamma}_4=0, \widehat{\mathcal{R}}_3\neq 0;$
$(\widetilde{\mathfrak{B}}_6):\ \widehat{\gamma}_5=0, \widehat{\mathcal{R}}_2\neq 0;$
$(\widetilde{\mathfrak{B}}_7):\ \widehat{\gamma}_4^2+\widehat{\gamma}_8^2=0, \widehat{\mathcal{R}}_3\neq 0;$
$(\widetilde{\mathfrak{B}}_8):\ \widehat{\gamma}_4^2+\widehat{\gamma}_9^2=0, \widehat{\mathcal{R}}_4\neq 0;$
$(\widetilde{\mathfrak{B}}_9):\ \widehat{\gamma}_6=0, \widehat{\mathcal{R}}_5\neq 0;$
$(\widetilde{\mathfrak{B}}_{10}):\ \widehat{\gamma}_6=0, \widehat{\mathcal{R}}_6\neq 0;$
$(\widetilde{\mathfrak{B}}_{11}):\ \widehat{\gamma}_6^2+\widehat{\gamma}_7^2=0, \widehat{\mathcal{R}}_3\neq 0;$
$(\widetilde{\mathfrak{B}}_{12}):\ \widehat{\beta}_1^2+\widehat{\gamma}_5^2=0;$
$(\widetilde{\mathfrak{C}}_1):\ N_7=0;$
$(\widetilde{\mathfrak{C}}_2):\ H_{12}\neq 0, H_2=0.$

Then according to DIAGRAMS *1 and 2, a quadratic system, corresponding to a point* $\tilde{a}\in\mathbb{R}^{12}$, *possesses:*

- *an invariant ellipse which is unique if and only if* $\tilde{a}\in\mathfrak{B}_i\cap\widetilde{\mathfrak{B}}_i$ $(i=1,2,\ldots,11)$*; moreover this ellipse is real (respectively complex) if the corresponding invariant polynomial* $\widehat{\mathcal{R}}_s\neq 0$, $(s=1,\ldots,6)$, *which belongs to the set of polynomials defining* $\widetilde{\mathfrak{B}}_i$ $(i=1,2,\ldots,11)$, *is negative (respectively positive);*
- *an infinite number of invariant ellipses if and only if either* $\tilde{a}\in\mathfrak{B}_{12}\cap\widetilde{\mathfrak{B}}_{12}$ *or* $\tilde{a}\in\mathfrak{C}_j\cap\widetilde{\mathfrak{C}}_j$ $(j=1,2)$. *The ellipses could be real or/and complex.*

Following [16] here we present the invariant polynomials which according to DIAGRAMS 1 and 2 are responsible for the existence and the number of invariant ellipses which systems (3) could possess.

First we single out the following five polynomials, basic ingredients in constructing invariant polynomials for systems (3):

$$
\begin{aligned}
&C_i(\tilde{a},x,y)=yp_i(x,y)-xq_i(x,y), \quad i=0,1,2,\\
&D_i(\tilde{a},x,y)=\frac{\partial p_i}{\partial x}+\frac{\partial q_i}{\partial y}, \quad i=1,2.
\end{aligned}
\tag{5}
$$

As it was shown in [30], these polynomials of degree one in the coefficients of systems (3), are GL-comitants of these systems. Let $f,g\in\mathbb{R}[\tilde{a},x,y]$ and

$$(f,g)^{(k)}=\sum_{h=0}^{k}(-1)^h\binom{k}{h}\frac{\partial^k f}{\partial x^{k-h}\partial y^h}\ \frac{\partial^k g}{\partial x^h\partial y^{k-h}}.$$

The polynomial $(f,g)^{(k)}\in\mathbb{R}[\tilde{a},x,y]$ is called *the transvectant of index k of* (f,g) (cf. [10, 17]).

Theorem 4 (see [33])**.** *Any GL-comitant of systems* (3) *can be constructed from the elements* (5) *by using the operations:* $+,\ -,\ \times$, *and by applying the differential operation* $(*,*)^{(k)}$.

Remark 5. *We point out that the elements* (5) *generate the whole set of GL-comitants and hence also the set of affine comitants as well as the set of T-comitants and CT-comitants (see [23] for detailed definitions).*

We construct the following GL-comitants of the second degree with respect to the coefficients of the initial systems:

$$
\begin{aligned}
&T_1 = (C_0, C_1)^{(1)}, \quad T_2 = (C_0, C_2)^{(1)}, \quad T_3 = (C_0, D_2)^{(1)}, \\
&T_4 = (C_1, C_1)^{(2)}, \quad T_5 = (C_1, C_2)^{(1)}, \quad T_6 = (C_1, C_2)^{(2)}, \\
&T_7 = (C_1, D_2)^{(1)}, \quad T_8 = (C_2, C_2)^{(2)}, \quad T_9 = (C_2, D_2)^{(1)}.
\end{aligned} \tag{6}
$$

Using these GL-comitants as well as the polynomials (5) we construct additional invariant polynomials. To be able to directly calculate the values of the invariant polynomials which we need, we define here for every canonical system a family of T-comitants expressed through C_i $(i = 0, 1, 2)$ and D_j $(j = 1, 2)$:

$$
\begin{aligned}
\hat{A} &= (C_1, T_8 - 2T_9 + D_2^2)^{(2)}/144, \\
\widehat{D} &= \big[2C_0(T_8 - 8T_9 - 2D_2^2) + C_1(6T_7 - T_6 - (C_1, T_5)^{(1)} \\
&\quad + 6D_1C_1D_2 - T_5) - 9D_1^2C_2\big]/36, \\
\widehat{E} &= [D_1(2T_9 - T_8) - 3(C_1, T_9)^{(1)} - D_2(3T_7 + D_1D_2)]/72, \\
\widehat{F} &= [6D_1^2(D_2^2 - 4T_9) + 4D_1D_2(T_6 + 6T_7) + 48C_0(D_2, T_9)^{(1)} \\
&\quad - 9D_2^2T_4 + 288D_1\widehat{E} - 24(C_2, \widehat{D})^{(2)} + 120(D_2, \widehat{D})^{(1)} \\
&\quad - 36C_1(D_2, T_7)^{(1)} + 8D_1(D_2, T_5)^{(1)}]/144, \\
\widehat{B} &= \Big\{16D_1(D_2, T_8)^{(1)}(3C_1D_1 - 2C_0D_2 + 4T_2) + 32C_0(D_2, T_9)^{(1)}(3D_1D_2 - 5T_6 + 9T_7) \\
&\quad + 2(D_2, T_9)^{(1)}(27C_1T_4 - 18C_1D_1^2 - 32D_1T_2 + 32(C_0, T_5)^{(1)}) \\
&\quad + 6(D_2, T_7)^{(1)}\Big[8C_0(T_8 - 12T_9) - 12C_1(D_1D_2 + T_7) + D_1(26C_2D_1 + 32T_5) \\
&\quad + C_2(9T_4 + 96T_3)\Big] + 6(D_2, T_6)^{(1)}\Big[32C_0T_9 - C_1(12T_7 + 52D_1D_2) - 32C_2D_1^2\Big] \\
&\quad + 48D_2(D_2, T_1)^{(1)}(2D_2^2 - T_8) - 32D_1T_8(D_2, T_2)^{(1)} + 9D_2^2T_4(T_6 - 2T_7) \\
&\quad - 16D_1(C_2, T_8)^{(1)}(D_1^2 + 4T_3) + 12D_1(C_1, T_8)^{(2)}(C_1D_2 - 2C_2D_1) \\
&\quad + 6D_1D_2T_4(T_8 - 7D_2^2 - 42T_9) + 12D_1(C_1, T_8)^{(1)}(T_7 + 2D_1D_2) + 96D_2^2[D_1(C_1, T_6)^{(1)} \\
&\quad + D_2(C_0, T_6)^{(1)}] - 16D_1D_2T_3(2D_2^2 + 3T_8) - 4D_1^3D_2(D_2^2 + 3T_8 + 6T_9) \\
&\quad + 6D_1^2D_2^2(7T_6 + 2T_7)252D_1D_2T_4T_9\Big\}/(2^83^3), \\
\widehat{K} &= (T_8 + 4T_9 + 4D_2^2)/72, \quad \widehat{H} = (8T_9 - T_8 + 2D_2^2)/72.
\end{aligned}
$$

In addition to (5) and (6) these polynomials will serve as bricks in constructing affine invariant polynomials for systems (3).

In paper [5] it was proved that the minimal polynomial basis of affine invariants up to degree 12 contains 42 elements, denoted by $A_1, \ldots, A_{42}$. Here, using the above bricks, we present some of these basic elements which are necessary for the construction of the invariant polynomials we need.

$$
\begin{aligned}
&A_1 = \hat{A}, \qquad A_2 = (C_2, \widehat{D})^{(3)}/12, \qquad A_3 = [[C_2, D_2)^{(1)}, D_2)^{(1)}, D_2)^{(1)}/48,\\
&A_4 = (\widehat{H}, \widehat{H})^{(2)}, \qquad A_5 = (\widehat{H}, \widehat{K})^{(2)}/2, \qquad A_6 = (\widehat{E}, \widehat{H})^{(2)}/2,\\
&A_7 = [[C_2, \widehat{E})^{(2)}, D_2)^{(1)}/8, \qquad A_8 = [[\widehat{D}, \widehat{H})^{(2)}, D_2)^{(1)}/48, \qquad A_9 = [[\widehat{D}, D_2)^{(1)}, D_2)^{(1)},\\
&A_{10} = [[\widehat{D}, \widehat{K})^{(2)}, D_2)^{(1)}/8, \qquad A_{11} = (\widehat{F}, \widehat{K})^{(2)}/4, \qquad A_{12} = (\widehat{F}, \widehat{H})^{(2)}/4,\\
&A_{13} = [[C_2, \widehat{H})^{(1)}, \widehat{H})^{(2)}, D_2)^{(1)}/24, \qquad A_{14} = (\widehat{B}, C_2)^{(3)}/36, \qquad A_{15} = (\widehat{E}, \widehat{F})^{(2)}/4,\\
&A_{17} = [[\widehat{D}, \widehat{D})^{(2)}, D_2)^{(1)}, D_2)^{(1)}/64, \qquad A_{18} = [[\widehat{D}, \widehat{F})^{(2)}, D_2)^{(1)}/16,\\
&A_{19} = [[\widehat{D}, \widehat{D})^{(2)}, \widehat{H})^{(2)}/16, \qquad A_{20} = [[C_2, \widehat{D})^{(2)}, \widehat{F})^{(2)}/16, \qquad A_{21} = [[\widehat{D}, \widehat{D})^{(2)}, \widehat{K})^{(2)}/16,\\
&A_{22} = \frac{1}{1152}[[C_2, \widehat{D})^{(1)}, D_2)^{(1)}, D_2)^{(1)}, D_2)^{(1)} D_2)^{(1)}, \qquad A_{23} = [[\widehat{F}, \widehat{H})^{(1)}, \widehat{K})^{(2)}/8,\\
&A_{24} = [[C_2, \widehat{D})^{(2)}, \widehat{K})^{(1)}, \widehat{H})^{(2)}/32, \qquad A_{31} = [[\widehat{D}, \widehat{D})^{(2)}, \widehat{K})^{(1)}, \widehat{H})^{(2)}/64,\\
&A_{32} = [[\widehat{D}, \widehat{D})^{(2)}, D_2)^{(1)}, \widehat{H})^{(1)}, D_2)^{(1)}/64, \qquad A_{33} = [[\widehat{D}, D_2)^{(1)}, \widehat{F})^{(1)}, D_2)^{(1)}, D_2)^{(1)}/128,\\
&A_{34} = [[\widehat{D}, \widehat{D})^{(2)}, D_2)^{(1)}, \widehat{K})^{(1)}, D_2)^{(1)}/64, \qquad A_{38} = [[C_2, \widehat{D})^{(2)}, \widehat{D})^{(2)}, \widehat{D})^{(1)}, \widehat{H})^{(2)}/64,\\
&A_{39} = [[\widehat{D}, \widehat{D})^{(2)}, \widehat{F})^{(1)}, \widehat{H})^{(2)}/64, \qquad A_{41} = [[C_2, \widehat{D})^{(2)}, \widehat{D})^{(2)}, \widehat{F})^{(1)}, D_2)^{(1)}/64,\\
&A_{42} = [[\widehat{D}, \widehat{F})^{(2)}, \widehat{F})^{(1)}, D_2)^{(1)}/16.
\end{aligned}
$$

In the above list the bracket "$[[$" means a succession of two or up to five parentheses "(" depending on the row it appears.

Using the elements of the minimal polynomial basis given above we construct the affine invariant polynomials:

$$
\begin{aligned}
\widehat{\gamma}_1(\tilde{a}) = {} & A_1^2(3A_6 + 2A_7) - 2A_6(A_8 + A_{12}),\\
\widehat{\gamma}_2(\tilde{a}) = {} & 9A_1^2A_2(23252A_3 + 23689A_4) - 1440A_2A_5(3A_{10} + 13A_{11})\\
& - 1280A_{13}(2A_{17} + A_{18} + 23A_{19} - 4A_{20}) - 320A_{24}(50A_8 + 3A_{10}\\
& + 45A_{11} - 18A_{12}) + 120A_1A_6(6718A_8 + 4033A_9 + 3542A_{11}\\
& + 2786A_{12}) + 30A_1A_{15}(14980A_3 - 2029A_4 - 48266A_5)\\
& - 30A_1A_7(76626A_1^2 - 15173A_8 + 11797A_{10} + 16427A_{11} - 30153A_{12})\\
& + 8A_2A_7(75515A_6 - 32954A_7) + 2A_2A_3(33057A_8 - 98759A_{12})\\
& - 60480A_1^2A_{24} + A_2A_4(68605A_8 - 131816A_9 + 131073A_{10} + 129953A_{11})\\
& - 2A_2(141267A_6^2 - 208741A_5A_{12} + 3200A_2A_{13}),\\
\widehat{\gamma}_3(\tilde{a}) = {} & 843696A_5A_6A_{10} + A_1(-27(689078A_8 + 419172A_9 - 2907149A_{10}\\
& - 2621619A_{11})A_{13} - 26(21057A_3A_{23} + 49005A_4A_{23} - 166774A_3A_{24}\\
& + 115641A_4A_{24})),
\end{aligned}
$$

$$
\begin{aligned}
\widehat{\gamma}_4(\tilde{a}) &= -488A_2^3A_4 + A_2(12(4468A_8^2 + 32A_9^2 - 915A_{10}^2 + 320A_9A_{11} - 3898A_{10}A_{11} \\
&\quad - 3331A_{11}^2 + 2A_8(78A_9 + 199A_{10} + 2433A_{11})) + 2A_5(25488A_{18} \\
&\quad - 60259A_{19} - 16824A_{21}) + 779A_4A_{21}) + 4(7380A_{10}A_{31} \\
&\quad - 24(A_{10} + 41A_{11})A_{33} + A_8(33453A_{31} + 19588A_{32} - 468A_{33} - 19120A_{34}) \\
&\quad + 96A_9(-A_{33} + A_{34}) + 556A_4A_{41} - A_5(27773A_{38} + 41538A_{39} \\
&\quad - 2304A_{41} + 5544A_{42})), \\
\widehat{\gamma}_5(\tilde{a}) &= A_{22}, \\
\widehat{\gamma}_6(\tilde{a}) &= A_1(64A_3 - 541A_4)A_7 + 86A_8A_{13} + 128A_9A_{13} - 54A_{10}A_{13} \\
&\quad - 128A_3A_{22} + 256A_5A_{22} + 101A_3A_{24} - 27A_4A_{24}, \\
\widehat{\gamma}_7(\tilde{a}) &= A_2\big[2A_3(A_8 - 11A_{10}) - 18A_7^2 - 9A_4(2A_9 + A_{10}) + 22A_8A_{22} + 26A_{10}A_{22}, \\
\widehat{\gamma}_8(\tilde{a}) &= A_6, \\
\widehat{\gamma}_9(\tilde{a}) &= 12A_1^2 + 12A_8 + 5A_{10} + 17A_{11}, \\
\widehat{\beta}_1(\tilde{a}) &= 3A_1^2 - 2A_8 - 2A_{12}, \\
\widehat{\beta}_2(\tilde{a}) &= 2A_{13}, \\
\widehat{\beta}_3(\tilde{a}) &= 8A_3 + 27A_4 - 54A_5, \\
\widehat{\beta}_4(\tilde{a}) &= A_4, \\
\widehat{\beta}_5(\tilde{a}) &= 8A_5 - 5A_4, \\
\widehat{\beta}_6(\tilde{a}) &= A_3, \\
\widehat{\beta}_7(\tilde{a}) &= 24A_3 + 11A_4 + 20A_5, \\
\widehat{\beta}_8(\tilde{a}) &= 41A_8 + 44A_9 + 32A_{10}, \\
\widehat{\mathcal{R}}_1(\tilde{a}) &= \theta A_6\big[5A_6(A_{10} + A_{11}) - 2A_7(12A_1^2 + A_8 + A_{12}) - 2A_1(A_{23} - A_{24}) \\
&\quad + 2A_5(A_{14} + A_{15}) + A_6(9A_8 + 7A_{12})\big], \\
\widehat{\mathcal{R}}_2(\tilde{a}) &= \widehat{\beta}_4\widehat{\beta}_6(2A_{10} - A_8 - A_9), \\
\widehat{\mathcal{R}}_3(\tilde{a}) &= \widehat{\beta}_2\big[A_2(80A_3 - 3A_4 - 54A_5) - 80A_{22} + 708A_{23} - 324A_{24}\big], \\
\widehat{\mathcal{R}}_4(\tilde{a}) &= T_{11}, \\
\widehat{\mathcal{R}}_5(\tilde{a}) &= 12A_1^2 + 12A_8 + 5A_{10} + 17A_{11}, \\
\widehat{\mathcal{R}}_6(\tilde{a}) &= 2A_{10} - A_8 - A_9, \\
\widehat{\mathcal{R}}_7(\tilde{a}) &= 4A_8 - 3A_9, \\
\nu_1 &= -A_6\big(A_1A_2 - 2A_{15}\big)\big(3A_1^2 - 2A_8 - 2A_{12}\big), \\
\nu_2 &= A_1\big(-461A_2A_4 + 183A_2A_5 - 296A_{22} + 122A_{24}\big) + A_4\big(467A_{14} + 922A_{15}\big) \\
&\quad + 2A_6\big(553A_8 + 183A_9 - 100A_{10} - 39A_{11} + 144A_{12}\big) \\
&\quad + A_7\big(5790A_1^2 - 1531A_8 - 140A_9 + 177A_{10} + 947A_{11} - 2791A_{12}\big), \\
\nu_3 &= A_4\big(18A_1^2 - 5A_8 + A_{10} + 3A_{11} - 9A_{12}\big),
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{N}(\tilde{a},x,y) &= (D_2^2+T_8-2T_9)/9,\\
\theta(\tilde{a}) &= 2A_5-A_4 \equiv \mathrm{Discrim}[\widetilde{N},x]/(16y^2),\\
\mathcal{F}(\tilde{a}) &= A_7,\\
\mathcal{T}_3(\tilde{a}) &= 8A_{15}-4A_1A_2,\\
H_2(\tilde{a},x,y) &= \big(C_1,-8\widehat{H}-\widetilde{N}\big)^{(1)}-2D_1\widetilde{N},\\
H_9(\tilde{a}) &= -\big[\big[\widehat{D},\widehat{D}\big)^{(2)},\widehat{D}\big)^{(1)},\widehat{D}\big)^{(3)},\\
H_{10}(\tilde{a}) &= \big[\big[\widehat{D},\widetilde{N}\big)^{(2)},D_2\big)^{(1)},\\
H_{11}(\tilde{a},x,y) &= -32\widehat{H}\Big[\big(C_2,\widehat{D}\big)^{(2)}+8\big(\widehat{D},D_2\big)^{(1)}\Big]+3\Big[\big(C_1,-8\widehat{H}-\widetilde{N}\big)^{(1)}-2D_1\widetilde{N}\Big]^2,\\
H_{12}(\tilde{a},x,y) &= \big(\widehat{D},\widehat{D}\big)^{(2)},\\
N_7(\tilde{a}) &= 12D_1\big(C_0,D_2\big)^{(1)}+2D_1^3+9D_1\big(C_1,C_2\big)^{(2)}+36\big[\big[C_0,C_1\big)^{(1)},D_2\big)^{(1)}.
\end{aligned}
$$

We remark that the last six invariant polynomials H_2, H_9 to H_{12}, and N_7 are constructed in [27], whereas $\mathcal{F}$ and $\mathcal{T}_3$ are defined in [32].

Next we construct the following T-comitants (for the definition of T-comitants see [24]) which are responsible for the existence of invariant straight lines of systems (3):

$$
\begin{aligned}
B_3(\tilde{a},x,y) &= (C_2,\widehat{D})^{(1)} = \mathrm{Jacob}(C_2,\widehat{D}),\\
B_2(\tilde{a},x,y) &= (B_3,B_3)^{(2)}-6B_3(C_2,\widehat{D})^{(3)},\\
B_1(\tilde{a}) &= \mathrm{Res}_x\left(C_2,\widehat{D}\right)/y^9 = -2^{-9}3^{-8}\left(B_2,B_3\right)^{(4)}.
\end{aligned}
\tag{7}
$$

Lemma 2 (see [23]). *For the existence of invariant straight lines in one (respectively 2; 3 distinct) directions in the affine plane it is necessary that $B_1=0$ (respectively $B_2=0$; $B_3=0$).*

At the moment we only have necessary and not necessary and sufficient conditions for the existence of an invariant straight line or for invariant lines in two or three directions.

Let us apply the translation $x=x'+x_0$, $y=y'+y_0$ to the polynomials $p(\tilde{a},x,y)$ and $q(\tilde{a},x,y)$. Then we obtain $\hat{p}(\hat{a}(a,x_0,y_0),x',y')=p(\tilde{a},x'+x_0,y'+y_0)$, $\hat{q}(\hat{a}(a,x_0,y_0),x',y')=q(\tilde{a},x'+x_0,y'+y_0)$. Let us construct the following polynomials

$$
\begin{aligned}
\Gamma_i(\tilde{a},x_0,y_0) \equiv \mathrm{Res}_{x'}\Big(C_i\big(\hat{a}(\tilde{a},x_0,y_0),x',y'\big),C_0\big(\hat{a}(\tilde{a},x_0,y_0),x',y'\big)\Big)/(y')^{i+1},\\
\Gamma_i(\tilde{a},x_0,y_0)\in\mathbb{R}[\tilde{a},x_0,y_0],\quad i=1,2.
\end{aligned}
$$

We denote

$$
\tilde{\mathcal{E}}_i(\tilde{a},x,y)=\Gamma_i(\tilde{a},x_0,y_0)\big|_{\{x_0=x,\,y_0=y\}}\in\mathbb{R}[\tilde{a},x,y],\quad i=1,2.
$$

Remark 6. *We note that the polynomials $\tilde{\mathcal{E}}_1(a,x,y)$ and $\tilde{\mathcal{E}}_2(a,x,y)$ are affine comitants of systems (3) and they are homogeneous polynomials in the coefficients a,b,c,d,e,f,g,h,k,l,m,n and non-homogeneous in x,y and $deg_{\tilde{a}}\tilde{\mathcal{E}}_1=3$, $deg_{(x,y)}\tilde{\mathcal{E}}_1=5$, $deg_{\tilde{a}}\tilde{\mathcal{E}}_2=4$, $deg_{(x,y)}\tilde{\mathcal{E}}_2=6$.*

Let $\mathcal{E}_i(\tilde{a}, X, Y, Z)$, $i = 1, 2$, be the homogenization of $\tilde{\mathcal{E}}_i(\tilde{a}, x, y)$, i.e.

$$\mathcal{E}_1(\tilde{a}, X, Y, Z) = Z^5 \tilde{\mathcal{E}}_1(\tilde{a}, X/Z, Y/Z), \quad \mathcal{E}_2(\tilde{a}, X, Y, Z) = Z^6 \tilde{\mathcal{E}}_1(\tilde{a}, X/Z, Y/Z)$$

The geometrical meaning of these affine comitants is given by the following lemma (see [23]):

Lemma 3 (see [23]). *(1) The straight line $\mathcal{L}(x, y) \equiv ux + vy + w = 0$, $u, v, w \in \mathbb{C}$, $(u, v) \neq (0, 0)$ is an invariant line for a quadratic system* (3) *if and only if the polynomial $\mathcal{L}(x, y)$ is a common factor of the polynomials $\tilde{\mathcal{E}}_1(\tilde{a}, x, y)$ and $\tilde{\mathcal{E}}_2(\tilde{a}, x, y)$ over $\mathbb{C}$, i.e.*

$$\tilde{\mathcal{E}}_i(\tilde{a}, x, y) = (ux + vy + w)\widetilde{W}_i(x, y), \quad i = 1, 2,$$

where $\widetilde{W}_i(x, y) \in \mathbb{C}[x, y]$.

(2) If $\mathcal{L}(x, y) = 0$ is an invariant straight line of multiplicity λ for a quadratic system (3)*, then $[\mathcal{L}(x, y)]^\lambda \mid \gcd(\tilde{\mathcal{E}}_1, \tilde{\mathcal{E}}_2)$ in $\mathbb{C}[x, y]$, i.e. there exist $W_i(\tilde{a}, x, y) \in \mathbb{C}[x, y]$, $i = 1, 2$, such that*

$$\tilde{\mathcal{E}}_i(\tilde{a}, x, y) = (ux + vy + w)^\lambda W_i((a), x, y), \quad i = 1, 2.$$

(3) If the line $l_\infty : Z = 0$ is of multiplicity $\lambda > 1$, then $Z^{\lambda-1} \mid \gcd(\mathcal{E}_1, \mathcal{E}_2)$.

The invariant polynomials $\widetilde{N}(\tilde{a}, x, y)$ and $\theta(\tilde{a})$, defined on page 18, are responsible for detecting parallel invariant lines as we can see in the following lemma.

Lemma 4 (see [23]). *A necessary condition for the existence of one couple (respectively two couples) of parallel invariant straight lines of a system* (3) *corresponding to $\tilde{a} \in \mathbb{R}^{12}$ is the condition $\theta(\tilde{a}) = 0$ (respectively $\widetilde{N}(\tilde{a}, x, y) = 0$).*

Now we introduce some important GL-comitants in the study of the invariant conics. Considering $C_2(\tilde{a}, x, y) = yp_2(\tilde{a}, x, y) - xq_2(\tilde{a}, x, y)$ as a cubic binary form of x and y we calculate

$$\eta(\tilde{a}) = \text{Discrim}[C_2, \xi], \quad M(\tilde{a}, x, y) = \text{Hessian}[C_2],$$

where $\xi = y/x$ or $\xi = x/y$.

Lemma 5 ([29]). *The number of infinite singularities (real and complex) of a quadratic system in* **QS** *is determined by the following conditions:*

(i) 3 *real if $\eta > 0$;*

(ii) 1 *real and* 2 *imaginary if $\eta < 0$;*

(iii) 2 *real if $\eta = 0$ and $M \neq 0$;*

(iv) 1 *real if $\eta = M = 0$ and $C_2 \neq 0$;*

(v) ∞ if $\eta = M = C_2 = 0$.

Moreover, for each one of these cases the quadratic systems (3) *can be brought via a linear transformation to one of the following 5 canonical systems:*

$$\begin{cases} \dot{x} &= a + cx + dy + gx^2 + (h-1)xy, \\ \dot{y} &= b + ex + fy + (g-1)xy + hy^2; \end{cases} \qquad (\mathbf{S}_I)$$

$$\begin{cases} \dot{x} &= a + cx + dy + gx^2 + (h+1)xy, \\ \dot{y} &= b + ex + fy - x^2 + gxy + hy^2; \end{cases} \qquad (\mathbf{S}_{II})$$

$$\begin{cases} \dot{x} &= a + cx + dy + gx^2 + hxy, \\ \dot{y} &= b + ex + fy + (g-1)xy + hy^2; \end{cases} \qquad (\mathbf{S}_{III})$$

$$\begin{cases} \dot{x} &= a + cx + dy + gx^2 + hxy, \\ \dot{y} &= b + ex + fy - x^2 + gxy + hy^2; \end{cases} \qquad (\mathbf{S}_{IV})$$

$$\begin{cases} \dot{x} &= a + cx + dy + x^2, \\ \dot{y} &= b + ex + fy + xy. \end{cases} \qquad (\mathbf{S}_V)$$

3 Configurations of invariant ellipses for the classes $\mathbf{QSE}_{(\eta<0)}$ and $\mathbf{QSE}_{(C_2=0)}$

Theorem 5. *Consider the classes* $\mathbf{QSE}_{(\eta<0)}$ *and* $\boldsymbol{QSE}_{(C_2=0)}$ *of all non-degenerate quadratic differential systems* (3) *possessing one real and two complex singularities at infinity, and the quadratic differential systems possessing the line at infinity filled up with singularities, respectively.*

(A) *These families are classified according to the configurations of invariant ellipses and of invariant straight lines of the systems, yielding 30 distinct such configurations for the class* $\boldsymbol{QSE}_{(\eta<0)}$ *and 5 for the class* $\boldsymbol{QSE}_{(C_2=0)}$. *This geometric classification appears in* Diagrams *3 and 4. More precisely:*

(A1) For the class $\boldsymbol{QSE}_{(\eta<0)}$, *there exist exactly 3 configurations of systems possessing an infinite number of ellipses. More precisely two of them contain only real ellipses and the third one contains simultaneously an infinite number of real and an infinite number of complex ellipses. The remaining 27 configurations possess exactly one invariant ellipse, (real for 21 of them) or complex (for another 6).*

(A2) For the class $\boldsymbol{QSE}_{(C_2=0)}$ *all the 5 configurations of systems possess an infinite number of ellipses (four of them with three simple invariant lines and one of them with a triple invariant line). More precisely three of the configurations contain only real ellipses, one contains only complex ones and the remaining configuration contains simultaneously an infinite number of real and an infinite number of complex ellipses.*

(B) *The bifurcation diagrams for systems in* $\boldsymbol{QSE}_{(\eta<0)}$ *and* $\boldsymbol{QSE}_{(C_2=0)}$ *done in the coefficient space* $\mathbb{R}^{12}$ *in terms of invariant polynomials appear in* Diagrams *5 to 7. In these diagrams we have necessary and sufficient conditions for the realization of each one of the configurations.*

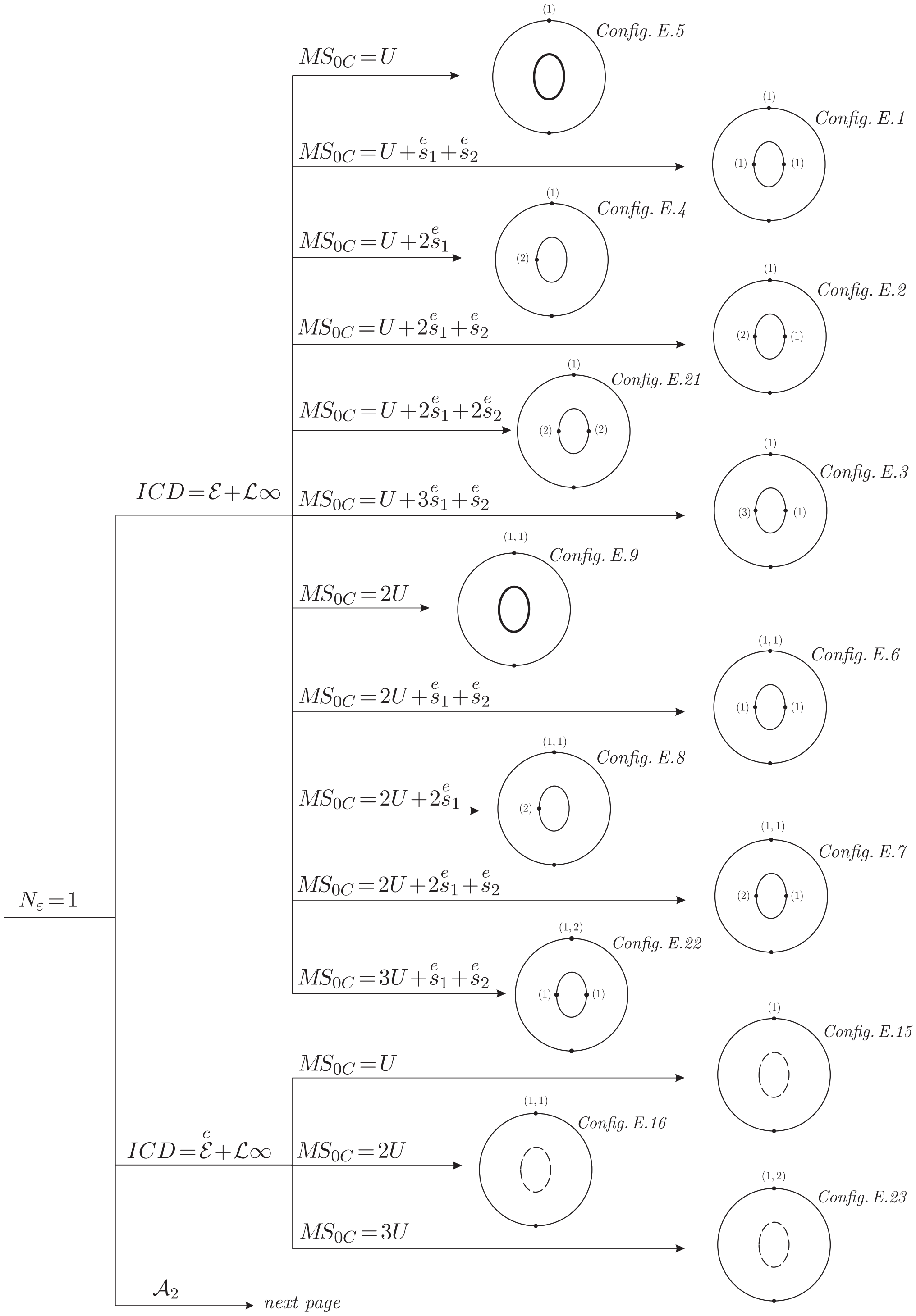

Diagram 3: **Configurations with one invariant ellipse**

Remark 7. *We note that on the expressions of the divisors ICD and ILD as well of the zero-cycles MS_{0C} and MS_{0C}^{Af} appearing in* Diagrams *3 and 4, we can read their types help in this classification and furthermore they are affinely invariant.*

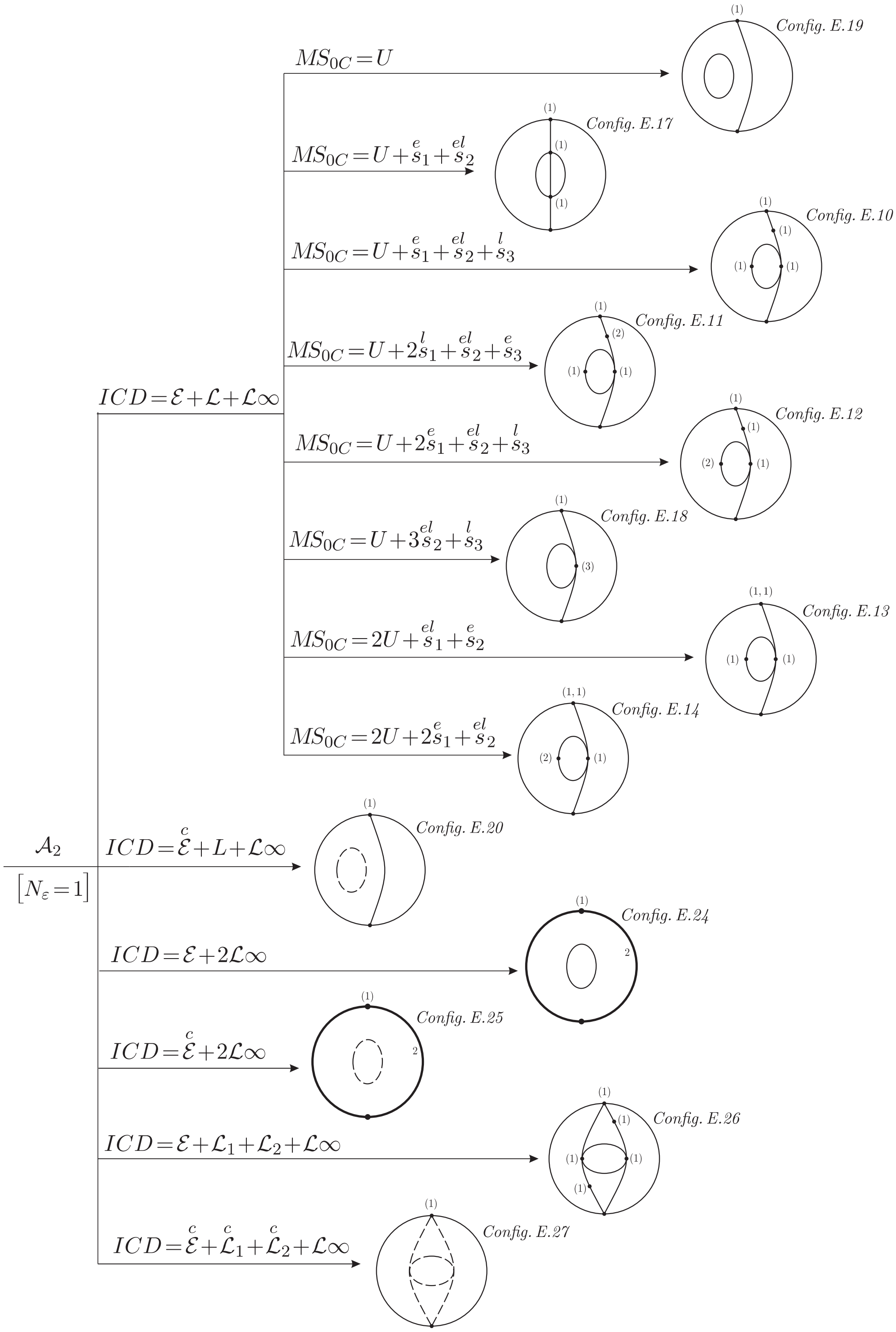

DIAGRAM 3: (*Cont.*) **Configurations with one invariant ellipse**

Proof of part (A). We prove part (A) under the assumption that part (B) is already proved. Later we prove part (B).

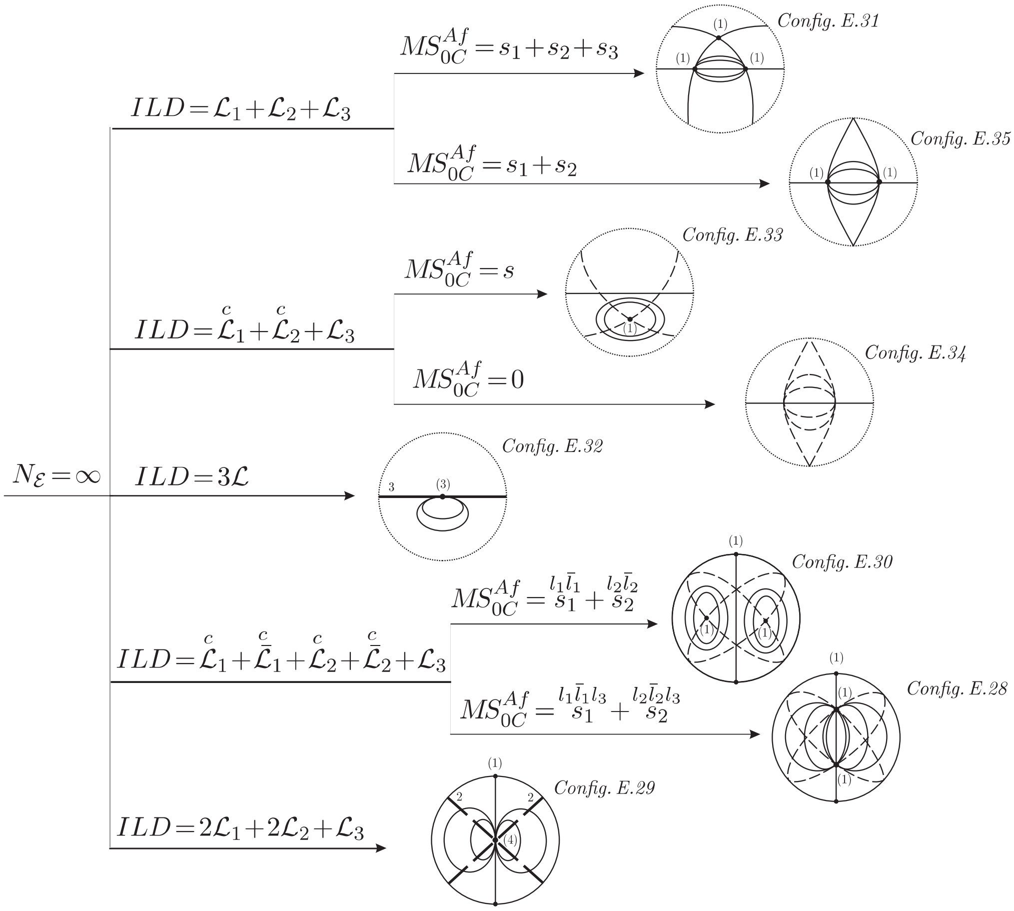

Diagram 4: **Configurations with a family of invariant ellipses**

We first need to make sure that the concepts introduced above gave us a sufficient number of invariants under the action of the affine group and time rescaling so as to be able to classify geometrically the classes $\mathbf{QSE}_{(\eta<0)}$ and $\mathbf{QSE}_{(C_2=0)}$ according to their configurations of their invariant ellipses and lines.

Fixing the values of N_ε and using the types of the divisors ICD in Diagram 3 (respectively ILD in Diagram 4) we split all the corresponding configurations in 8 (respectively in 5) groups. We observe that some groups have only one configuration. For the groups which possess more than one configuration we use the types of zero-cycles MS_{0C} and MS_{0C}^{Af}, correspondingly. This suffices for distinguishing all the configurations.

As a result we obtain the 35 geometric configurations displayed in Diagrams 3 and 4. This proves statement (A) of this theorem. ∎

Proof of part (B). According to [16] a quadratic system could have an invariant ellipse only if $\widehat{\gamma}_1 = \widehat{\gamma}_2 = 0$ and either $\eta < 0$ or $C_2 = 0$. We examine the cases $\eta < 0$ and $C_2 = 0$ separately.

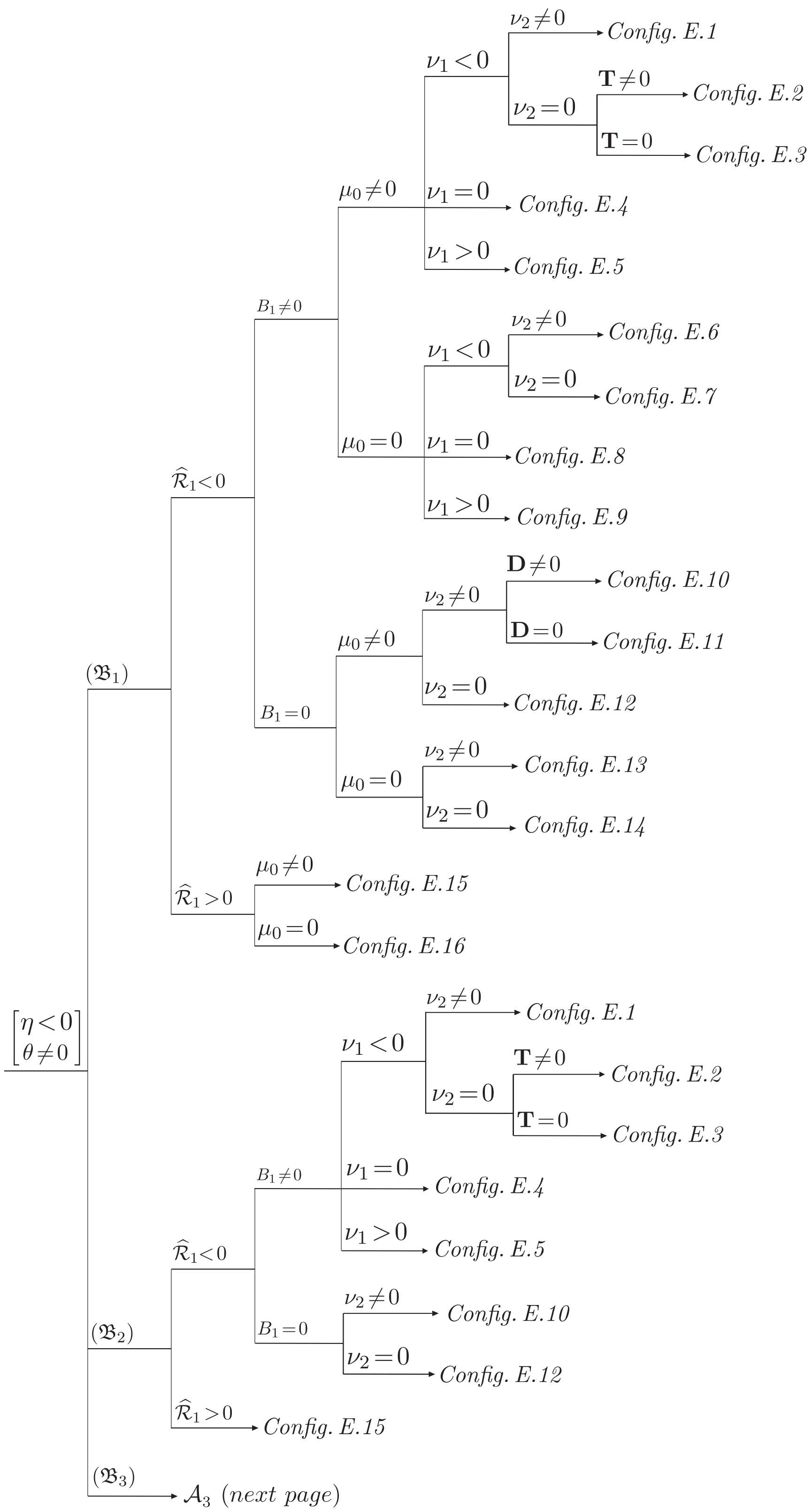

Diagram 5: **Bifurcation diagram in $\mathbb{R}^{12}$ of the configurations: Case $\eta<0$, $\theta\neq0$**

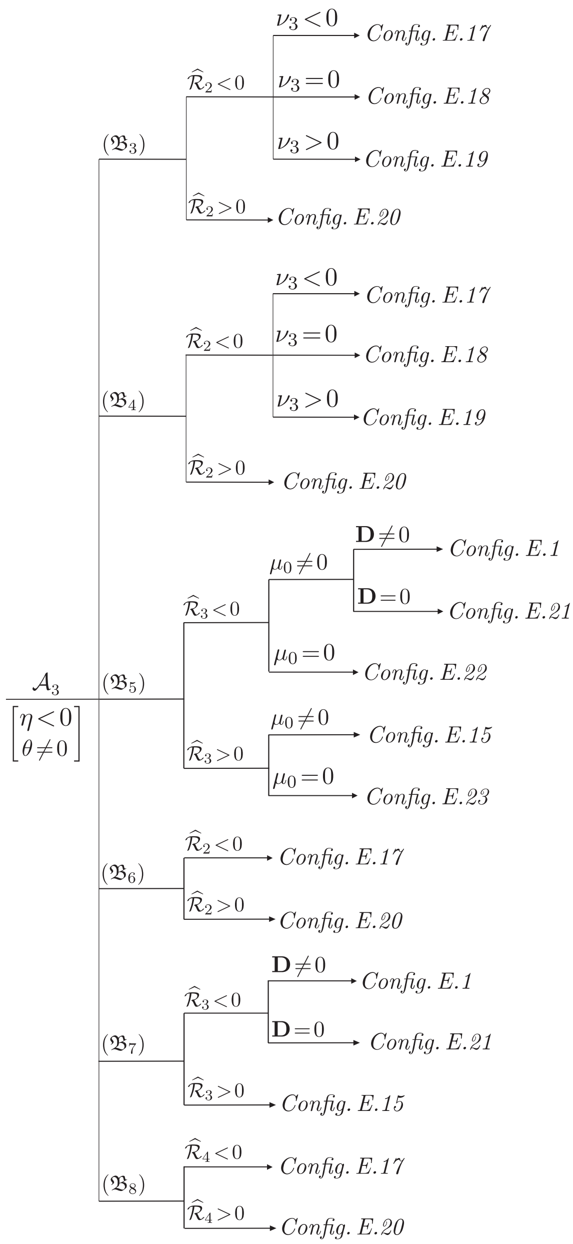

DIAGRAM 5: (*Cont.*) **Bifurcation diagram in $\mathbb{R}^{12}$ of the configurations: Case $\eta < 0$, $\theta \neq 0$**

3.1 The case $\eta < 0$

According to Lemma 5 a quadratic system with the condition $\eta < 0$ could be brought via an affine transformation and time rescaling to the following canonical form:

$$\begin{aligned}\dot{x} =& a + cx + dy + gx^2 + (h+1)xy,\\ \dot{y} =& b + ex + fy - x^2 + gxy + hy^2,\end{aligned} \tag{8}$$

with $C_2 = x(x^2 + y^2)$, i.e. this system possesses at infinity one real and two complex infinity singularities. Following DIAGRAM 1 (see also [16]) we discuss two subcases: $\theta \neq 0$ and $\theta = 0$.

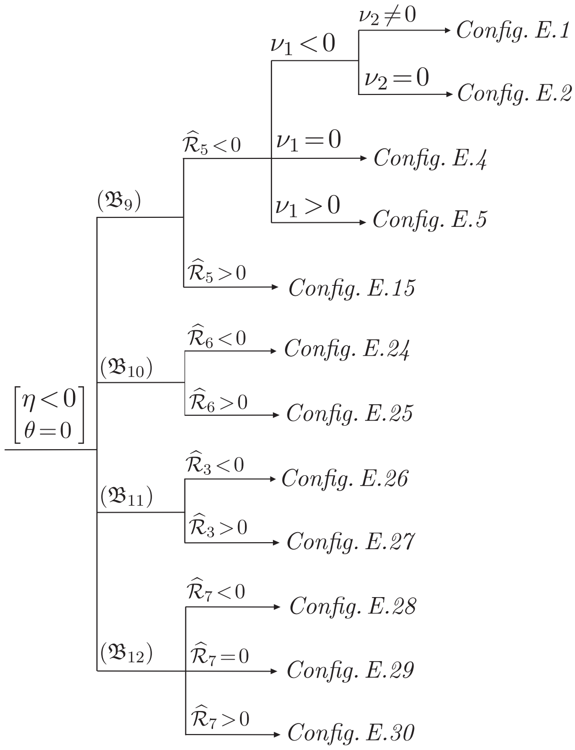

DIAGRAM 6: **Bifurcation diagram in $\mathbb{R}^{12}$ of the configurations: Case $\eta < 0$, $\theta = 0$**

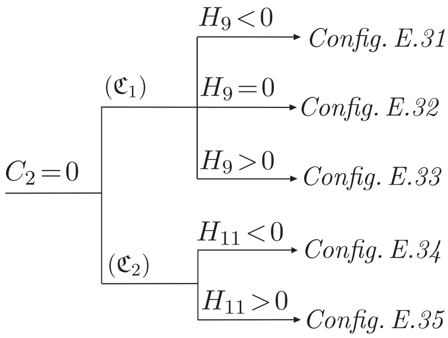

DIAGRAM 7: **Bifurcation diagram in $\mathbb{R}^{12}$ of the configurations: Case $C_2 = 0$**

3.1.1 The subcase $\theta \neq 0$

We examine step by step each one of the possibilities presented in Corollary 1.

3.1.1.1 The possibility $(\mathfrak{B}_1)$: $\widehat{\beta}_1\widehat{\beta}_2\widehat{\beta}_3 \neq 0$**.** As it was proved in [16] in this case by an affine transformation and time rescaling, systems (8) could be brought to the canonical form

$$\begin{aligned} \dot{x} &= a + dy + gx^2 + (h+1)xy, \\ \dot{y} &= \frac{ah}{g} - dx - x^2 + gxy + hy^2, \quad g \neq 0, \end{aligned} \tag{9}$$

which possesses an invariant conic

$$\Phi(x,y)=\frac{a}{g}+x^2+y^2=0. \tag{10}$$

This conic is irreducible if and only if $a\neq 0$. For the above systems we calculate

$$\begin{gathered}\theta=(h+1)\big[g^2+(h-1)^2\big]/2,\quad \widehat{\beta}_1=-d^2\big[g^2+(h-1)^2\big]\big[9g^2+(3h+1)^2\big]/16,\\ \widehat{\beta}_2=-g\big[9g^2+(3h+1)^2\big]/2,\quad \widehat{\beta}_3=(3h-1)\big[9g^2+(3h+5)^2\big]/2,\end{gathered} \tag{11}$$

and therefore we conclude that for the above systems the condition $\theta\widehat{\beta}_1\widehat{\beta}_2\widehat{\beta}_3\widehat{\mathcal{R}}_1\neq 0$ is equivalent to the condition

$$adg(h+1)(3h-1)\neq 0. \tag{12}$$

We observe that

$$\widehat{\mathcal{R}}_1=3agd^2(1+h)^2\big[g^2+(h-1)^2\big]^4\big[9g^2+(3h+1)^2\big]/128\ \Rightarrow\ \operatorname{sign}(ag)=\operatorname{sign}(\widehat{\mathcal{R}}_1).$$

3.1.1.1.1 The case $\widehat{\mathcal{R}}_1<0$. Then $ag<0$ and clearly the ellipse (10) is real.

Taking into account Lemma 2 we examine if systems (9) could possess at least one invariant line. Calculations yield

$$B_1=-\frac{a^2}{g^2}(g^2+h^2)\big[g^2+(h-1)^2\big]^2\big[a(h+1)^2+d^2g\big], \tag{13}$$

and we consider two subcases: $B_1\neq 0$ and $B_1=0$.

1) *The subcase $B_1\neq 0$.* Then by Lemma 2 systems (9) could not possess invariant lines. For these systems we calculate $\mu_0=-h\big[g^2+(h+1)^2\big]$ and we examine two possibilities: $\mu_0\neq 0$ and $\mu_0=0$.

a) *The possibility $\mu_0\neq 0$.* Then by Lemma 1 the systems have finite singularities of total multiplicity 4. We detect that two of these singularities are located on the ellipse (10), more exactly such singularities are $M_{1,2}\big(x_{1,2},y_{1,2}\big)$ with

$$x_{1,2}=-\frac{d(h+1)\pm\sqrt{Z_1}}{g^2+(h+1)^2},\quad y_{1,2}=\frac{dg^2\mp(h+1)\sqrt{Z_1}}{g\big[g^2+(h+1)^2\big]},\quad Z_1=-g\Big[a\big[g^2+(1+h)^2\big]+d^2g\Big]. \tag{14}$$

Other two singularities of systems (9) are $M_{3,4}\big(x_{3,4},y_{3,4}\big)$ (generically located outside the ellipse) with

$$x_{3,4}=-\frac{dg\pm\sqrt{Z_2}}{2g},\quad y_{3,4}=\frac{dg\pm\sqrt{Z_2}}{2h},\quad Z_2=g(d^2g+4ah). \tag{15}$$

On the other hand for systems (9) we calculate

$$\nu_1=-d^4\big[g^2+(h-1)^2\big]^2\big[9g^2+(3h+1)^2\big]Z_1/256. \tag{16}$$

We observe that

$$\operatorname{sign}(\nu_1)=-\operatorname{sign}(Z_1),$$

and this means that this invariant polynomial is responsible for what kind of singularities are $M_{1,2}$: are they real or complex, distinct or coinciding.

***a*.1)** *The case* $\nu_1 < 0$. Then $Z_1 > 0$ and we obtain that the singularities $M_{1,2}$ located on the invariant ellipse are real. We need to determine the conditions when at least one of the singularities $M_{3,4}$ will also lie on the ellipse. For this, considering (10), we calculate

$$\begin{aligned}\Phi(x,y)|_{\{x=x_{3,4},\ y=y_{3,4}\}} &= \frac{d^2g\left(g^2+h^2\right) \pm d\left(g^2+h^2\right)\sqrt{g\left(4ah+d^2g\right)}+2ah\left(g^2+h^2+h\right)}{2gh^2}\\ &\equiv \Omega_{3,4}(a,g,h).\end{aligned}$$

It is clear that at least one of the singularities $M_3(x_3,y_3)$ or $M_4(x_4,y_4)$ belongs to the ellipse (10) if and only if

$$\Omega_3\Omega_4 = \frac{aZ_3}{g^2h^2} = 0, \qquad Z_3 = d^2g\left(g^2+h^2\right)+a\left(g^2+h^2+h\right)^2.$$

On the other hand for systems (9) we have

$$\nu_2 = -105d\left[9g^2+(3h+1)^2\right]Z_3, \tag{17}$$

and clearly by (12) the condition $\nu_2 = 0$ is equivalent to $Z_3 = 0$. So we conclude that the following remark is valid:

Remark 8. *Assume that for systems* (9) *the conditions* (12) *and* $h \neq 0$ *(i.e.* $\mu_0 \neq 0$*) hold. Then at least one of the singularities* M_3, M_4 *belongs to the ellipse if and only if* $\nu_2 = 0$.

Next we examine two subcases: $\nu_2 \neq 0$ and $\nu_2 = 0$.

$\boldsymbol{\alpha}$) *The subcase* $\nu_2 \neq 0$. In this case we have *Config. E.1* since another singularity belongs to ellipse if and only if $\nu_2 = 0$ (example: $a = 1, d = -1, g = -1, h = -2$).

$\boldsymbol{\beta}$) *The subcase* $\nu_2 = 0$. In this case we have $Z_3 = 0$, i.e. at least one of the two other singular points will also lie on the ellipse. Moreover $g^2+h^2+h \neq 0$ otherwise we obtain a contradiction with the conditions stated at (12). So the condition $Z_3 = 0$ implies $a = -\dfrac{d^2g\left(g^2+h^2\right)}{\left(g^2+h^2+h\right)^2}$. In this case two singularities coalesce, namely $M_4 \equiv M_2$ and considering the coordinates of $M_i, i = 1,2,3,4$, we obtain three singularities

$$(x_1,y_1) = \left(-\frac{d[h+(2+h)(g^2+h^2)]}{(g^2+h^2+h)[g^2+(h+1)^2]},\ \frac{dg\left(g^2+h^2-1\right)}{\left(g^2+h^2+h\right)\left[g^2+(h+1)^2\right]}\right)$$

and

$$(x_2,y_2) = \left(-\frac{dh}{g^2+h^2+h},\ \frac{dg}{g^2+h^2+h}\right), \quad (x_3,y_3) = \left(-\frac{d\left(g^2+h^2\right)}{g^2+h^2+h},\ \frac{dg\left(g^2+h^2\right)}{h\left(g^2+h^2+h\right)}\right).$$

Therefore we have located on the ellipse a double singularity M_2 and a simple singularity M_1. On the other hand we have

$$\Phi(x,y)|_{\{x=x_3,y=y_3\}} = \frac{d^2(g^2+h^2)(g^2+h^2-h)}{h^2(g^2+h+h^2)} \equiv \frac{d^2(g^2+h^2)Z_3'}{h^2(g^2+h+h^2)}.$$

Thus we conclude that the singularity M_3 will belong to the ellipse if and only if $Z_3' = 0$. Now taking into consideration Proposition 1 (see Table 1), for systems (9) in this case we calculate

$$\mathbf{D} = 0,\ \mathbf{T} = \frac{12d^6g^2\left(Z_3'\right)^2}{-\left(g^2+h^2+h\right)^4}(gy-hx-x)^2(gx+hy)^2\left[gx\left(g^2+h^2+1\right)+hy\left(g^2+h^2-1\right)\right]^2,$$

and due to (12) the condition $\mathbf{T} = 0$ is equivalent to $Z_3' = 0$, i.e. the invariant polynomial $\mathbf{T}$ indicates if the third singularity belongs to the ellipse or not. We discuss two possibilities:

***β*.1)** *The possibility* $\mathbf{T} \neq 0$. In this case we obtain *Config. E.2* (example: $a = 5/9, d = -1, g = -1, h = -2$).

***β*.2)** *The possibility* $\mathbf{T} = 0$. In this case we have $Z_3' = 0$, i.e. $g^2 + h^2 = h$. Substituting this expression in the coordinates (x_3, y_3) we obtain that M_3 coincides with M_2. So we deduce that we have one triple and one simple singularities located on the ellipse. As a result we arrive at *Config. E.3* (example: $a = -1, d = -1, g = 4/17, h = 1/17$).

***a*.2)** *The case* $\nu_1 = 0$. In this case we have $Z_1 = 0$ (see (14)), i.e. the two singularities which belong to the ellipse coalesce. On the other hand the two singularities which are located outside the ellipse remains outside the ellipse because the condition $Z_1 = 0$ implies $a = -\dfrac{d^2 g}{g^2 + (h+1)^2}$ and for this value of the parameter a we obtain $Z_3 = \dfrac{d^2 g^3}{g^2 + (h+1)^2} \neq 0$. In such a way we get *Config. E.4* (example: $a = 1/2, d = -1, g = -1, h = -2$).

***a*.3)** *The case* $\nu_1 > 0$. Then $Z_1 < 0$, i.e. the two singularities which belong to the ellipse are complex. We note that the condition $Z_1 < 0$ implies $Z_3 \neq 0$, because if $Z_3 = 0$ we found $Z_1 = \frac{d^2 g^4}{(g^2+h^2+h)^2} > 0$. This leads to *Config. E.5* (example: $a = 1/4, d = -1, g = -1, h = -2$).

We claim that in this configuration the invariant ellipse is a limit cycle drown in diagram in boldface (see Remark 2 (b)). Indeed taking into consideration Theorem 3 (see statement **(B_1)**) we conclude that in the case under examination for the existence of limit cycles the following conditions must be satisfied:

$$\eta < 0, \quad \theta \widehat{\beta}_1 \widehat{\beta}_2 \widehat{\beta}_3 \neq 0, \; \widehat{\gamma}_1 = \widehat{\gamma}_2 = 0, \; \widehat{\mathcal{R}}_1 < 0, \quad \mathcal{T}_3 \mathcal{F} < 0. \tag{18}$$

Clearly all the conditions are satisfied except the last one. So it remains to verify that $\mathcal{T}_3\mathcal{F} < 0$ is fulfilled, too. For systems (9) we calculate

$$\begin{aligned} \mathcal{T}_3\mathcal{F} =& -\frac{1}{8} d^2 g \left[(9g^2 + (3h+1)^2\right]^2 \left(ag^2 + ah^2 + 2ah + a + d^2 g\right), \\ \nu_1 =& \frac{1}{256} d^4 g \left[(g^2 + (h-1)^2\right]^2 \left[(9g^2 + (3h+1)^2\right]^2 \left(ag^2 + ah^2 + 2ah + a + d^2 g\right), \end{aligned}$$

and evidently the condition $\nu_1 > 0$ implies $\mathcal{T}_3\mathcal{F} < 0$. This completes the proof of our claim.

***b*)** *The possibility* $\mu_0 = 0$. This condition implies $h = 0$ and the condition (12) becomes $adg \neq 0$. In this case we obtain $\mu_1 = dg(g^2+1)x \neq 0$. According to Lemma 1 we conclude that exactly one of the four finite singularities has gone to infinity and coalesced with the real infinity singularity. So we obtain one real infinite singularity of multiplicity two which is of type $(1,1)$ (i.e. one finite and one infinity singularities coalesced, see Remark 2).

For $h = 0$ considering the coordinates of $M_{1,2}(x_{1,2}, y_{1,2})$ (see (14)) we obtain that these two singularities remain located on the ellipse (10). On the other hand from (15) it is not so difficult to determine that the singularity M_3 has gone to infinity and a straightforward calculation gives us the coordinates of the fourth singularity: $M_4\left(0, -\dfrac{a}{d}\right)$.

Again we consider the value of ν_1 and we examine three cases:

***b*.1)** *The case* $\nu_1 < 0$. Then we have $Z_1 > 0$ and this implies the existence of two real distinct singularities located on the ellipse. On the other hand considering (10) we have

$$\Phi(x,y)|_{\{x=x_4,y=y_4\}} = \frac{a\left(d^2+ag\right)}{d^2 g},$$

and since $a \neq 0$ the singularity M_4 belongs to the ellipse if and only if $d^2 + ag = 0$. Calculating $\nu_2 = -105dg^3\left(9g^2+1\right)\left(d^2+ag\right)$ we conclude that the singularity M_4 will belong to the ellipse if and only if $\nu_2 = 0$. So we discuss two subcases: $\nu_2 \neq 0$ and $\nu_2 = 0$.

$\boldsymbol{\alpha}$) *The subcase* $\nu_2 \neq 0$. Then the singularity M_4 remains outside the ellipse and we arrive at *Config. E.6* (example: $a = 5/8, d = -1, g = -1$).

$\boldsymbol{\beta}$) *The subcase* $\nu_2 = 0$. This implies $a = -\dfrac{d^2}{g}$ and we obtain that the singularity M_4 coincides with M_2. As a result we arrive at *Config. E.7* (example: $a = -1, d = -1, g = 1$).

***b*.2)** *The case* $\nu_1 = 0$. In this case we have $Z_1 = 0$, i.e. $a = -\dfrac{d^2 g}{g^2+1}$ (see (14)) and therefore the two singularities which belong to the ellipse coalesce. On the other hand we calculate $\nu_2 = -\dfrac{105d^3g^3\left(9g^2+1\right)}{g^2+1} \neq 0$ and this means that M_4 remains outside the ellipse. So we arrive at *Config. E.8* (example: $a = 1/2, d = -1, g = -1$).

***b*.3)** *The case* $\nu_1 > 0$. In this case we have $Z_1 < 0$, i.e. the two singularities which belong to the ellipse are complex. On the other hand this fact implies that $\nu_2 \neq 0$. Therefore we have *Config. E.9* (example: $a = 1/4, d = -1, g = -1$).

We claim that in this configuration the invariant ellipse is a limit cycle, too (see Remark 2 (b)). For this it is sufficient to show that the conditions (18) are satisfied in this particular case, when $\mu_0 = 0$ (i.e. $h = 0$). Indeed, for systems (9) with $h = 0$ we obtain

$$\begin{aligned} \mathcal{T}_3\mathcal{F} &= -\frac{1}{8}d^2 g\left(9g^2+1\right)^2\left(ag^2+a+d^2g\right), \\ \nu_1 &= \frac{1}{256}d^4 g\left(g^2+1\right)^2\left(9g^2+1\right)^2\left(ag^2+a+d^2g\right) \end{aligned}$$

and clearly the condition $\nu_1 > 0$ implies $\mathcal{T}_3\mathcal{F} < 0$, i.e. our claim is proved.

2) *The subcase* $B_1 = 0$. Considering the condition (12) we obtain that $B_1 = 0$ (see (13)) is equivalent to $a = -\dfrac{d^2 g}{(h+1)^2}$ which implies the existence of the invariant line

$$\mathcal{L}(x,y) = (h+1)x + d = 0. \tag{19}$$

On the other hand for this value of the parameter a we obtain

$$B_2 = -\frac{648d^4\left[g^2+(h-1)^2\right]^2\left(g^2+h^2\right)x^4}{(h+1)^4},$$

which is nonzero due to condition (12). It follows from Lemma 2 and Lemma 4 that the conditions $B_1 = 0$, $B_2 \neq 0$ and $\theta \neq 0$ implies that there exists at most one simple invariant straight line of systems (9). On the other hand for these systems we have $\mu_0 = -h\left[g^2+(h+1)^2\right]$ and we examine two possibilities: $\mu_0 \neq 0$ and $\mu_0 = 0$.

***a*) *The possibility* $\mu_0 \neq 0$.** Then the condition $\mu_0 = -h\big[g^2 + (h+1)^2\big] \neq 0$ gives $h \neq 0$ and considering condition (12) by Lemma 1 the systems (9) have finite singularities of total multiplicity 4. Taking into account the coordinates of the singularities $M_{i,j}(x_{i,j}, y_{i,j})$ $(i = j = 1, 2, 3, 4)$ mentioned earlier (see page 27) in this particular case these singularities have the following real coordinates

$$\begin{gathered}(x_1, y_1) = \left(-\frac{d}{h+1}, 0\right), \quad (x_2, y_2) = \left(\frac{d\left[g^2 - (h+1)^2\right]}{(h+1)\left[g^2 + (h+1)^2\right]}, \frac{2dg}{g^2 + (h+1)^2}\right), \\ (x_3, y_3) = \left(-\frac{dh}{h+1}, \frac{dg}{h+1}\right), \quad (x_4, y_4) = \left(-\frac{d}{h+1}, \frac{dg}{h(h+1)}\right).\end{gathered} \tag{20}$$

We observe that due to $a = -\dfrac{d^2 g}{(h+1)^2}$ the invariant ellipse for systems (9) becomes

$$\Phi(x, y) = x^2 + y^2 - \frac{d^2}{(h+1)^2}.$$

As it was shown earlier, the singularities $M_{1,2}(x_{1,2}, y_{1,2})$ are located on the ellipse and the singularities $M_{3,4}(x_{3,4}, y_{3,4})$ are generically located outside the ellipse. We also determine that the singularities M_1 and M_4 are located on the invariant line. On the other hand in generic case the singularities M_2 and M_3 could not belong to the line since calculations yield

$$\mathcal{L}(x_2, y_2) = \frac{2dg^2}{g^2 + (h+1)^2}, \qquad \mathcal{L}(x_3, y_3) = d(1-h).$$

Due to the condition (12) we get $\mathcal{L}(x_2, y_2) \neq 0$, i.e. the ellipse and the invariant line have M_1 as the unique common point at which a line is tangent to the ellipse.

Considering Remark 8 we conclude that one of the singularities M_3 or M_4 belongs to the ellipse if and only if $\nu_2 = 0$. So, in what follows we discuss two cases: $\nu_2 \neq 0$ and $\nu_2 = 0$.

***a.1*) *The case* $\nu_2 \neq 0$.** Then according to Remark 8 neither the singularity M_3 nor M_4 could belong to the ellipse. On the other hand the singularity M_4 is located on the invariant line whereas the singularity M_3 belongs to the invariant line if and only if $\mathcal{L}(x_3, y_3) = d(1-h) = 0$. Due to $d \neq 0$ we obtain the condition $h = 1$. We observe that this condition is governed by the invariant polynomial $\mathbf{D}$ because for systems (9) in the case $a = -\dfrac{d^2 g}{(h+1)^2}$ we calculate

$$\nu_2 = \frac{105 d^3 g^3 \left(g^2 + h^2 - 1\right)\left[9g^2 + (3h+1)^2\right]}{(h+1)^2}, \quad \mathbf{D} = -\frac{192 d^8 g^6 (h-1)^2 \left(g^2 + h^2 - 1\right)^2}{(h+1)^8}, \tag{21}$$

and due to the condition $\nu_2 \neq 0$ we obtain that the condition $\mathbf{D} = 0$ is equivalent to $h = 1$. We examine two subcases: $\mathbf{D} \neq 0$ and $\mathbf{D} = 0$.

***α*) *The subcase* $\mathbf{D} \neq 0$.** Then $h \neq 1$, i.e. the singularity M_3 remains outside the invariant curves and this leads to *Config. E.10* (example: $d = -1, g = 1, h = -2$).

***β*) *The subcase* $\mathbf{D} = 0$.** In this case we have $h = 1$ and considering (15) we obtain that M_3 coalesces with M_4 which is located on the invariant line and we arrive at *Config. E.11* (example: $d = 1, g = 1, h = 1$).

a.2) *The case* $\nu_2 = 0$. Due to (12), from (21) we obtain that the condition $\nu_2 = 0$ gives $g^2+h^2-1 = 0$. This implies $\mathbf{D} = 0$, and moreover we have $h \neq 1$ due to $g \neq 0$. We observe that setting $g^2 = 1-h^2$ in the expressions of the coordinates of (x_2, y_2) from (20) we obtain that $(x_2, y_2) = (x_3, y_3)$. So on the ellipse we get a double singularity and this leads to *Config. E.12* (example: $d = -1, g = \sqrt{3}/2, h = -1/2$).

b) *The possibility* $\mu_0 = 0$. Then the condition $\mu_0 = -h\big[g^2 + (h+1)^2\big] = 0$ implies $h = 0$ and we obtain $\mu_1 = dg(g^2+1)x \neq 0$. According to Lemma 1 we conclude that exactly one of the four finite singularities has gone to infinity and coalesced with the real infinite singularity. So we obtain one real infinite singularity of multiplicity two which is of type $(1,1)$ (see Remark 2). Considering the coordinates of the finite singularities given in (20) we observe that M_4 has gone to infinity along the invariant line $\mathcal{L} = 0$ and the remaining real finite singularities are

$$(x_1, y_1) = (-d, 0), \quad (x_2, y_2) = \left(\frac{d\,(g^2-1)}{g^2+1}, \frac{2dg}{g^2+1}\right), \quad (x_3, y_3) = (0, dg). \tag{22}$$

In order to determine the position of the singularity M_3 we calculate

$$\mathcal{L}(x_3, y_3) = d, \quad \Phi(x_3, y_3) = d^2(g^2-1).$$

Due to the condition (12) we obtain $\mathcal{L}(x_3, y_3) \neq 0$, i.e. M_3 could not belong to the invariant line. On the other hand $\Phi(x_3, y_3) = 0$ if and only if $g^2 - 1 = 0$. We observe that systems (9) in the case under examination (i.e. $h = 0$ and $a = -d^2 g$) become

$$\dot{x} = -d^2 g + dy + gx^2 + xy, \quad \dot{y} = -dx + gxy - x^2. \tag{23}$$

We determine that the condition $g^2 - 1 = 0$ is equivalent to $\nu_2 = 0$ because for the above systems we have

$$\nu_2 = 105 d^3 g^3 \left(9g^2+1\right)\left(g^2-1\right).$$

So we discuss two cases: $\nu_2 \neq 0$ and $\nu_2 = 0$.

b.1) *The case* $\nu_2 \neq 0$. In this case the singularity M_3 remains outside the invariant curves and this leads to *Config. E.13* (example: $d = -1, g = -2, h = 0$).

b.2) *The case* $\nu_2 = 0$. This implies $g = \pm 1$. However we can consider $g = 1$ due to the rescaling $(x, y, t) \mapsto (x, -y, -t)$ in systems (23) which changes the sign of the parameter g. In this case considering (22) we obtain $(x_3, y_3) = (x_2, y_2)$ and as a result we get *Config. E.14* (example: $d = -1, g = 1, h = 0$).

3.1.1.1.2 The case $\widehat{\mathcal{R}}_1 > 0$.

This condition implies $ag > 0$ and clearly the ellipse (10) is complex. On the other hand considering (13) we observe that for systems (9) the conditions (12) and $ag > 0$ imply $B_1 \neq 0$. Then by Lemma 2 systems (9) could not possess invariant lines.

For these systems we calculate $\mu_0 = -h\big[g^2 + (h+1)^2\big]$ and we examine two subcases: $\mu_0 \neq 0$ and $\mu_0 = 0$.

1) *The subcase* $\mu_0 \neq 0$. Then by Lemma 1, systems (9) have finite singularities of total multiplicity 4 and their coordinates are given in (14). We observe that the condition $ag > 0$ implies $Z_1 < 0$, i.e.

the singularities $M_{1,2}(x_{1,2}, y_{1,2})$ are complex and as it was proved earlier they belong to the complex ellipse.

On the other hand the condition $ag > 0$ implies $Z_3 = d^2g\left(g^2+h^2\right) + a\left(g^2+h^2+h\right)^2 \neq 0$. This fact implies that the singularities $M_{3,4}(x_{3,4}, y_{3,4})$ remain outside the complex ellipse. Therefore the unique possible configuration is *Config. E.15* (example: $a = -1, d = -1, g = -1, h = -2$).

2) *The subcase* $\mu_0 = 0$. This condition implies $h = 0$. In this case we obtain $\mu_1 = dg(g^2+1)x \neq 0$ and by Lemma 1 only one finite singularity coalesced with the real infinite singularity which has multiplicity $(1,1)$ (see Remark 2). Therefore we arrive at *Config. E.16* (example: $a = -1, d = -1, g = -1$).

Thus, we have all the configurations indicated in DIAGRAM 5 in the block corresponding to the possibility $(\mathfrak{B}_1)$.

3.1.1.2 The possibility $(\mathfrak{B}_2)$: $\widehat{\beta}_1\widehat{\beta}_2 \neq 0, \widehat{\beta}_3 = 0$. According to [16] in this case by an affine transformation and time rescaling systems (8) could be brought to the canonical form (9) with $h = 1/3$, i.e. we get a subfamily of (9) which was investigated in the previous subsection. As it was shown, for $h \neq 1/3$ systems (9) possess 16 configurations *Config. E.1 - Config. E.16*. Moreover it is necessary to highlight that the value $h = 1/3$ is not a bifurcation value for distinguishing these configurations.

It remains to find out which conditions defining each one of the configurations are compatible in this case. We claim that the configurations (i) *Config. E.6 - Config. E.9*, *Config. E.13*, *Config. E.14*, *Config. E.16* and (ii) *Config. E.11*, could not be realizable for systems (9) with $h = 1/3$.

Indeed, for each one of the configuration from the group (i) condition $\mu_0 = 0$ is necessary. However, for $h = 1/3$ we have $\mu_0 = -\left(9g^2+16\right)/27 \neq 0$, i.e. the configurations from the group (i) could not be realized for $h = 1/3$.

Secondly, as it was shown in the previous subsection, a system (9) possesses the configuration *Config. E.11* if and only if the following conditions hold:

$$B_1 = 0, \quad \mu_0 \neq 0, \quad \nu_2 \neq 0, \quad \mathbf{D} = 0.$$

However, in the case $h = 1/3$ the condition $B_1 = 0$ yields $a = -\left(9d^2g\right)/16$ and then we calculate

$$\nu_2 = \frac{105d^3g^3\left(9g^2-8\right)\left(9g^2+4\right)}{16}, \quad \mathbf{D} = -\frac{-27d^8g^6\left(9g^2-8\right)^2}{256}.$$

Evidently, the condition $\nu_2 \neq 0$ implies $\mathbf{D} \neq 0$, and this completes the proof of our claim.

Therefore in this case (i.e. for $h = 1/3$) we have *Config. E.1* (example: $a = 1, d = -1, g = -1$), *Config. E.2* (example: $a = 90/169, d = -1, g = -1$), *Config. E.3* (example: $a = \sqrt{2}/4, d = -1, g = -\sqrt{2}/3$), *Config. E.4* (example: $a = 9/25, d = -1, g = -1$), *Config. E.5* (example: $a = 1/4, d = -1, g = -1$), *Config. E.10* (example: $a = -1, d = -1, g = 1/2$), *Config. E.12* (example: $a = -1, d = -1, g = 2\sqrt{2}/3$) and *Config. E.15* (example: $a = -1, d = -1, g = -1$).

Next we discuss the existence of limit cycle for this subfamily of systems (9) defined by $h = 1/3$. We observe that due to $\widehat{\beta}_3 = 0$ the conditions (18) are not satisfied. On the other hand according to Theorem 3, statement $(\boldsymbol{B_2})$ we have the following necessary and sufficient conditions for existence of limit cycles:

$$\eta < 0, \quad \theta\widehat{\beta}_1\widehat{\beta}_2 \neq 0, \ \widehat{\beta}_3 = \widehat{\gamma}_1 = \widehat{\gamma}_2 = \widehat{\gamma}_3 = 0, \ \widehat{\mathcal{R}}_1 < 0, \quad \mathcal{T}_3\mathcal{F} < 0. \tag{24}$$

Since for the configuration *Config. E.5* the conditions $\widehat{\mathcal{R}}_1 < 0$ and $\nu_1 > 0$ hold, considering Remark 1 we deduce that so far for systems (9) with $h = 1/3$ all the above conditions with the exeption of $\mathcal{T}_3\mathcal{F} < 0$ are fulfilled. We now calculate

$$\begin{aligned} \mathcal{T}_3\mathcal{F} =& -\frac{1}{72}d^2g\left(9g^2+4\right)^2\left(9ag^2+16a+9d^2g\right), \\ \nu_1 =& \frac{d^4g\left(9g^2+4\right)^4\left(9ag^2+16a+9d^2g\right)}{186624}, \end{aligned}$$

and evidently the condition $\nu_1 > 0$ implies $\mathcal{T}_3\mathcal{F} < 0$. So the conditions (24) are satisfied and the ellipse from *Config. E.5* is a limit cycle.

It is not too difficult to determine that for the remaining configurations (i.e. excluding the configurations of the groups (i) and (ii) defined above) all the corresponding conditions are compatible and this is confirmed by the examples presented above.

Thus, for all the realizable configurations for systems (9) with $h = 1/3$ we obtain the conditions presented in DIAGRAM 5 in the block corresponding to the possibility $(\mathfrak{B}_2)$.

3.1.1.3 The possibility $(\mathfrak{B}_3)$: $\widehat{\beta}_1 \neq 0, \widehat{\beta}_2 = 0, \widehat{\beta}_5 \neq 0$. As it was proved in [16], in this case by an affine transformation and time rescaling systems (8) could be brought to the canonical form

$$\dot{x} = dy + (h+1)xy, \qquad \dot{y} = b - dx - x^2 + hy^2, \tag{25}$$

which possesses an invariant conic

$$\Phi(x,y) = \frac{b}{h} + x^2 + y^2 = 0, \quad h \neq 0. \tag{26}$$

This conic is irreducible if and only if $b \neq 0$. For the above systems we calculate

$$\begin{aligned} \theta = (h+1)(h-1)^2/2, \quad & \widehat{\beta}_1 = -d^2(h-1)^2(3h+1)^2/16, \\ \widehat{\beta}_5 = -2(h+1)(3h-1), \quad & \widehat{\mathcal{R}}_2 = bh(h+1)^2(h-1)^2(3h+1)^4/8, \end{aligned} \tag{27}$$

and therefore we conclude that for the above systems the condition $\theta\widehat{\beta}_1\widehat{\beta}_5\widehat{\mathcal{R}}_2 \neq 0$ is equivalent to the condition

$$bdh(h-1)(h+1)(3h-1)(3h+1) \neq 0. \tag{28}$$

On the other hand we have

$$\widehat{\mathcal{R}}_2 = bh(h+1)^2(h-1)^2(3h+1)^4/8 \ \Rightarrow \ \mathrm{sign}\,(bh) = \mathrm{sign}\,(\widehat{\mathcal{R}}_2).$$

3.1.1.3.1 The case $\widehat{\mathcal{R}}_2 < 0$. Then $bh < 0$ and clearly the ellipse (26) is real.

We observe that systems (25) possess the invariant line

$$\mathcal{L}(x,y) = (h+1)x + d = 0. \tag{29}$$

Then by Lemma 2 the condition $B_1 = 0$ is satisfied. Moreover, since $\theta \neq 0$, by Lemmas 2 and 4 systems (25) could possess another invariant straight line only if $B_2 = 0$. We calculate

$$B_2 = -648b^2(h-1)^4x^4, \tag{30}$$

and due to condition (28) we have $B_2 \neq 0$. So systems (25) possess exactly one invariant straight line $\mathcal{L}(x,y) = 0$.

For these systems we calculate $\mu_0 = -h(h+1)^2$ and due to condition (28) we have $\mu_0 \neq 0$. Considering Lemma 1 the systems (25) have finite singularities of total multiplicity 4. We observe that two of these singularities are located on the ellipse (26) as well as on the invariant line (29). More precisely these are the singularities $M_{1,2}\big(x_{1,2}, y_{1,2}\big)$ with

$$x_{1,2} = -\frac{d}{h+1}, \quad y_{1,2} = \pm\frac{\sqrt{Z_1'}}{h+1}, \quad Z_1' = -\Big[d^2 + \frac{b}{h}(h+1)^2\Big]. \tag{31}$$

Other two singularities of systems (25) are $M_{3,4}\big(x_{3,4}, y_{3,4}\big)$ (generically located outside both invariant curves) with

$$x_{3,4} = \frac{-d \pm \sqrt{Z_2'}}{2}, \quad y_{3,4} = 0, \quad Z_2' = 4b + d^2. \tag{32}$$

On the other hand for systems (25) we calculate

$$\nu_3 = -2h^2(h+1)^2(3h+1)^2 Z_1'. \tag{33}$$

We observe that

$$\operatorname{sign}(\nu_3) = -\operatorname{sign}(Z_1'),$$

and this means that the invariant polynomial ν_3 determines if the singularities $M_{1,2}$ are either real or complex, distinct or coinciding.

1) *The subcase* $\nu_3 < 0$. Then $Z_1' > 0$ and the singularities $M_{1,2}$ are real. We need to determine the conditions when at least one of the singularities $M_{3,4}$ located outside the invariant curves coincides with one of their points. In this sense considering (26) and (29) we calculate

$$\begin{aligned}
\Phi(x,y)|_{\{x=x_{3,4},\ y=y_{3,4}\}} &= \frac{2bh + 2b + d^2h \mp dh\sqrt{Z_2'}}{2h} \equiv \Omega_{3,4}'(b,d,h),\\
\mathcal{L}(x,y)|_{\{x=x_{3,4},\ y=y_{3,4}\}} &= \frac{d(1-h) \pm (h+1)\sqrt{Z_2'}}{2} \equiv \mathcal{L}_{3,4}'(b,d,h).
\end{aligned}$$

It is clear that at least one of the singularities $M_3(x_3, y_3)$ or $M_4(x_4, y_4)$ belongs to the ellipse (26) or to the line (29) if and only if the conditions

$$\Omega_3'\Omega_4' = \frac{b\big[b(h+1)^2 + d^2h\big]}{h^2} = \frac{-bZ_1'}{h^2} = 0 \quad \text{or} \quad \mathcal{L}_3'\mathcal{L}_4' = -\big[d^2h + b(h+1)^2\big] = Z_1' = 0, \tag{34}$$

are satisfied, respectively.

We observe that the conditions $Z_1' \neq 0$ and (28) imply $\Omega_3'\Omega_4'\mathcal{L}_3'\mathcal{L}_4' \neq 0$. Therefore none of the points $M_{3,4}$ could belong to the ellipse or to the line. So we arrive at *Config. E.17* (example: $b = -1, d = -1, h = 1/4$).

2) *The subcase* $\nu_3 = 0$. Then $Z_1' = 0$ which implies $b = -\dfrac{d^2h}{(h+1)^2}$ and therefore the singularities M_1 and M_2 coalesce. It is clear that in this case the invariant line and the ellipse have a unique common point at which they are tangent and this point is a double singularity. However according to (34) in this case we have $\Omega_3'\Omega_4' = 0$ and $\mathcal{L}_3'\mathcal{L}_4' = 0$, and this imply that the singularity M_3 also belongs to both curves. More exactly we obtain $M_3 = M_2 = M_1$ which leads to a triple singularity. On the other hand due to the condition (28) we obtain

$$\Phi(x,y)|_{\{x=x_4,\ y=y_4\}} = \frac{d^2(h-1)}{h+1} \neq 0, \quad \mathcal{L}(x,y)|_{\{x=x_4,\ y=y_4\}} = d(1-h) \neq 0,$$

i.e. the singularity M_4 remains outside both invariant curves. So the only possible configuration is *Config. E.18* (example: $b = 2, d = -1, h = -2$).

3) *The subcase* $\nu_3 > 0$. Then $Z_1' < 0$ and this implies that the singularities M_1 and M_2 located at the intersections of the invariant ellipse with the invariant line are complex. On the other hand we observe that due to conditions (28) and (34) we have $\Omega_3'\Omega_4'\mathcal{L}_3'\mathcal{L}_4' \neq 0$ and therefore none of the points $M_{3,4}$ could belong to the ellipse or to the line. It is not difficult to convince ourselves that in the case under examination we obtain *Config. E.19* (example: $b = -1/8, d = -1, h = 1/4$).

We claim that in this configuration the invariant ellipse is not a limit cycle. Indeed since for systems (25) we have $\mathcal{T}_3\mathcal{F} = 0$, by Theorem 3 (see statement **(B)**) we conclude that our claim is valid.

3.1.1.3.2 The case $\widehat{\mathcal{R}}_2 > 0$.

Then $bh > 0$ and in this case the ellipse (26) is complex. According to (31) the condition $bh > 0$ implies $Z_1' < 0$, i.e. evidently the singularities $M_{1,2}$ located on the ellipse also are complex. Therefore we arrive at *Config. E.20* (example: $b = 1, d = -1, h = 1/6$).

Thus we have all the configurations indicated in Diagram 5 in the block corresponding to the possibility $(\mathfrak{B}_3)$.

3.1.1.4 The possibility $(\mathfrak{B}_4)$: $\widehat{\beta}_1 \neq 0, \widehat{\beta}_2 = \widehat{\beta}_5 = 0$.

According to [16] in this case by an affine transformation and time rescaling systems (8) could be brought to the canonical form (25) with $h = 1/3$, i.e. we get a subfamily of (25) which was investigated in the previous subsection. As it was shown, for $h \neq 1/3$ systems (25) possess four configurations, namely *Config. E.17*, *Config. E.18*, *Config. E.19* and *Config. E.20*. Moreover it is necessary to point out that the value $h = 1/3$ is not a bifurcation value for the corresponding configurations.

It is not too difficult to determine that all the configurations are realizable in the case $h = 1/3$, too. In fact, for *Config. E.17* we take $b = -11/16$ and $d = -1$, for *Config. E.18* we put $b = -3/16$ and $d = -1$, for *Config. E.19* we write $b = -3/32$ and $d = -1$ and finally for *Config. E.20* we consider $b = 1$ and $d = -1$.

So we obtain the condition presented in Diagram 5 in the block corresponding to the possibility $(\mathfrak{B}_4)$.

3.1.1.5 The possibility $(\mathfrak{B}_5)$: $\widehat{\beta}_1 = 0, \widehat{\beta}_6 \neq 0, \widehat{\beta}_2 \neq 0$.

As it was proved in [16] in this case by an affine transformation and time rescaling systems (8) could be brought to the canonical form

$$\dot{x} = a + gx^2 + (h+1)xy, \qquad \dot{y} = \frac{ah}{g} - x^2 + gxy + hy^2, \tag{35}$$

which possesses an invariant conic (of the elliptic type)

$$\Phi(x,y)=\frac{a}{g}+x^2+y^2=0,\quad g\neq 0. \tag{36}$$

This conic is irreducible if and only if $a\neq 0$. For the above systems we calculate

$$\begin{aligned}\theta&=\frac{1}{2}(h+1)\left[g^2+(h-1)^2\right],\quad \widehat{\beta}_6=(3h+1)\left[9g^2+(3h+1)^2\right]/8,\\ \widehat{\beta}_2&=-g\left[g^2+(3h+1)^2\right]/2,\quad \widehat{\mathcal{R}}_3=160ag\left(g^2+h^2\right)\left[g^2+(3h+1)^2\right],\end{aligned} \tag{37}$$

and therefore we conclude that for the above systems the condition $\theta\widehat{\beta}_2\widehat{\beta}_6\widehat{\mathcal{R}}_3\neq 0$ is equivalent to the condition

$$ag(h+1)(3h+1)\neq 0. \tag{38}$$

Taking into account Lemma 2 we examine if systems (35) could possess at least one invariant line. Calculations yield

$$B_1=-\frac{a^3(h+1)^2\left[g^2+(h-1)^2\right]^2\left(g^2+h^2\right)}{g^2}, \tag{39}$$

and due to condition (38) we obtain $B_1\neq 0$. In this case by Lemma 2 we can conclude that systems (35) could not possess invariant lines.

On the other hand we have

$$\widehat{\mathcal{R}}_3=160ag\left(g^2+h^2\right)\left[g^2+(3h+1)^2\right]\ \Rightarrow\ \operatorname{sign}(ag)=\operatorname{sign}(\widehat{\mathcal{R}}_3).$$

3.1.1.5.1 The case $\widehat{\mathcal{R}}_3<0$. Then $ag<0$ and clearly the ellipse (36) is real.

For systems (35) we calculate $\mu_0=-h\big[g^2+(h+1)^2\big]$ and we examine two subcases: $\mu_0\neq 0$ and $\mu_0=0$.

1) *The subcase* $\mu_0\neq 0$. Then by Lemma 1 the systems have finite singularities of total multiplicity 4. We detect that two of these singularities are located on the ellipse (36), more exactly such singularities are $M_{1,2}\big(x_{1,2},y_{1,2}\big)$ with

$$x_{1,2}=\pm\frac{\sqrt{Z_1}}{g^2+(h+1)^2},\quad y_{1,2}=\pm\frac{(h+1)\sqrt{Z_1}}{g\big[g^2+(h+1)^2\big]},\quad Z_1=-ag\big[g^2+(h+1)^2\big]. \tag{40}$$

Other two singularities of systems (35) are $M_{3,4}\big(x_{3,4},y_{3,4}\big)$ (generically located outside the ellipse) with

$$x_{3,4}=\pm\frac{\sqrt{Z_2}}{g},\quad y_{3,4}=\mp\frac{\sqrt{Z_2}}{h},\quad Z_2=agh. \tag{41}$$

Since $\operatorname{sign}(\widehat{\mathcal{R}}_3)=-\operatorname{sign}(Z_1)$, the condition $\widehat{\mathcal{R}}_3<0$ implies $Z_1>0$. In this case we have two real distinct singularities on the ellipse, namely $M_{1,2}$.

We need to determine the conditions when at least one of the singularities $M_{3,4}$ located outside the ellipse coincide with its points. In this order considering (36) we calculate

$$\Phi(x,y)|_{\{x=x_{3,4},\ y=y_{3,4}\}}=\frac{aZ_3}{gh},\quad Z_3=g^2+h(h+1).$$

It is clear that at least one of the singularities $M_3(x_3, y_3)$ or $M_4(x_4, y_4)$ belongs to the ellipse (36) if and only if $Z_3 = 0$.

On the other hand for systems (35) we have

$$\mathbf{D} = \frac{768a^4h\left[g^2 + (h+1)^2\right] Z_3^4}{g^4},$$

and clearly due to the conditions (38) and $\mu_0 \neq 0$ the condition $\mathbf{D} = 0$ is equivalent to $Z_3 = 0$.

a) *The possibility* $\mathbf{D} \neq 0$. Then $Z_3 \neq 0$ and the singularities $M_{3,4}$ remain outside the ellipse and we have *Config. E.1* (example: $a = 1, g = -3/2, h = -2$).

b) *The possibility* $\mathbf{D} = 0$. Then $Z_3 = 0$, i.e. $g^2 + h(h+1) = 0$. In order to use this relation, due to $h \neq 0$ we apply the following parametrization: $g = g_1 h$ and then $h = -1/(g_1^2 + 1)$. Considering the coordinates (40) and (41) we obtain

$$x_{1,2} = \pm\frac{\sqrt{ag_1}}{g_1}, \quad y_{1,2} = \mp\sqrt{ag_1}\,; \quad x_{3,4} = \mp\frac{\sqrt{ag_1}}{g_1}, \quad y_{3,4} = \pm\sqrt{ag_1},$$

and we observe that M_3 coincides with M_2 and M_4 coincides with M_1. As a result we arrive at *Config. E.21* (example: $a = -1, g = 1/4, h = -(2 + \sqrt{3})/4$).

2) *The subcase* $\mu_0 = 0$. In this subcase we get $h = 0$ and this implies $\mu_1 = 0$ and $\mu_2 = ag\left(g^2 + 1\right) x^2 \neq 0$ due to the condition (38). Therefore by Lemma 1 exactly two finite singularities have gone to infinity. More exactly according to the factorization of μ_2 by the same lemma we deduce that both points coalesced with the infinite singularity $[0 : 1 : 0]$. So we obtain a triple singularity at infinity of the type $(1, 2)$ (see Remark 2), and this leads to *Config. E.22* (example: $a = 1, g = -1$).

3.1.1.5.2 The case $\widehat{\mathcal{R}}_3 > 0$. This condition implies $ag > 0$ and clearly the ellipse (36) is complex. For systems (35) we have $\mu_0 = -h\big[g^2 + (h+1)^2\big]$ and we examine two subcases: $\mu_0 \neq 0$ and $\mu_0 = 0$.

1) *The subcase* $\mu_0 \neq 0$. Then by Lemma 1 the systems (35) have finite singularities of total multiplicity 4 and their coordinates are given in (40). We observe that the condition $ag > 0$ implies $Z_1 < 0$, i.e. the singularities $M_{1,2}\big(x_{1,2}, y_{1,2}\big)$ are complex and as it was proved earlier they belong to a complex ellipse.

It is not to difficult to determine that the condition $Z_3 = 0$ in this case implies $h < 0$ and due to $ag > 0$ we obtain that the singularities $M_{3,4}$ are also complex. Moreover the condition $Z_3 = 0$ forces them to coalesce with the two complex singularities on the ellipse. Therefore on the complex ellipse we get two double complex singularities. This is however irrelevant in view of the definition of a configuration (see Definition 4).

Thus we conclude that in both cases $Z_3 \neq 0$ and $Z_3 = 0$ we arrive at the same configuration, namely *Config. E.15* (examples: $a = -1, g = -3/2, h = -2$ and $a = -1, g = -249/512, h = -(256 + \sqrt{3535})/512$, respectively).

2) *The subcase* $\mu_0 = 0$. In this case we have $h = 0$ and as it was shown in the case $\widehat{\mathcal{R}}_3 < 0$ we get at infinity a triple singularity of the type $(1, 2)$ (see Remark 2), and this leads to *Config. E.23* (example: $a = -1, g = -1$).

In this way we have all the configurations indicated in DIAGRAM 5 in the block corresponding to the possibility $(\mathfrak{B}_5)$.

3.1.1.6 The possibility $(\mathfrak{B}_6)$: $\widehat{\beta}_1 = 0, \widehat{\beta}_6 \neq 0, \widehat{\beta}_2 = 0$.

As it was proved in [16] in this case by an affine transformation and time rescaling systems (8) could be brought to the canonical form

$$\dot{x} = (h+1)xy, \qquad \dot{y} = b - x^2 + hy^2, \tag{42}$$

which possesses an invariant conic (of the elliptic type)

$$\Phi(x,y) = \frac{b}{h} + x^2 + y^2 = 0, \quad h \neq 0. \tag{43}$$

This conic is irreducible if and only if $b \neq 0$. For the above systems we calculate

$$\theta = (h+1)(h-1)^2/2, \quad \widehat{\beta}_6 = (3h+1)^3/8, \quad \widehat{\mathcal{R}}_2 = bh(3h+1)^4\left(h^2-1\right)^2/8, \tag{44}$$

and therefore we conclude that for the above systems the condition $\theta\widehat{\beta}_6\widehat{\mathcal{R}}_2 \neq 0$ is equivalent to the condition

$$bh(h+1)(h-1)(3h+1) \neq 0. \tag{45}$$

On the other hand we have

$$\operatorname{sign}(bh) = \operatorname{sign}(\widehat{\mathcal{R}}_2).$$

3.1.1.6.1 The case $\widehat{\mathcal{R}}_2 < 0$.

This condition implies $bh < 0$ and clearly the ellipse (43) is real.

Taking into account Lemma 2 we examine if systems (42) could possess at least one invariant line. Calculations yield $B_1 = 0$ and $B_2 = -648b^2(h-1)^4x^4$ which is nonzero due to condition (45). It follows from Lemma 2 and Lemma 4 that the conditions $B_1 = 0$, $B_2 \neq 0$ and $\theta \neq 0$ imply that there exists exactly one simple invariant straight line of systems (42), namely

$$\mathcal{L}(x,y) = x = 0. \tag{46}$$

Moreover for these systems we have $\mu_0 = -h(h+1)^2$ which is nonzero due to condition (45). Therefore in this case the coordinates of the finite singularities for the systems (42) could be obtained from the coordinates described in (31) and (32) setting $d = 0$:

$$\begin{gathered} x_{1,2} = 0, \quad y_{1,2} = \pm\sqrt{Z_1}, \quad Z_1 = -\frac{b}{h}, \\ x_{3,4} = \pm\sqrt{Z_2}, \quad y_{3,4} = 0, \quad Z_2 = b. \end{gathered} \tag{47}$$

We observe that the singularities $M_{1,2}$ belong to the ellipse as well as to the invariant line, and the singularities $M_{3,4}$ are generically located outside both invariant curves.

Since $\operatorname{sign}(\widehat{\mathcal{R}}_2) = -\operatorname{sign}(Z_1)$, the condition $\widehat{\mathcal{R}}_2 < 0$ implies $Z_1 > 0$. In this case the singularities $M_{1,2}$ are real and distinct.

We need to determine the conditions when at least one of the singularities $M_{3,4}$, in general located outside the invariant curves, will lie on these curves. In this order considering (43), (46) and (47) we calculate

$$\Phi(x,y)|_{\{x=x_{3,4},\ y=y_{3,4}\}} = \frac{b(h+1)}{h} \equiv \Omega, \quad \mathcal{L}(x,y)|_{\{x=x_{3,4},\ y=y_{3,4}\}} = \pm\sqrt{b} \equiv \mathcal{L}_{3,4}.$$

It is clear that at least one of the singularities M_3 or M_4 belongs to the ellipse (43) and the invariant line (46) if and only if $\Omega = 0$ or $\mathcal{L}_3\mathcal{L}_4 = 0$, respectively. Due to conditions (45) none of these conditions could hold. As a result we arrive at *Config. E.17* (example: $b = 1, h = -2$).

3.1.1.6.2 The case $\widehat{\mathcal{R}}_2 > 0$. This condition implies $bh > 0$ and clearly the ellipse (43) is complex. According to (47) the condition $bh > 0$ implies $Z_1 < 0$, i.e. evidently the singularities $M_{1,2}$ located on the ellipse are also complex. Thus we arrive at *Config. E.20* (example: $b = 1, h = 1/2$).

So we have all the configurations indicated in DIAGRAM 5 in the block corresponding to the possibility $(\mathfrak{B}_6)$.

3.1.1.7 The possibility $(\mathfrak{B}_7)$: $\widehat{\beta}_1 = \widehat{\beta}_6 = 0, \widehat{\beta}_2 \neq 0$. As it was proved in [16] in this case by an affine transformation and time rescaling systems (8) could be brought to the canonical form

$$\dot{x} = a + gx^2 + \frac{2xy}{3}, \quad \dot{y} = -\frac{a}{3g} - x^2 - \frac{y^2}{3} + gxy, \tag{48}$$

which possesses an invariant conic (of the elliptic type)

$$\Phi(x, y) = \frac{a}{g} + x^2 + y^2 = 0, \quad g \neq 0. \tag{49}$$

This conic is irreducible if and only if $a \neq 0$. For the above systems we calculate

$$\theta = \left(9g^2 + 16\right)/27, \quad \widehat{\beta}_2 = -g^3/2, \quad \widehat{\mathcal{R}}_3 = 160ag^3\left(9g^2 + 1\right)/9,$$

and therefore we conclude that for the above systems the condition $\theta\widehat{\beta}_2\widehat{\mathcal{R}}_3 \neq 0$ is equivalent to the condition $ag \neq 0$. Moreover we clearly have $\operatorname{sign}(ag) = \operatorname{sign}(\widehat{\mathcal{R}}_3)$.

On the other hand for systems (48) we calculate

$$B_1 = -\frac{4a^3\left(9g^2 + 1\right)\left(9g^2 + 16\right)^2}{6561g^2} \neq 0, \tag{50}$$

due to $ag \neq 0$. Therefore by Lemma 2 we conclude that systems (48) could not possess invariant lines.

3.1.1.7.1 The case $\widehat{\mathcal{R}}_3 < 0$. Then $ag < 0$ and clearly the ellipse (49) is real.

For systems (48) we calculate $\mu_0 = \left(9g^2 + 4\right)/27 \neq 0$. Then by Lemma 1 the systems have finite singularities of total multiplicity 4. We detect that two of these singularities are located on the ellipse (49), more exactly such singularities are $M_{1,2}\big(x_{1,2}, y_{1,2}\big)$ with

$$x_{1,2} = \pm\frac{3\sqrt{Z_1}}{9g^2 + 4}, \quad y_{1,2} = \pm\frac{2\sqrt{Z_1}}{g\big(9g^2 + 4\big)}, \quad Z_1 = -ag\big(9g^2 + 4\big). \tag{51}$$

Other two singularities of systems (48) are $M_{3,4}\big(x_{3,4}, y_{3,4}\big)$ (generically located outside the ellipse) with

$$x_{3,4} = \pm\frac{\sqrt{-3ag}}{3g}, \quad y_{3,4} = \pm\sqrt{-3ag}. \tag{52}$$

Since $\operatorname{sign}(\widehat{\mathcal{R}}_3) = -\operatorname{sign}(Z_1)$, the condition $\widehat{\mathcal{R}}_3 < 0$ implies $Z_1 > 0$. In this case we have two real distinct singularities on the ellipse, namely $M_{1,2}$.

We need to determine the conditions when at least one of the singularities $M_{3,4}$, in general located outside the ellipse, will lie on the ellipse. In this order considering (49) we calculate

$$\Phi(x,y)|_{\{x=x_{3,4},\ y=y_{3,4}\}} = aZ_3, \quad Z_3 = -\frac{9g^2-2}{3g}.$$

It is clear that at least one of the singularities $M_3(x_3, y_3)$ or $M_4(x_4, y_4)$ belongs to the ellipse (49) if and only if $Z_3 = 0$. We observe that the invariant polynomial $\mathbf{D}$ is responsible for this condition because for systems (48) we have

$$\mathbf{D} = -\frac{256a^4\left(9g^2+4\right)Z_3^4}{59049g^4}, \quad ag \neq 0.$$

1) *The subcase* $\mathbf{D} \neq 0$. Then $Z_3 \neq 0$ and the singularities $M_{3,4}$ remain outside the ellipse and we have *Config. E.1* (example: $a = -1, g = 1/4$).

2) *The subcase* $\mathbf{D} = 0$. Then $Z_3 = 0$, i.e. $9g^2 - 2 = 0$. In this case we have $g = \pm\sqrt{2}/3$. Without loss of generality we can assume $g = \sqrt{2}/3$ since the rescaling $(x, y, t) \to (-x, y, t)$ simultaneously change the signs of the parameters a and g of the systems (48). So $g = \sqrt{2}/3$ and for the coordinates of the singularities $M_{1,2}$ and $M_{3,4}$ of these systems we have

$$x_{1,2} = \pm\frac{\sqrt{-a}}{\sqrt[4]{2}} = x_{3,4}, \quad y_{1,2} = \pm\sqrt[4]{2}\sqrt{-a} = y_{3,4}.$$

So we obtain that the singularity M_3 (respectively M_4) coalesces with M_1 (respectively M_2). As a result we have two double singularities located on the ellipse which leads to *Config. E.21* (example: $a = -75, g = \sqrt{2}/3$).

3.1.1.7.2 The case $\widehat{\mathcal{R}}_3 > 0$. This condition implies $ag > 0$ and clearly the ellipse (49) is complex. Since for systems (48) we have $\mu_0 \neq 0$, by Lemma 1 these systems have finite singularities of total multiplicity 4 and their coordinates are given in (51) and (52). We observe that the condition $ag > 0$ implies $Z_1 < 0$, i.e. the singularities $M_{1,2}\big(x_{1,2}, y_{1,2}\big)$ are complex and as it was proved earlier they belong to the complex ellipse.

It is not to difficult to determine that the singularities $M_{3,4}$ are also complex. Moreover the condition $\mathbf{D} = 0$ forces them to coalesce with the two complex singularities on the ellipse as we discussed in the case $\widehat{\mathcal{R}}_3 < 0$. So we get two double complex singularities located on the complex ellipse.

Thus considering Definition 4 we conclude that in both cases, i.e. $\mathbf{D} \neq 0$ and $\mathbf{D} = 0$, we arrive at the same configuration, namely *Config. E.15* (examples: $a = -1, g = -1$ and $a = 27, g = \sqrt{2}/3$, respectively).

Therefore we have all the configurations indicated in DIAGRAM 5 in the block corresponding to the possibility $(\mathfrak{B}_7)$.

3.1.1.8 The possibility $(\mathfrak{B}_8)$: $\widehat{\beta}_1 = \widehat{\beta}_6 = \widehat{\beta}_2 = 0$. As it was proved in [16] in this case by an affine transformation and time rescaling, systems (8) could be brought to the canonical form

$$\dot{x} = 2xy/3, \qquad \dot{y} = b - x^2 - y^2/3, \tag{53}$$

which possesses an invariant conic

$$\Phi(x, y) = -3b + x^2 + y^2 = 0. \tag{54}$$

This conic is irreducible if and only if $b \neq 0$. For the above systems we calculate

$$\theta = 16/27, \quad \widehat{\mathcal{R}}_4 = -32b\left(3x^2 + y^2\right)/9, \tag{55}$$

and therefore we conclude that for systems (53) the condition $\widehat{\mathcal{R}}_4 \neq 0$ is equivalent to the condition $b \neq 0$. On the other hand we have $\mathrm{sign}\,(b) = -\mathrm{sign}\,(\widehat{\mathcal{R}}_4)$.

3.1.1.8.1 The case $\widehat{\mathcal{R}}_4 < 0$. This condition implies $b > 0$ and clearly the ellipse (54) is real.

We observe that systems (53) possess the invariant line $x = 0$. Then by Lemma 2 the condition $B_1 = 0$ is satisfied. Moreover by this lemma systems (53) could not possess another invariant line because $B_2 = -2048b^2x^4 \neq 0$ due to the condition $b \neq 0$. So systems (53) possess exactly one invariant line $x = 0$. For these systems we have $\mu_0 = 4/27 \neq 0$.

On the other hand in this case the coordinates of the finite singularities for the systems (53) could be obtained from the coordinates described in (47) setting $h = -1/3$:

$$x_{1,2} = 0, \quad y_{1,2} = \pm\sqrt{3b}\,; \qquad x_{3,4} = \pm\sqrt{b}\,, \quad y_{3,4} = 0. \tag{56}$$

We detect that the singularities $M_{1,2}$ belong to the ellipse as well as to the invariant line $x = 0$. Since $b > 0$ all four singularities are real and $M_{3,4}$ are located outside the invariant curves. As a result we arrive at *Config. E.17* (example: $b = 1$).

3.1.1.8.2 The case $\widehat{\mathcal{R}}_4 > 0$. This condition implies $b < 0$ and clearly the ellipse (54) is complex. According to (56) the condition $b < 0$ implies that the singularities $M_{1,2}$ located on the ellipse also are complex. Thus we arrive at *Config. E.20* (example: $b = -1$).

Then we have all the configurations indicated in Diagram 5 in the block corresponding to the possibility $(\mathfrak{B}_8)$.

3.1.2 The subcase $\theta = 0$

We examine step by step each one of the possibilities presented in Corollary 1 and corresponding to this case.

3.1.2.1 The possibility $(\mathfrak{B}_9)$: $\widetilde{N} \neq 0, \widehat{\beta}_1 \neq 0, \widehat{\beta}_2 \neq 0$. As it was proved in [16] in this case by an affine transformation and time rescaling systems (8) could be brought to the canonical form

$$\begin{aligned} \dot{x} &= a + dy + gx^2, \\ \dot{y} &= -\frac{a}{g} - dx - x^2 + gxy - y^2, \quad g \neq 0, \end{aligned} \tag{57}$$

which possesses an invariant conic

$$\Phi(x, y) = \frac{a}{g} + x^2 + y^2 = 0. \tag{58}$$

This conic is irreducible if and only if $a \neq 0$. For the above systems we calculate

$$\begin{gathered}\widetilde{N} = \left(g^2+4\right) x^2, \quad \widehat{\beta}_1 = -d^2\left(g^2+4\right)\left(9g^2+4\right)/16,\\ \widehat{\beta}_2 = -g\left(g^2+4\right)/2, \quad \widehat{\mathcal{R}}_5 = 12ag\left(g^2+4\right),\end{gathered} \tag{59}$$

and therefore we conclude that for systems (57) the condition $\widetilde{N}\widehat{\beta}_1\widehat{\beta}_2\widehat{\mathcal{R}}_5 \neq 0$ is equivalent to the condition $adg \neq 0$. On the other hand we observe that $\operatorname{sign}(ag) = \operatorname{sign}(\widehat{\mathcal{R}}_5)$.

Taking into account Lemma 2 we calculate

$$B_1 = -\frac{a^2d^2\left(g^2+1\right)\left(g^2+4\right)^2}{g} \neq 0, \tag{60}$$

due to condition $adg \neq 0$, and we conclude that systems (57) could not possess invariant lines.

3.1.2.1.1 The case $\widehat{\mathcal{R}}_5 < 0$. Then $ag < 0$ and clearly the ellipse (58) is real.

For systems (57) we calculate $\mu_0 = g^2 \neq 0$ and therefore by Lemma 1 the systems have finite singularities of total multiplicity 4.

We detect that two of these singularities are located on the ellipse (58), more exactly the singularities $M_{1,2}\big(x_{1,2}, y_{1,2}\big)$, with

$$x_{1,2} = \pm\frac{\sqrt{Z_1}}{g}, \quad y_{1,2} = \frac{d}{g}, \quad Z_1 = -(ag+d^2). \tag{61}$$

Other two singularities of systems (57) are $M_{3,4}\big(x_{3,4}, y_{3,4}\big)$ (generically located outside the ellipse), with

$$x_{3,4} = -\frac{dg \pm \sqrt{Z_2}}{2g}, \quad y_{3,4} = -\frac{1}{2}\left(dg \pm \sqrt{Z_2}\right), \quad Z_2 = g\left(d^2g - 4a\right). \tag{62}$$

On the other hand for systems (57) we calculate

$$\nu_1 = -d^4g^2\left(g^2+4\right)^2\left(9g^2+4\right)^2 Z_1/256. \tag{63}$$

We observe that

$$\operatorname{sign}(\nu_1) = -\operatorname{sign}(Z_1),$$

and this means that the singularities $M_{1,2}$ are real (respectively, complex; coinciding) if $\nu_1 < 0$ (respectively, $\nu_1 > 0$ and $\nu_1 = 0$).

1) *The subcase* $\nu_1 < 0$. Then the singularities $M_{1,2}$ located on the invariant ellipse are real. We need to determine the conditions when at least one of the singularities $M_{3,4}$ located outside the ellipse lies on the ellipse. For this, considering (58) and (62), we calculate

$$\Phi(x, y)\big|_{\{x=x_{3,4},\ y=y_{3,4}\}} = \frac{d^2\left(g^3+g\right) - 2ag^2 \pm d\left(g^2+1\right)\sqrt{g\left(d^2g-4a\right)}}{2g} \equiv \Omega_{3,4}(a, d, g).$$

It is clear that at least one of the singularities $M_3(x_3, y_3)$ or $M_4(x_4, y_4)$ belongs to the ellipse (58) if and only if

$$\Omega_3\Omega_4 = \frac{aZ_3}{g} = 0, \qquad Z_3 = ag^3 + d^2\left(g^2+1\right).$$

On the other hand for systems (57) we have

$$\nu_2 = -105dg\left(9g^2+4\right)Z_3,$$

and clearly since $adg \neq 0$ the condition $\nu_2 = 0$ is equivalent to $Z_3 = 0$.

Next we examine two possibilities: $\nu_2 \neq 0$ and $\nu_2 = 0$.

a) *The possibility* $\nu_2 \neq 0$. In this case we have *Config. E.1* since other singularities could belong to ellipse if and only if $\nu_2 = 0$ (example: $a = 5/4, d = 1, g = -1$).

b) *The possibility* $\nu_2 = 0$. In this case we have $Z_3 = 0$ and since $g \neq 0$ this condition gives $a = -\dfrac{d^2\left(g^2+1\right)}{g^3}$. Therefore we obtain the following coordinates of the singularities $M_i, i = 1, 2, 3, 4$:

$$(x_1, y_1) = \left(\frac{d}{g^2}, \frac{d}{g}\right), \quad (x_2, y_2) = \left(-\frac{d}{g^2}, \frac{d}{g}\right),$$
$$(x_3, y_3) = \left(-\frac{d\left(g^2+1\right)}{g^2}, -\frac{d\left(g^2+1\right)}{g}\right), \quad (x_4, y_4) = \left(\frac{d}{g^2}, \frac{d}{g}\right).$$

As we can observe, the singularity M_4 coalesced with M_1. Therefore on the ellipse we have a double singularity M_1 and a simple singularity M_2. On the other hand we have

$$\Phi(x,y)|_{\{x=x_3, y=y_3\}} = \frac{d^2\left(g^4+3g^2+2\right)}{g^2} \neq 0,$$

due to $d \neq 0$. Hence the singularity M_3 remains outside the ellipse and we arrive at *Config. E.2* (example: $a = -1, d = \sqrt{2}/2, g = 1$).

2) *The subcase* $\nu_1 = 0$. In this case we have $Z_1 = 0$, i.e. $a = -d^2 g$ and this implies $Z_3 = d^2 \neq 0$. Therefore the two singularities which belong to the ellipse coalesce, whereas other two singularities remain outside the ellipse. In this way we get *Config. E.4* (example: $a = 1, d = 1, g = -1$).

3) *The subcase* $\nu_1 > 0$. Then we have $Z_1 < 0$, i.e. the two singularities which belong to the ellipse are complex. We note that the condition $Z_1 < 0$ implies $Z_3 \neq 0$, because if $Z_3 = 0$ we found $Z_1 = d^2/g^2 > 0$. This leads to *Config. E.5* (example: $a = 1/2, d = 1, g = -1$).

We claim that in this configuration the invariant ellipse is a limit cycle. Indeed, taking into consideration Theorem 3 (see statement $(\boldsymbol{B_3})$) we conclude that in the case under examination, for the existence of limit cycles the following conditions must be satisfied:

$$\eta < 0,\ \theta = \widehat{\gamma}_1 = \widehat{\gamma}_2 = \widehat{\gamma}_6 = 0, \widehat{\beta}_1\widehat{\beta}_2 \neq 0,\ \widehat{\mathcal{R}}_5 < 0,\ \mathcal{T}_3\mathcal{F} < 0.$$

Clearly all the conditions are satisfied except the last one. So it remains to verify that $\mathcal{T}_3\mathcal{F} < 0$ is fulfilled, too. For systems (57) we calculate

$$\nu_1 = \frac{1}{256}d^4g^2\left(g^2+4\right)^2\left(9g^2+4\right)^2\left(ag+d^2\right), \quad \mathcal{T}_3\mathcal{F} = -\frac{1}{8}d^2g^2\left(9g^2+4\right)^2\left(ag+d^2\right),$$

and evidently the condition $\nu_1 > 0$ implies $\mathcal{T}_3\mathcal{F} < 0$. This completes the proof of our claim.

3.1.2.1.2 The case $\widehat{\mathcal{R}}_5 > 0$. This condition implies $ag > 0$ and clearly the ellipse (58) is complex.

As we discuss on the case $\widehat{\mathcal{R}}_5 < 0$ for these systems we have $\mu_0 \neq 0$. Then by Lemma 1, systems (57) have finite singularities of total multiplicity 4 and the coordinates are given in (61) and (62). We observe that the condition $ag > 0$ implies $Z_1 < 0$, i.e. the singularities $M_{1,2}\big(x_{1,2}, y_{1,2}\big)$ are complex and as it was proved earlier they belong to the complex ellipse.

On the other hand the condition $ag > 0$ yields $Z_3 = ag^3 + d^2\left(g^2+1\right) \neq 0$. This fact implies that the singularities $M_{3,4}\big(x_{3,4}, y_{3,4}\big)$ remain outside the complex ellipse. Therefore the unique possible configuration is *Config. E.15*, detected before (example: $a = -1, d = 1, g = -1$).

Thus, we have all the configurations indicated in Diagram 6 in the block corresponding to the possibility $(\mathfrak{B}_9)$.

3.1.2.2 The possibility $(\mathfrak{B}_{10})$: $\widetilde{N} \neq 0, \widehat{\beta}_1 \neq 0, \widehat{\beta}_2 = 0$. As it was proved in [16] in this case by an affine transformation and time rescaling systems (8) could be brought to the 2-parameter family of systems

$$\dot{x} = dy, \qquad \dot{y} = b - dx - x^2 - y^2, \qquad d \neq 0, \tag{64}$$

which is a subfamily of (25) defined by the condition $h = -1$. Clearly these systems possess the same invariant ellipse (26) which in this particular case takes the form

$$\Phi(x, y) = -b + x^2 + y^2 = 0. \tag{65}$$

This conic is irreducible if and only if $b \neq 0$.

We claim that the infinite invariant line $Z = 0$ for systems (64) is of multiplicity 2. Indeed considering Lemma 3 for these systems we calculate:

$$\gcd(\mathcal{E}_1, \mathcal{E}_2) = dZ,$$

and by Lemma 3, statement *(3)*, the line $Z = 0$ is a double one. This completes the proof of our claim.

For the above systems we calculate

$$\widetilde{N} = 16bd^2x^2, \quad \widehat{\beta}_1 = -d^2, \quad \widehat{\mathcal{R}}_6 = -4b, \tag{66}$$

and therefore we conclude that for these systems the condition $\widetilde{N}\widehat{\beta}_1\widehat{\mathcal{R}}_6 \neq 0$ is equivalent to the condition $bd \neq 0$. On the other hand we have $\operatorname{sign}(b) = -\operatorname{sign}(\widehat{\mathcal{R}}_6)$.

3.1.2.2.1 The case $\widehat{\mathcal{R}}_6 < 0$. Then $b > 0$ and clearly the ellipse (65) is real.

Considering Lemma 2 we examine if systems (64) could possess at least one invariant affine line. Calculations yield $B_1 = 0$, however we claim that this condition is implied by the existence of the double line at the infinity. Indeed, the coefficients of systems (64) could be perturbed with a small parameter $0 < \varepsilon \ll 1$ as follows:

$$\dot{x} = (d + \varepsilon x)y, \qquad \dot{y} = b - dx - x^2 - y^2.$$

Evidently, these systems possess the invariant line $d+\varepsilon x=0$ and hence by Lemma 2 we have $B_1=0$ (this could also be checked directly).

On the other hand, by Lemma 2, systems (64) could not possess any finite invariant lines because $B_2=-10368b^2x^4\neq 0$, due to condition $bd\neq 0$.

For these systems we calculate $\mu_0=\mu_1=0$ and $\mu_2=d^2(x^2+y^2)\neq 0$. According to Lemma 1, two finite singularities coalesced with infinite singularities, namely with the complex singularities $[\pm i:1:0]$.

Therefore on the line $Z=0$ we get two double complex infinite singularities. This is however irrelevant in view of the definition of a configuration (see Definition 4). So the unique real singularity at infinity is of multiplicity one.

On the other hand, by Lemma 1, the systems (64) have finite singularities of total multiplicity 2. We detect that these singularities are located outside the ellipse (65) and their coordinates are $M_{1,2}\big(x_{1,2},y_{1,2}\big)$, with

$$x_{1,2}=-\frac{d\pm\sqrt{4b+d^2}}{2},\quad y_{1,2}=0. \tag{67}$$

For these singularities we calculate

$$\Phi(x,y)|_{\{x=x_{1,2},\ y=y_{1,2}\}}=\frac{d(d\pm\sqrt{4b+d^2})}{2}\equiv\Omega_{1,2}(b,d),\quad \Omega_1\Omega_2=-bd^2\neq 0.$$

We conclude that neither M_1 nor M_2 could belong to the ellipse. As a result we arrive at *Config. E.24* (example: $b=1,d=-1$).

3.1.2.2.2 The case $\widehat{\mathcal{R}}_6>0$. Then $b<0$ and clearly the ellipse (65) is complex. Since none of the singularities $M_{1,2}$ could be on the ellipse, we obtain *Config. E.25* (example: $b=-1,d=-1$).

So we have all the configurations indicated in DIAGRAM 6 in the block corresponding to the possibility ($\mathfrak{B}_{10}$).

3.1.2.3 The possibility ($\mathfrak{B}_{11}$) : $\widetilde{N}\neq 0,\widehat{\beta}_1=0$. As it was proved in [16] in this case by an affine transformation and time rescaling, systems (8) could be brought to the canonical form (9) with $h=-1$ and $d=0$. So we consider the following systems

$$\dot{x}=a+gx^2,\quad \dot{y}=-\frac{a}{g}-x^2+gxy-y^2,\quad g\neq 0, \tag{68}$$

which possess the invariant ellipse (10), i.e.

$$\Phi(x,y)=\frac{a}{g}+x^2+y^2=0, \tag{69}$$

which is irreducible if and only if $a\neq 0$.

We observe that systems (68) possess the invariant lines $\mathcal{L}_{1,2}(x,y)=a+gx^2=0$, i.e.

$$\mathcal{L}_1(x,y)=x-\frac{\sqrt{-ag}}{g}=0,\quad \mathcal{L}_2(x,y)=x+\frac{\sqrt{-ag}}{g}=0. \tag{70}$$

Then by Lemma 2 the condition $B_1 = 0$ is satisfied. Moreover by this lemma the systems (68) could possess an invariant line in another direction only if $B_2 = 0$. However, for these systems we have

$$B_2 = -\frac{648a^2\left(g^2+1\right)\left(g^2+4\right)^2 x^4}{g^2} \neq 0. \tag{71}$$

So we conclude that systems (68) possess exactly two invariant lines (70), which are distinct due to $ag \neq 0$ and they could be real or complex, depending on $\operatorname{sign}(ag)$.

For the above systems we calculate

$$\widetilde{N} = \left(g^2+4\right)x^2, \quad \widehat{\mathcal{R}}_3 = 160ag\left(g^2+1\right)\left(g^2+4\right), \tag{72}$$

and therefore we conclude that $\operatorname{sign}(ag) = \operatorname{sign}(\widehat{\mathcal{R}}_3)$.

3.1.2.3.1 The case $\widehat{\mathcal{R}}_3 < 0$. Then $ag < 0$ which implies that the ellipse (69) as well as the invariant lines (70) are real.

For systems (68) we calculate $\mu_0 = g^2 \neq 0$ due to the condition $g \neq 0$. Then by Lemma 1 the systems have finite singularities of total multiplicity 4.

We detect that two of these singularities are located on the ellipse (69), more exactly such singularities are $M_{1,2}\big(x_{1,2}, y_{1,2}\big)$, with

$$x_{1,2} = \pm\frac{\sqrt{-ag}}{g}, \quad y_{1,2} = 0. \tag{73}$$

Other two singularities of systems (68) are $M_{3,4}\big(x_{3,4}, y_{3,4}\big)$ (generically located outside the ellipse), with

$$x_{3,4} = \pm\frac{\sqrt{-ag}}{g}, \quad y_{3,4} = \pm\sqrt{-ag}. \tag{74}$$

We also detect that the singularities $M_{1,3}$ (respectively, $M_{2,4}$) belong to the line $\mathcal{L}_1(x,y) = 0$ (respectively $\mathcal{L}_2(x,y) = 0$).

For the singularities $M_{3,4}$ we calculate

$$\Phi(x,y)|_{\{x=x_{3,4},\ y=y_{3,4}\}} = -ag \neq 0,$$

and we conclude that neither $M_3(x_3,y_3)$ nor $M_4(x_4,y_4)$ belong to the ellipse (69). According to (74) we observe that $y_3 > 0$ and $y_4 < 0$, which leads to *Config. E.26* (example: $a = 1, g = -1$).

3.1.2.3.2 The case $\widehat{\mathcal{R}}_3 > 0$. Then $ag > 0$ and this implies that the ellipse (69) as well as the invariant lines (70) and the four singularities of systems (68) are complex. Therefore we obtain *Config. E.27* (example: $a = -1, g = -1$).

Therefore we have all the configurations indicated in DIAGRAM 6 in the block corresponding to the possibility $(\mathfrak{B}_{11})$.

3.1.2.4 The possibility $(\mathfrak{B}_{12}): \ \widetilde{N} = 0.$ As it was proved in [16] in this case by an affine transformation and time rescaling systems (8) could be brought to the systems

$$\dot{x} = 2xy, \quad \dot{y} = b - x^2 + y^2, \tag{75}$$

which possess the family of invariant ellipses

$$\Phi(x, y) = b + qx + x^2 + y^2 = 0, \quad q \in \mathbb{R}, \tag{76}$$

depending on the parameter q and having the corresponding determinant $\Delta = (4b - q^2)/4$. So for any fixed value of the parameter q, the ellipses from the family (76) are irreducible if and only if $\Delta \neq 0$.

Since for systems (75) we have $B_1 = B_2 = B_3 = 0$, by Lemma 2 these systems could possess invariant lines in three different directions. We verify that these systems indeed possess the following five invariant lines:

$$\mathcal{L}_1(x, y) = x = 0, \quad \mathcal{L}_{2,4}(x, y) = (x - iy)^2 - b = 0, \quad \mathcal{L}_{3,5}(x, y) = (x + iy)^2 - b = 0. \tag{77}$$

Since $\mu_0 = -4 \neq 0$ the above systems possess finite singularities of total multiplicity four and their coordinates are

$$x_{1,2} = 0, \quad y_{1,2} = \pm\sqrt{-b}, \qquad x_{3,4} = \pm\sqrt{b}, \quad y_{3,4} = 0.$$

We observe that if $b \neq 0$ systems (75) have two real and two complex singularities. Moreover we have that the real singularities are located on the real invariant line $\mathcal{L}_1(x, y) = 0$ if $b < 0$ (namely $M_{1,2}$) and outside this invariant line if $b > 0$ (in this case the real singularities are M_3 and M_4). In the case $b = 0$ all four singularities coincide and we have one real singularity of multiplicity four.

On the other hand for systems (75) we calculate $\widehat{\mathcal{R}}_7 = 32b$, which implies $\operatorname{sign}(b) = \operatorname{sign}(\widehat{\mathcal{R}}_7)$. It is clear that if $\widehat{\mathcal{R}}_7 \leq 0$ the invariant ellipses from the family (76) are real and if $\widehat{\mathcal{R}}_7 > 0$ they could be real or complex, depending on the parameter q in (76). So, if $\widehat{\mathcal{R}}_7 < 0$ we arrive at *Config. E.28* (example: $b = -75, q \in \mathbb{R}$) and if $\widehat{\mathcal{R}}_7 = 0$ the complex invariant lines (77) become double and we obtain *Config. E.29* (example: $b = 0, q \neq 0$). In the case $\widehat{\mathcal{R}}_7 > 0$ we observe that for any fixed value of parameter b, any ellipse from the family (76) is invariant for systems (75). In other words, for any $b > 0$ we have that such systems possess simultaneously an infinite number of real ellipses as well as an infinite number of complex ellipses. Therefore, taking into consideration Remark 3 we arrive at *Config. E.30* (example: $b = 1/8, |q| > \sqrt{2}/2$).

Thus, we have all the configurations indicated in Diagram 6 in the block corresponding to the possibility $(\mathfrak{B}_{12})$.

3.2 The case $C_2 = 0$

According to Lemma 5 a quadratic system with the condition $C_2 = 0$ could be brought via an affine transformation and time rescaling to the following canonical form:

$$\dot{x} = a + cx + dy + x^2, \quad \dot{y} = b + ex + fy + xy, \tag{78}$$

with $C_2 \equiv 0$, i.e. the line at infinity of this system is filled up with singularities. Following [16] (see Diagram 1) and Corollary 1 we discuss two possibilities.

3.2.1 The possibility ($\mathfrak{C}_1$): $H_{10} \neq 0$.

As it was proved in [16] in this case we have the systems

$$\dot{x} = a + y + x^2, \quad \dot{y} = xy, \tag{79}$$

which possess the family of invariant ellipses

$$\Phi(x, y) = a + 2y + x^2 + m^2 y^2 = 0, \quad m \neq 0. \tag{80}$$

We observe that for any fixed value of the parameter a, the ellipses from the family (80) are irreducible if and only if $\Delta = am^2 - 1 \neq 0$.

Since for systems (79) we have $B_1 = B_2 = B_3 = 0$, by Lemma 2 these systems could possess invariant lines in three different directions. We verify that these systems indeed possess the following three invariant lines:

$$\mathcal{L}_1(x, y) = y = 0, \quad \mathcal{L}_{2,3}(x, y) = y \pm \sqrt{-a}\, x + a = 0. \tag{81}$$

It is clear that the invariant lines $\mathcal{L}_2(x, y)$ and $\mathcal{L}_3(x, y)$ are real, coinciding or complex if $a < 0$, $a = 0$ or $a > 0$, respectively.

Since $\mu_0 = 0$ and $\mu_1 = x \neq 0$, Lemma 1 tells us that there exist exactly three finite singularities. Their coordinates are given by

$$x_{1,2} = \pm\sqrt{-a}, \quad y_{1,2} = 0, \qquad x_3 = 0, \quad y_3 = -a.$$

For each fixed value of the parameter a we observe that if $a < 0$ (respectively, $a > 0$) the singularities $M_{1,2}$ are real (respectively, complex) and they verify the equalities

$$\Phi(x_1, y_1) = \mathcal{L}_1(x_1, y_1) = \mathcal{L}_2(x_1, y_1) = 0, \quad \Phi(x_2, y_2) = \mathcal{L}_1(x_2, y_2) = \mathcal{L}_3(x_2, y_2) = 0,$$

respectively. Moreover we detect that the singularity M_3 belongs to the invariant lines $\mathcal{L}_2(x, y)$ and $\mathcal{L}_3(x, y)$ and this singularity belongs to the irreducible invariant ellipse $\Phi(x, y) = 0$ if and only if $a = 0$. In such a case (i.e. when $a = 0$) all three singularities coincide (as well the three invariant lines $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$) and we have one real singularity of multiplicity three located on a triple invariant line.

On the other hand for systems (79) we calculate $H_9 = 2304a^3$, which implies $\operatorname{sign}(a) = \operatorname{sign}(H_9)$. It is clear that if $H_9 \leq 0$ then the irreducible conics from the family (80) are real and if $H_9 > 0$ they could be real or complex, depending on the parameter m in (80). So, if $H_9 < 0$ we arrive at *Config. E.31* (example: $a = -75, m \in \mathbb{R}\backslash\{0\}$) and if $H_9 = 0$ we obtain *Config. E.32* (example: $a = 0, m \in \mathbb{R}\backslash\{0\}$). In the case $H_9 > 0$, as we discussed earlier, for any fixed value of the parameter a, systems (79) possess simultaneously an infinite number of real ellipses as well as an infinite number of complex ellipses. Then, based on Remark 3 we obtain *Config. E.33* (example: $a = 1/2, |m| < \sqrt{2}$).

Thus, we have all the configurations indicated in Diagram 7 in the block corresponding to the possibility ($\mathfrak{C}_1$).

3.2.2 The possibility $(\mathfrak{C}_2)$: $H_{10} = 0$.

As it was proved in [16] in this case we have the systems

$$\dot{x} = a + x^2, \qquad \dot{y} = xy, \qquad a \neq 0, \tag{82}$$

which possess the family of invariant ellipses

$$\Phi(x, y) = a + x^2 + m^2y^2 = 0, \quad m \in \mathbb{R}\backslash\{0\}. \tag{83}$$

Since for this family of ellipses we have $\Delta = am^2 \neq 0$ due to $am \neq 0$, we deduce that the family (83) cannot contain any reducible conic.

For systems (82) we calculate $B_1 = B_2 = B_3 = 0$ and hence, by Lemma 2, these systems could possess invariant lines in three different directions. We verify that these systems possess the following three invariant lines:

$$\mathcal{L}_1(x, y) = y = 0, \quad \mathcal{L}_{2,3}(x, y) = x \mp \sqrt{-a} = 0. \tag{84}$$

It is clear that the invariant lines $\mathcal{L}_2(x, y)$ and $\mathcal{L}_3(x, y)$ are real if $a < 0$ and complex if $a > 0$.

Since $\mu_0 = \mu_1 = 0$ and $\mu_2 = ax^2 \neq 0$, Lemma 1 tells us that there exist exactly two finite singularities. Their coordinates are

$$x_{1,2} = \pm\sqrt{-a}, \quad y_{1,2} = 0.$$

We observe that if $a < 0$ (respectively $a > 0$) the singularities $M_{1,2}$ are real (respectively complex) and they are located on each one of the invariant ellipse of the family $\Phi(x, y) = 0$ from (83). Moreover we observe that

$$\mathcal{L}_1(x_1, y_1) = \mathcal{L}_2(x_1, y_1) = 0, \quad \mathcal{L}_1(x_2, y_2) = \mathcal{L}_3(x_2, y_2) = 0.$$

On the other hand for systems (82) we calculate $H_{11} = -192ax^4$, which implies $\operatorname{sign}(a) = -\operatorname{sign}(H_{11})$. So, we arrive at *Config. E.34* if $H_{11} < 0$ (example: $a = 27, m \in \mathbb{R}$) and *Config. E.35* if $H_{11} > 0$ (example: $a = -75, m \in \mathbb{R}$).

Thus, we have all the configurations indicated in DIAGRAM 7 in the block corresponding to the possibility $(\mathfrak{C}_2)$.

Since all the affine invariant subsets in $\mathbb{R}^{12}$ defined in Corollary 1 are examined, we conclude that Theorem 5 is proved.

4 Concluding comments

Now we present some coonclusions about the 35 configurations obtained and their realization. DIAGRAMS 5, 6 and 7 give an algorithm to compute for any system possessing an invariant ellipse, presented in any normal form, its configuration. Moreover DIAGRAMS 5, 6 and 7 are the bifurcation diagrams of the configurations of such systems, done in the 12-parameter space of the coefficients of these systems.

4.1 Concluding comments for $\eta < 0$

According to Theorem 5, each non-degenerate quadratic system in the class $\mathbf{QSE}_{(\eta<0)}$ possesses either exactly one invariant ellipse or a family of invariant ellipses. This class yields 30 distinct configurations which can be split into the following cases according to their geometry:

$\boldsymbol{\alpha_1}$) Fourteen configurations with exactly one ellipse and no invariant lines other that a line at infinity, which is simple. Among these we only have three cases where the ellipse is complex. The configurations are split into subsets by the total multiplicity of the real singularities located on them, whose maximum is 5 in the case of real ellipses and 3 in the case of complex ellipses.

We point out that in this class we have the only two configurations with limit cycles occurring in the family **QSE**. In both configurations we only have one real singular point located on the configuration, on the line of infinity. The two configurations whit the ellipse as a limit cycle are distinguished by the multiplicity of this singularity which could be one or two.

$\boldsymbol{\alpha_2}$) Eleven configurations with exactly one ellipse and invariant lines of total multiplicity 2, including the line at infinity. Among these we only have two configurations with complex ellipses, distinguished by the number of invariant lines, which could be 2 or 1. The remaining configurations are distinguished by the number of invariant lines (1 or 2) and by the geometry of the positions of the invariant lines with respect to the ellipses as well as the multiplicities of the real singularities located on the configurations.

$\boldsymbol{\alpha_3}$) Two configurations of systems possessing exactly one invariant ellipse (real or complex) and three simple invariant lines, including the line at infinity, the two affine lines being real or complex.

$\boldsymbol{\alpha_4}$) Three configurations, each one of them possessing an infinite family of invariant ellipses. Two of them possess only real ellipses (*Config. E.28* and *Config. E.29*) and one of them (*Config. E.30*) possesses simultaneously an infinity of real ellipses and an infinity of complex ellipses (according to Remark 3 we only placed the real ellipses on the drawing of this configuration). All three configurations possess invariant lines of total multiplicity 6, including the line at infinity and they are distinguished by the number of singular points located on the real invariant lines of the configurations.

4.2 Concluding comments for $C_2 = 0$

According to Theorem 5, each non-degenerate quadratic system in the class $\mathbf{QSE}_{(C_2=0)}$ possesses an infinite family of invariant ellipses and in addition its line at infinity is filled up with singularities. This class yields five distinct configurations which can be split into the following cases according to their geometry:

$\boldsymbol{\beta_1}$) Three configurations possessing an infinity of real ellipses (*Config. E.31*, *Config. E.33* and *Config. E.35*), have affine invariant lines (real) of total multiplicity 3. These configurations are distinguished by the number of singular points located on the these invariant lines.

$\boldsymbol{\beta_2}$) One configuration (*Config. E.34*) has an infinity of only complex ellipses and three invariant lines two of them complex parallel lines.

$\boldsymbol{\beta_3}$) One configuration (*Config. E.33*) possesses simultaneously an infinity of real ellipses and an infinity of complex ellipses (according to Remark 3 we only placed the real ellipses on the drawing of this configuration). This configuration also possesses three invariant lines, two of them complex intersecting at a real singular point.

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References

[1] S. Abhyankar; *What is the difference between a Parabola and a Hyperbola?*, The Mathematical Intelligencer **10**, No. 4 (1988), 36–43.

[2] J. C. Artés, J. Llibre, D. Schlomiuk and N. Vulpe; *Geometric configurations of singularities of planar polynomial differential systems - A global classification in the quadratic case.* To be published by Birkhäuser, 2019.

[3] V. Baltag; *Algebraic equations with invariant coefficients in qualitative study of the polynomial homogeneous differential systems*, Bull. of Acad. of Sci. of Moldova. Mathematics **2** (2003), 31–46.

[4] V. A. Baltag, N. I. Vulpe; *Total multiplicity of all finite critical points of the polynomial differential system*, Planar nonlinear dynamical systems (Delft, 1995), Differ. Equ. Dyn. Syst. **5** (1997), 455–471.

[5] D. Bularas, Iu. Calin, L. Timochouk; N. Vulpe; *T-comitants of quadratic systems: a study via the translation invariants*, Delft University of Technology, Faculty of Technical Mathematics and Informatics, Report no. 96–90, 1996.

[6] L. Cairó and J. Llibre; *Darbouxian first integrals and invariants for real quadratic systems having an invariant conic*, J. Phys. A: Math. Gen. **35** (2002), 589 - 608.

[7] S. Chandrasekhar; *An introduction to the study of stellar structure*, Chicago: University of Chicago Press, 1939.

[8] C. Christopher, J. Llibre, J. V. Pereira; *Multiplicity of invariant algebraic curves in polynomial vector fields*, Pacific J. Math. **229** (2007), 63–117.

[9] G. Darboux; *Mémoire sur les équations différentielles du premier ordre et du premier degré.* Bull. Sci. Math. **2** (1878), 60–96; 123–144; 151–200.

[10] J. Hv. Grace, A. Young; *The algebra of invariants.* New York: Stechert, 1941.

[11] D. Hilbert; *Mathematical problems*, Bull. Amer. Math. Soc. **8** (1902), 437–479.

[12] J. P. Jouanolou; *Equations de Pfaff algébriques*, in: Lecture Notes in Math., Vol. 708, Springer, New York, 1979.

[13] J. Llibre,, D. Schlomiuk; *The geometry of quadratic differential systems with a weak focus of third order*, Canad.J. Math. **56** (2004), No. 2, 310–343.

[14] J. Llibre, X. Zhang; *Darboux theory of integrability in $\mathbb{C}^n$ taking into account the multiplicity.* J. Differential Equations **246** (2009), No. 2, 541–551.

[15] A. J. Lotka; *Analytical note on certain rhythmic relations in organic systems*, Proc. Natl. Acad. Sci. USA **6** (1920), 410–415.

[16] R. D. S. Oliveira, A. C. Rezende, D. Schlomiuk, N. Vulpe; *The family of quadratic differential systems with irreducible invariant ellipses: a complete classification in the space* $\mathbb{R}^{\mathbf{12}}$. To be published in 2019 as a preprint.

[17] P. J. Olver; *Classical Invariant Theory*, London Mathematical Society Student Sexts **44**, Cambridge University Press, 1999.

[18] H. Poincaré; *Mémoire sur les courbes définies par les équations différentielles*, J. Math. Pures Appl. (4) **1** (1885), 167–244; O euvres de Henri Poincaré, Vol. **1**, Gauthier–Villard, Paris, 1951, pp 95–114.

[19] H. Poincaré; *Sur l'intégration algébrique des équations différentielles*, C. R. Acad. Sci. Paris, **112** (1891), 761–764.

[20] H. Poincaré; *Sur l'intégration algébrique des équations différentielles du premier ordre et du premier degré*, I. Rend. Circ.Mat. Palermo **5** (1891), 169–191.

[21] J. R. Roth; *Periodic small-amplitude solutions to Volterra's problem of two conflicting populations and their application to the plasma continuity equations*, J. Math. Phys. **10** (1969), 1–43.

[22] D. Schlomiuk, J. Pal; *On the geometry in the neighborhood of infinity of quadratic differential systems with a weak focus*, Qual. Theory Dyn. Syst. **2** (2001), No. 1, 1–43.

[23] D. Schlomiuk, N. Vulpe; *Planar quadratic differential systems with invariant straight lines of at least five total multiplicity*, Qual. Theory Dyn. Syst. **5** (2004), 135–194.

[24] D. Schlomiuk, N. Vulpe; *Geometry of quadratic differential systems in the neighbourhood of the line at infinity*, J. Differential Equations **215** (2005), 357–400.

[25] D. Schlomiuk, N. Vulpe; *Integrals and phase portraits of planar quadratic differential systems with invariant lines of at least five total multiplicity*, Rocky Mountain J. Math. **38** (2008), 1–60.

[26] D. Schlomiuk, N. Vulpe; *Planar quadratic differential systems with invariant straight lines of total multiplicity four*, Nonlinear Anal. **68** (2008), No. 4, 681–715.

[27] D. Schlomiuk, N. Vulpe; *The full study of planar quadratic differential systems possessing a line of singularities at infinity*, J. Dynam. Differential Equations **20** (2008), No. 4, 737–775.

[28] D. Schlomiuk, N. Vulpe; *Integrals and phase portraits of planar quadratic differential systems with invariant lines of total multiplicity four*, Bull. of Acad. of Sci. of Moldova. Mathematics, **56** (2008), No. 1, 27–83.

[29] D. Schlomiuk, N. Vulpe; *Global classification of the planar Lotka–Volterra differential systems according to their configurations of invariant straight lines*, J. Fixed Point Theory Appl. **8** (2010), No. 1, 177–245.

[30] K. S. Sibirskii; *Introduction to the algebraic theory of invariants of differential equations*, Translated from the Russian. Nonlinear Science: Theory and Applications. Manchester University Press, Manchester, 1988.

[31] V. Volterra; *Leçons sur la théorie mathématique de la lutte pour la vie*, Paris: Gauthier Villars, 1931.

[32] N. Vulpe; *Characterization of the finite weak singularities of quadratic systems via invariant theory*, Nonlinear Anal. **74** (2011), 6553–6582.

[33] N.I. Vulpe; *Polynomial bases of comitants of differential systems and their applications in qualitative theory*, (Russian) "Shtiintsa", Kishinev, 1986.

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