The Lefschetz Coincidence Class of $p$ Maps

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October 15, 2012

Abstract

Let $X$ be an arbitrary topological space and let $Y$ be a closed connected oriented $n$-dimensional manifold. In this work we consider $p$ maps $f_1, \ldots, f_p : X \to Y$, $p \geq 2$, define a Lefschetz class $L(f_1, \ldots, f_p) \in H^{n(p-1)}(X; \mathbb{Q})$ and prove that $L(f_1, \ldots, f_p) \neq 0$ implies $f_1(x) = f_2(x) = \cdots = f_p(x)$ for some $x \in X$. In the particular case where $Y$ is a homology sphere are presented some formulas to calculate $L(f_1, \ldots, f_p)$.

1 Introduction

Given $f, g : X \to Y$ two maps between topological spaces, the coincidence set of $f$ and $g$ is defined by:

$$\text{Coin}(f, g) = \{x \in X \mid f(x) = g(x)\}.$$  

Obviously, $\text{Coin}(f, g) \neq \emptyset$ if and only if the intersection of the set $\{(f(x), g(x)) \mid x \in X\}$ with the diagonal $\Delta_Y = \{(y, y) \mid y \in Y\}$ is a nonempty set.

Let $j : Y \times Y \to (Y \times Y, Y \times Y - \Delta_Y)$ be the inclusion map and let $(f, g) : X \to Y \times Y$ be defined by $(f, g)(x) = (f(x), g(x))$, $\forall x \in X$. Thus, if $\text{Coin}(f, g) = \emptyset$, in any cohomology theory, the induced homomorphism $[j \circ (f, g)]^* : H^*(Y \times Y, Y \times Y - \Delta_Y) \to H^*(X)$ is trivial. In fact, in this case, $j \circ (f, g)$ is factored by

$$
\begin{array}{c}
X \xrightarrow{(f, g)} Y \times Y \xrightarrow{j} (Y \times Y, Y \times Y - \Delta_Y) \\
Y \times Y - \Delta_Y \xrightarrow{\downarrow} (Y \times Y - \Delta_Y, Y \times Y - \Delta_Y)
\end{array}
$$

Thus, if $[j \circ (f, g)]^*(u) \neq 0$ for some cohomology class $u \in H^*(Y \times Y, Y \times Y - \Delta_Y)$ then $\text{Coin}(f, g) \neq \emptyset$, i.e, there exists $x_0 \in X$ such that $f(x_0) = g(x_0)$.

Throughout this paper, homology and cohomology will considered with rational coefficients. If $Y$ is a closed connected oriented $n$-dimensional manifold, there exists a
cohomology class \( \mu \in H^n(Y \times Y, Y \times Y - \Delta_Y) \), called the Thom class of the manifold \( Y \), which is related to the fundamental class \( \zeta \in H_n(Y) \) of \( Y \). The cohomology class \( L(f, g) = [j \circ (f, g)]^*(\mu) \in H^n(X) \) is called the Lefschetz class of the pair \( (f, g) \). Also if \( X \) is a closed connected oriented \( n \)-dimensional manifold the Lefschetz number of the pair \( (f, g) \) is defined by

\[
\Lambda(f, g) = \sum_i (-1)^i \text{trace}_i(f^*g^i)
\]  

(1.1)

where \( g^i = D^{-1}_Y \circ g_* \circ D_X \). Here, \( D_X \) and \( D_Y \) are the Poincaré duality isomorphisms of \( X \) and \( Y \), respectively.

We recall that

\[
\Lambda(f, g) = [\zeta', L(f, g)],
\]

where \( \zeta' \in H_n(X) \) is the fundamental class of \( X \) and \([, ,]\) denotes the Kronecker product (see [3], Theorem 14.4, p. 396). Therefore, if \( \Lambda(f, g) \neq 0 \) there exists a coincidence of \( f \) and \( g \).

This work is divided in four sections. The first one is dedicated to preliminary notions. In section 2, we consider \( Y \) a closed connected oriented \( n \)-dimensional manifold, \( X \) an arbitrary topological space and given \( p \) maps \( f_1, \ldots, f_p : X \to Y \), \( p \geq 2 \), define a Lefschetz class \( L(f_1, \ldots, f_p) \in H^{n(p-1)}(X) \) and prove that \( L(f_1, \ldots, f_p) \neq 0 \) implies \( \text{Coin}(f_1, \ldots, f_p) \neq \emptyset \). In section 3, we calculate the Lefschetz class when \( Y \) is a homology sphere space. In section 4, we give formulas for \( L(f_1, \ldots, f_p) \) which are more computable, but which are applied only in special cases.

2 The Lefschetz class of \( p \) maps

We recall that the product of two pairs of topological spaces \((X, A)\) and \((Y, B)\) is defined by

\[
(X, A) \times (Y, B) := (X \times Y, X \times A \cup Y).
\]

Inductively, given \((X_i, A_i)\), \( i = 1, \ldots, k \), \( k \) pairs of topological spaces we define

\[
\prod_{i=1}^{k} (X_i, A_i) = \bigcup_{i=1}^{k} X_i \cup Y_i,
\]

where

\[
Y_i = X_1 \times \cdots \times X_{i-1} \times A_i \times X_{i+1} \times \cdots \times X_k.
\]
Also, given a pair \((Z, W)\), we denote \((Z, W)^k := (Z, W) \times \cdots \times (Z, W)\).

From now on, \(Y\) denotes a closed connected oriented \(n\)-manifold.

Let \(\zeta \in H_n(Y)\) be the fundamental class of \(Y\), let \(\mu \in H^n(Y \times Y, Y \times Y - \Delta_Y)\) be the Thom class related to such orientation and let \(j : Y \times Y \to (Y \times Y, Y \times Y - \Delta_Y)\) be the inclusion map. Given maps \(f_1, \ldots, f_p : X \to Y\), \(p \geq 2\), the coincidence set of \(f_1, \ldots, f_p\) is defined by

\[
\text{Coin}(f_1, \ldots, f_p) = \{x \in X \mid f_1(x) = f_2(x) = \cdots = f_p(x)\}.
\]

For each \(i = 1, \ldots, p - 1\), let us consider \(h_i = (f_i, f_{i+1}) : X \to Y \times Y\). It is clear that

\[
\text{Coin}(f_1, f_2, \ldots, f_p) = \text{Coin}(h_1, \ldots, h_{p-1}).
\]

We define the Lefschetz class of \(f_1, \ldots, f_p\) by

\[
L(f_1, f_2, \ldots, f_p) = [(j \times \cdots \times j) \circ (h_1, \ldots, h_{p-1})]^* (\mu \times \cdots \times \mu) \in H^n(X). \tag{1}
\]

**Theorem 2.1.** If \(L(f_1, f_2, \ldots, f_p) \neq 0\) then \(\text{Coin}(f_1, \ldots, f_p) \neq \emptyset\).

*Proof.* Suppose \(\text{Coin}(f_1, \ldots, f_p) = \emptyset\). Then, for every \(x \in X\), \(f_i(x) \neq f_j(x)\), for some \(i \neq j\). Therefore, for each \(x \in X\),

\[
(h_1(x), \ldots, h_{p-1}(x)) \in \bigcup_{i=1}^{p-1} Z_i,
\]

where \(Z_i = \prod_{j=1}^{p-1} A_i^j\), with \(A_i^j = Y \times Y - \Delta_Y\) if \(j = i\) and \(A_i^j = Y \times Y\) if \(j \neq i\).

We may see that \((Y \times Y, Y \times Y - \Delta_Y)^{p-1} = ((Y \times Y)^{p-1}, \bigcup_{i=1}^{p-1} Z_i)\).

Thus, the composed map \((j \times \cdots \times j) \circ (h_1, \ldots, h_{p-1})\) factors as in the following diagram:

\[
\begin{array}{ccc}
X^{(h_1, \ldots, h_{p-1})} & \xrightarrow{(Y \times Y)^{p-1} \times \cdots \times j} & (Y \times Y, Y \times Y - \Delta_Y)^{p-1} \\
\downarrow & & \uparrow \\
\bigcup_{i=1}^{p-1} Z_i & \xrightarrow{(\bigcup_{i=1}^{p-1} Z_i) \times \bigcap_{i=1}^{p-1} Z_i} & \bigcup_{i=1}^{p-1} Z_i
\end{array}
\]

Therefore, \(L(f_1, f_2, \ldots, f_p) = [(j \times \cdots \times j) \circ (h_1, \ldots, h_{p-1})]^* (\mu \times \cdots \times \mu) = 0\). 

The Lefschetz class has the following property.
Proposition 2.2. If \( L(f_1, \ldots, f_p) \neq 0 \) then \( L(f_i, f_j) \neq 0 \), for all \( 1 \leq i < j \leq p \).

Proof. Let us to prove the case \( p = 3 \). In this case,

\[
L(f_1, f_2, f_3) = (f_1, f_2)^*(e_n \times e_n) + (-1)^n(f_1, f_3)^*(e_n \times e_n) + (f_2, f_3)^*(e_n \times e_n)
\]

\[
L(f_1, f_2) = f_1^*(e_n) + (-1)^n f_2^*(e_n)
\]

\[
L(f_1, f_3) = f_1^*(e_n) + (-1)^n f_3^*(e_n)
\]

\[
L(f_2, f_3) = f_2^*(e_n) + (-1)^n f_3^*(e_n)
\]

Suppose that \( L(f_1, f_2) = 0 \). Then,

\[
(f_1^*(e_n) + (-1)^n f_2^*(e_n)) - f_2^*(e_n) = 0 \tag{2.1}
\]

\[
(f_1^*(e_n) + (-1)^n f_3^*(e_n)) - f_3^*(e_n) = 0 \tag{2.2}
\]

From equation (2.1) we conclude that \((f_1, f_2)^*(e_n \times e_n) = 0\) and from equation (2.2) we conclude that 
\((-1)^n(f_1, f_3)^*(e_n \times e_n) + (f_2, f_3)^*(e_n \times e_n) = 0\). Therefore,

\[
L(f_1, f_2, f_3) = (f_1, f_2)^*(e_n \times e_n) + (-1)^n(f_1, f_3)^*(e_n \times e_n) + (f_2, f_3)^*(e_n \times e_n) = 0.
\]

Similarly, \( L(f_1, f_3) = 0 \) or \( L(f_2, f_3) = 0 \) implies \( L(f_1, f_2, f_3) = 0 \).

Therefore,

\[
L(f_1, f_2, f_3) \neq 0 \Rightarrow L(f_1, f_2) \neq 0, \ L(f_1, f_3) \neq 0 \text{ and } L(f_2, f_3) \neq 0.
\]

For the general case, it is enough to show that \( L(f_1, \ldots, f_p) \neq 0 \) implies \( L(\hat{f}_j) \neq 0 \), where \( \hat{f}_j = (f_1, \ldots, f_{j-1}, f_{j+1}, \ldots, f_p) \), for all \( j = 1, \ldots, p \). In fact to prove this, it is used the same idea of the previous case.

3 The Lefschetz class for a homology sphere space \( Y \)

Throughout this section \( Y \) denotes an \( n \)-homology sphere space.

Let \( e_n \in H^n(Y) \) be such that \([\xi, e_n] = 1\) and let \( e_0 \in H^0(Y) \) be the identity element of the \( \mathbb{Q} \)-algebra \( H^*(Y) \). Recall that \( j : Y \times Y \to (Y \times Y, Y \times Y - \Delta_Y) \) denotes the inclusion map and \( \mu \in H^n(Y \times Y, Y \times Y - \Delta_Y) \) denotes the Thom class. Thus

\[
\text{Proposition 3.1. } j^*(\mu) = e_n \times e_0 + (-1)^n e_0 \times e_n.
\]

Now, we can describe \( L(f_1, \ldots, f_p) \) in the setting that \( Y \) is a homology sphere space.
Proposition 3.2. Let $Y$ be an $n$-homology sphere. Then

$$L(f_1, \ldots, f_p) = \hat{f}_p^* (e) + (-1)^n \hat{f}_{p-1}^* (e) + (-1)^{2n} \hat{f}_{p-2}^* (e) + \cdots$$

$$+ (-1)^{(p-2)n} \hat{f}_2^* (e) + (-1)^{(p-1)n} \hat{f}_1^* (e)$$

where $\hat{f}_j = (f_1, \ldots, f_{j-1}, f_{j+1}, \ldots, f_p)$, $j = 1, \ldots, p$, and $e = e_n \times \cdots \times e_n$.

Proof.

$$L(f_1, \ldots, f_p) =$$

$$(h_1, \ldots, h_{p-1})^* (j^* (\mu) \times \cdots \times j^* (\mu)) =$$

$$h_1^* (j^* (\mu)) \sim h_2^* (j^* (\mu)) \sim \cdots \sim h_{p-1}^* (j^* (\mu)) =$$

$$(f_1^* (e_n) + (-1)^n f_2^* (e_n)) \sim (f_2^* (e_n) + (-1)^n f_3^* (e_n)) \sim \cdots \sim (f_{p-1}^* (e_n) + (-1)^n f_p^* (e_n)).$$

It is not difficult to see that the above product is exactly the same as

$$\hat{f}_p^* (e) + (-1)^n \hat{f}_{p-1}^* (e) + (-1)^{2n} \hat{f}_{p-2}^* (e) + \cdots + (-1)^{(p-2)n} \hat{f}_2^* (e) + (-1)^{(p-1)n} \hat{f}_1^* (e).$$

Corollary 3.3. Let $f_1, \ldots, f_p : Y^{p-1} \to Y$ be maps, where $Y$ is a $n$-homology sphere. Then,

$$L(f_1, \ldots, f_p) = \left( \sum_{i=1}^{p} (-1)^{n(p-i)} \text{deg}(\hat{f}_i) \right) e$$

where $\hat{f}_i = (f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_p)$, $i = 1, \ldots, p$, and $e = e_n \times \cdots \times e_n$.

Therefore, $L(f_1, \ldots, f_p) \neq 0$ if and only if $\sum_{i=1}^{p} (-1)^{n(p-i)} \text{deg}(\hat{f}_i) \neq 0$. Thus, we define the Lefschetz number for $f_1, \ldots, f_p$ by

$$\Lambda(f_1, \ldots, f_p) = \sum_{i=1}^{p} (-1)^{n(p-i)} \text{deg}(\hat{f}_i).$$

4 Related Results

Throughout this section $Y$ denotes an $n$-homology sphere space.
4.1 Particular case 1

Let $f_1, f_2, f_3 : Y \times Y \to Y$ be maps. By using the induced homomorphisms $f_1^*, f_2^*, f_3^* : H^n(Y) \to H^n(Y \times Y)$, we will calculate the Lefschetz number, $\Lambda(f_1, f_2, f_3)$. From Corollary 3.3,

$$
\Lambda(f_1, f_2, f_3) = \deg(f_1, f_2) + (-1)^n \deg(f_1, f_3) + (-1)^{2n} \deg(f_2, f_3)
$$

$$
= \deg(f_1, f_2) + (-1)^n \deg(f_1, f_3) + \deg(f_2, f_3).
$$

Suppose that the homomorphisms $f_1^*, f_2^*, f_3^* : H^n(Y) \to H^n(Y \times Y)$ are given by

$$
f_1^*(e_n) = a_{11}e_1 + a_{12}e_2
$$

$$
f_2^*(e_n) = a_{21}e_1 + a_{22}e_2
$$

$$
f_3^*(e_n) = a_{31}e_1 + a_{32}e_2
$$

where $e_1 = e_n \times e_0$ and $e_2 = e_0 \times e_n$. We have that $e_1 \sim e_1 = e_2 \sim e_2 = 0$, $e_1 \sim e_2 = e_n \times e_n$ and $e_2 \sim e_1 = (-1)^n e_n \times e_n$. It follows that

$$
(f_1, f_2)^*(e_n \times e_n) = f_1^*(e_n) \sim f_2^*(e_n)
$$

$$
= (a_{11}e_1 + a_{12}e_2) \sim (a_{21}e_1 + a_{22}e_2)
$$

$$
= (a_{11}a_{22} + (-1)^n a_{12}a_{21})e_n \times e_n.
$$

Similarly,

$$
(f_1, f_3)^*(e_n \times e_n) = (a_{11}a_{32} + (-1)^n a_{12}a_{31})e_n \times e_n
$$

and

$$
(f_2, f_3)^*(e_n \times e_n) = (a_{21}a_{32} + (-1)^n a_{22}a_{31})e_n \times e_n
$$

Thus, we conclude that

$$
\Lambda(f_1, f_2, f_3) = (a_{11}a_{22} + (-1)^n a_{12}a_{21}) - (a_{11}a_{32} + (-1)^n a_{12}a_{31}) + (a_{21}a_{32} + (-1)^n a_{22}a_{31}).
$$

Note that for $n$ odd, $\Lambda(f_1, f_2, f_3)$ is the determinant of the matrix

$$
\begin{pmatrix}
1 & a_{11} & a_{12} \\
1 & a_{21} & a_{22} \\
1 & a_{31} & a_{32}
\end{pmatrix}
$$

Thus, we have the following result.
Corollary 4.1. Let $n$ be an odd number and let $f_1, f_2, f_3 : Y \times Y \to Y$ be maps with $f_1^*(e_n) = a_{11}e_1 + a_{12}e_2$, $f_2^*(e_n) = a_{21}e_1 + a_{22}e_2$ and $f_3^*(e_n) = a_{31}e_1 + a_{32}e_2$. If
\[
\det \begin{pmatrix}
1 & a_{11} & a_{12} \\
1 & a_{21} & a_{22} \\
1 & a_{31} & a_{32}
\end{pmatrix} \neq 0
\]
then there exists $x_0 \in Y \times Y$ such that $f_1(x_0) = f_2(x_0) = f_3(x_0)$.

We recall that for any map $f : X \to Z$ between topological spaces and for any $k, q$ positive integers given, we have that
\[
f_*(u) \smile \beta = f_*(u \smile f^*(\beta)),
\]
for every $u \in H_{k+q}(X)$ and $\beta \in H^k(Z)$.

Let us consider now $f : Y \times Y \to Y \times Y$ a map and let $d \in \mathbb{Z}$ be the degree of $f$, i.e., $f_*(\zeta \times \zeta) = d \cdot \zeta \times \zeta$, where $\zeta \in H_n(Y)$ is the fundamental class of $Y$. Thus
\[
f_*(\zeta \times \zeta \smile f^*(\beta)) = d \cdot (\zeta \times \zeta) \smile \beta, \text{ for all } \beta \in H^k(Y \times Y).
\]
Denoting by $\Delta_k$ the determinant of $f_* : H_k(Y \times Y) \to H_k(Y \times Y)$, which is equal to the determinant of $f^* : H^k(Y \times Y) \to H^k(Y \times Y)$, we have that
\[
d^{\beta_k} = \Delta_k \Delta_{2n-k},
\]
where $\beta_k$ is the $k$-th Betti number of $Y \times Y$. In particular, for $k = n$,
\[
d^2 = \Delta_n^2.
\]
Therefore, $d = \pm \Delta_n$.

Example. Let $n$ be an even integer and let $f_1, f_2 : Y \times Y \to Y$ be maps. Let $e_1 = e_n \times e_0$ and $e_2 = e_0 \times e_n$ be generators of $H^n(Y \times Y)$. Suppose
\[
\begin{align*}
f_1^*(e_n) &= a_{11}e_1 + a_{12}e_2 \\
f_2^*(e_n) &= a_{21}e_1 + a_{22}e_2
\end{align*}
\]
By the previous remark we have that $(f_1, f_2)^*(e_n \times e_n) = \pm (a_{11}a_{22} - a_{12}a_{21}) e_n \times e_n$.

On the other hand,
\[
(f_1, f_2)^*(e_n \times e_n) = f_1^*(e_n) \smile f_2^*(e_n) = (a_{11}a_{22} + (-1)^n a_{12}a_{21}) e_n \times e_n.
\]
Since $n$ is an even number,

$$(f_1, f_2)^*(e_n \times e_n) = (a_{11}a_{22} + a_{12}a_{21}) \ e_n \times e_n,$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is the matrix of $(f_1, f_2)^* : H^n(Y \times Y) \to H^n(Y \times Y)$.

Moreover, since $e_n \sim e_n = 0$,

$$0 = f_1^*(e_n \sim e_n) = f_1^*(e_n) - f_1^*(e_n) = (a_{11}a_{12} + (-1)^n a_{11}a_{12}) \ e_n \times e_n = 2a_{11}a_{12} \ e_n \times e_n,$$

which implies $2a_{11}a_{12} = 0$. Therefore, $a_{11} = 0$ or $a_{12} = 0$. In a similar way we have

$$0 = f_2^*(e_n \sim e_n) = f_2^*(e_n) - f_2^*(e_n) = (a_{21}a_{22} + (-1)^n a_{21}a_{22}) \ e_n \times e_n = 2a_{21}a_{22} \ e_n \times e_n,$$

which implies that $a_{21} = 0$ or $a_{22} = 0$.

Then the matrix of $(f_1, f_2)^* : H^n(Y \times Y) \to H^n(Y \times Y)$ is one of the following type:

$$A = \begin{pmatrix} 0 & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & 0 \end{pmatrix}.$$

We remark that in the first and fourth cases we have

$$(f_1, f_2)^*(e_n \times e_n) = -\det(A) \ e_n \times e_n$$

and in the second and third cases we have

$$(f_1, f_2)^*(e_n \times e_n) = \det(A) \ e_n \times e_n.$$

From the above remarks, it follows that:

**Corollary 4.2.** Let $f_1, f_2 : Y \times Y \to Y$ be maps with $f_1^*(e_n) = a_{11}e_1 + a_{12}e_2$ and $f_2^*(e_n) = a_{21}e_1 + a_{22}e_2$. If

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \neq 0$$

then, for all $p \in Y$, there exists $x_p \in Y \times Y$ such that $f_1(x_p) = f_2(x_p) = p$.

**Proof.** Given $p \in Y$, let us consider the constant map $f_3(x) = p$, for all $x \in Y \times Y$. In this case,

$$\Lambda(f_1, f_2, f_3) = \deg(f_1, f_2) = \pm \det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

It follows that if $a_{11}a_{22} - a_{12}a_{21} \neq 0$, then there exists $x_0 \in Y \times Y$ such that

$$f_1(x_0) = f_2(x_0) = p.$$
4.2 Particular case 2

Let $f_1, f_2, f_3, f_4 : Y \times Y \times Y \to Y$ be maps. Let us calculate the Lefschetz number, $\Lambda(f_1, f_2, f_3, f_4)$, in terms of the induced homomorphisms $f_1^*, f_2^*, f_3^*, f_4^* : H^n(Y) \to H^n(Y \times Y \times Y)$. From Corollary 3.3,

$$
\Lambda(f_1, f_2, f_3) = \deg(f_1, f_2, f_3) + (-1)^n \deg(f_1, f_2, f_4)
+ (-1)^{2n} \deg(f_1, f_3, f_4) + (-1)^{3n} \deg(f_2, f_3, f_4)
= \deg(f_1, f_2, f_3) + (-1)^n \deg(f_1, f_2, f_4)
+ \deg(f_1, f_3, f_4) + (-1)^n \deg(f_2, f_3, f_4).
$$

Suppose that the homomorphisms $f_1^*, f_2^*, f_3^*, f_4^* : H^n(Y) \to H^n(Y \times Y \times Y)$ are given by

$$
\begin{align*}
    f_1^*(e_n) &= a_{11}e_1 + a_{12}e_2 + a_{13}e_3 \\
    f_2^*(e_n) &= a_{21}e_1 + a_{22}e_2 + a_{23}e_3 \\
    f_3^*(e_n) &= a_{31}e_1 + a_{32}e_2 + a_{33}e_3 \\
    f_4^*(e_n) &= a_{41}e_1 + a_{42}e_2 + a_{43}e_3
\end{align*}
$$

where $e_1 = e_n \times e_0 \times e_0$, $e_2 = e_0 \times e_n \times e_0$ and $e_3 = e_0 \times e_0 \times e_n$. Then, we have

$$
\begin{align*}
    \deg(f_1, f_2, f_3) &= a_{11}(a_{22}a_{33} + (-1)^n a_{23}a_{32}) + a_{12}((-1)^n a_{21}a_{33} + a_{23}a_{31}) \\
    &\quad + a_{13}(a_{21}a_{32} + (-1)^n a_{22}a_{31}) \\
    \deg(f_1, f_2, f_4) &= a_{11}(a_{22}a_{43} + (-1)^n a_{23}a_{42}) + a_{12}((-1)^n a_{21}a_{43} + a_{23}a_{41}) \\
    &\quad + a_{13}(a_{21}a_{42} + (-1)^n a_{22}a_{41}) \\
    \deg(f_1, f_3, f_4) &= a_{11}(a_{32}a_{43} + (-1)^n a_{33}a_{42}) + a_{12}((-1)^n a_{31}a_{43} + a_{33}a_{41}) \\
    &\quad + a_{13}(a_{31}a_{42} + (-1)^n a_{32}a_{41}) \\
    \deg(f_2, f_3, f_4) &= a_{21}(a_{32}a_{43} + (-1)^n a_{33}a_{42}) + a_{22}((-1)^n a_{31}a_{43} + a_{33}a_{41}) \\
    &\quad + a_{23}(a_{31}a_{42} + (-1)^n a_{32}a_{41})
\end{align*}
$$

Therefore, if $n$ is odd then

$$
\Lambda(f_1, f_2, f_3, f_4) = \det\begin{pmatrix}
1 & a_{11} & a_{12} & a_{13} \\
1 & a_{21} & a_{22} & a_{23} \\
1 & a_{31} & a_{32} & a_{33} \\
1 & a_{41} & a_{42} & a_{43}
\end{pmatrix}
$$
References


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