On formal local cohomology modules with respect to a pair of ideals

V. H. Jorge Pérez[1, *] and T.H. Freitas [2 †]

[1] Universidade de São Paulo - ICMC, Caixa Postal 668, 13560-970, São Carlos-SP, Brazil (*e-mail: vhjperez@icmc.usp.br*).

[2] Universidade de São Paulo - ICMC, Caixa Postal 668, 13560-970, São Carlos-SP, Brazil (*e-mail: tfreitas@icmc.usp.br*).

Abstract

We introduce a generalization of formal local cohomology module, which we call a formal local cohomology module with respect to a pair of ideals and study its various properties. We analyze their structure, the upper and lower vanishing and non-vanishing. There are various exact sequences concerning the formal cohomology modules. Among them a MayerVietoris sequence for two ideals with respect to pairs ideals. We also give another proof the generalized version of the local duality theorem.

1 Introduction

Throughout this paper R is a commutative Noetherian (non-zero identity) ring and $\mathfrak{a}, \mathfrak{b}, I, J$ be ideals of R. For an R-module M, its well known, for $i \in \mathbb{N}$, $H^i_{\mathfrak{a}}(M)$ denote the i-th local cohomology module of M with respect to $\mathfrak{a}$ (see [3], [8] for more details).

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When $(R, \mathfrak{m}, k)$ be a local ring and M an R-module, Schenzel in [15], defined an object of study as follows. Let $\underline{x} = x_1, \ldots, x_r$ a system of elements of R and $\mathfrak{b} = \mathrm{R}ad(\underline{x}R)$ and $\check{C}_{\underline{x}}$ denote the Čech complex of R with respect to $\underline{x}$. The projective system of R-modules $\{M/\mathfrak{a}^n M\}_{n\in\mathbb{N}}$ induces a projective system of R-complexes $\{\check{C}_{\underline{x}} \otimes M/\mathfrak{a}^n M\}$. Consider the projective limit $\varprojlim(\check{C}_{\underline{x}} \otimes M/\mathfrak{a}^n M)$.

For an integer $i \in \mathbb{Z}$, the cohomology module $H^i(\varprojlim(\check{C}_{\underline{x}} \otimes M/\mathfrak{a}^n M))$ is called the i-th $\mathfrak{a}$-formal cohomology with respect to $\mathfrak{b}$, denoted by $\check{\mathfrak{F}}^i_{\mathfrak{a},\mathfrak{b}}(M)$. In the case of $\mathfrak{b} = \mathfrak{m}$ we speak simply about the ith $\mathfrak{a}$-formal cohomology.

Now, consider the family of local cohomology modules $\{H^i_{\mathfrak{b}}(M/\mathfrak{a}^n M)\}_{n\in\mathbb{N}}$. For every n, there is a natural homomorphism $H^i_{\mathfrak{b}}(M/\mathfrak{a}^{n+1}M) \to H^i_{\mathfrak{b}}(M/\mathfrak{a}^n M)$ such that the family forms a projective system. Their projective limit $\varprojlim H^i_{\mathfrak{b}}(M/\mathfrak{a}^n M)$ is called the i-th formal local of M with respect to $\mathfrak{b}$ denoted by $\mathfrak{F}^i_{\mathfrak{a},\mathfrak{b}}(M)$. In [15] too, when $\mathfrak{b} = \mathfrak{m}$, Schenzel has proved the following isomorphism $\check{\mathfrak{F}}^i_{\mathfrak{a},\mathfrak{m}}(M) \cong \mathfrak{F}^i_{\mathfrak{a},\mathfrak{m}}(M)$, showing the relation between formal local cohomology and projective limits of some local cohomology modules.

In [18], Takahashi, Yoshino and Yoshizawa introduced a generalization of the notion of local cohomology module, call a local cohomology module with respect to a pair of ideals (I, J), and obtained various results, important for our purpose. More accurately, for R-module M (not necessarily finitely generated), the set of elements of M

$$\Gamma_{I,J}(M) = \{x \in M \mid I^n x \subseteq Jx \quad for \quad n \gg 1\}$$

is a left exact functor, additive and covariant, from the category of all R-modules, called (I, J)-torsion functor. For an integer i, the i-th right derived functor of $\Gamma_{I,J}$ is denoted by $H^i_{I,J}$ and will be call to as i-th local cohomology functor with respect to (I, J). For an R-module M, $H^i_{I,J}(M)$ refer as the i-th local cohomology module of M, with respect to (I, J) and $\Gamma_{I,J}(M)$ as the (I, J)- torsion part of M. When $J = 0$, the $H^i_{I,J}$ coincides with the usual local homology functor H^i_I.

In this paper too, the authors introduce a generalization of Čech complexes, as follows. For an element $x \in R$, let $S_{a,J}$ the set multiplicatively closed subset of R, consisting of all elements of the form $x^n + j$ where $j \in J$ and $n \in \mathbb{N}$. For an R-module M, let $M_{x,J} = S^{-1}_{x,J}M$. The complex $\check{C}_{\underline{x},J}$ is defined as

$$\check{C}_{\underline{x},J} : \ 0 \to R \to R_{x,J} \to 0$$

where R is sitting in the 0th position and $R_{x,J}$ in the 1st position in the complex. For a system of elements of R $\underline{x} = x_1, \dots, x_s$, define a complex $\check{C}_{\underline{x},J} = \bigotimes_{i=1}^{s} \check{C}_{\underline{x}_i,J}$. If $J = 0$ this definition coincides with the usual $\check{C}$ech complex with respect to $\underline{x} = x_1, \dots, x_s$.

Now, we are able to introduce the new object of study and proof some results.

2 Formal local cohomology with respect to a pair of ideals

Again as done above, consider $\underline{x} = x_1, \dots, x_s$ is a system of elements of R which generate the ideal I. Let $\check{C}_{\underline{x},J}$ the $\check{C}$ech complex of R with respect to (I, J). For an R-module M finitely generated and an ideal $\mathfrak{a}$ the projective system of $\{M/\mathfrak{a}^n M\}_{n\in\mathbb{N}}$ induces a projective system of R-complexes $\{\check{C}_{\underline{x},J} \otimes M/\mathfrak{a}^n M\}$. Let the projective limit $\varprojlim \big(\check{C}_{\underline{x},J} \otimes M/\mathfrak{a}^n M\big)$.

Definition 2.1. *Using the construction above, for an integer $i \in \mathbb{Z}$, the cohomology module $H^i(\varprojlim \big(\check{C}_{\underline{x},J} \otimes M/\mathfrak{a}^n M)\big)$ is called the i-th $\mathfrak{a}$-formal cohomology with respect to (I, J), denoted by $\check{\mathfrak{F}}^i_{\mathfrak{a},I,J}(M)$.*

Note that, if $J = 0$, $\check{C}_{\underline{x},J}$ coincides with the usual $\check{C}$ech complex $\check{C}_{\underline{x}}$ with respect to $\underline{x} = x_1, \dots, x_s$. Therefore $\check{\mathfrak{F}}^i_{\mathfrak{a},I,0}(M) \cong \check{\mathfrak{F}}^i_{\mathfrak{a},I}(M)$. Now,if $J = 0$ and $I = \mathfrak{m}$ we have $\check{\mathfrak{F}}^i_{\mathfrak{a},\mathfrak{m},0}(M) \cong \check{\mathfrak{F}}^i_{\mathfrak{a},\mathfrak{m}}(M)$. This new definition is a natural generalization of $\mathfrak{a}$-formal cohomology with respect to $\mathfrak{b}$ and $\mathfrak{a}$-formal cohomology, both introduced by Schenzel in [15] and discussed by Mafi, Asgharzadeh and Divaaani-Aazar, Eghbali and Chu in other papers.

Proposition 2.2. *Let R be a local Noetherian ring, $\underline{x} = x_1, \dots, x_s$ elements of R, $I = (\underline{x})$ ideal of R and M finitely generated R-module. If M is J-torsion R-module, then $\check{\mathfrak{F}}^i_{\mathfrak{a},I,J}(M) \cong \check{\mathfrak{F}}^i_{\mathfrak{a},I}(M)$.*

Proof. By Takahashi and etal, in [18, Corollary 2.5], we have

$$(\check{C}_{\underline{x},J} \otimes M/\mathfrak{a}^n M)) \cong (\check{C}_{\underline{x}} \otimes M/\mathfrak{a}^n M)).$$

Applying the inverse limit we obtain

$$\varprojlim(\check{C}_{\underline{x},J} \otimes M/\mathfrak{a}^n M)) \cong \varprojlim(\check{C}_{\underline{x}} \otimes M/\mathfrak{a}^n M)).$$

Therefore,

$$\begin{aligned}
\check{\mathfrak{F}}^i_{\mathfrak{a},I,J}(M) &= H^i(\varprojlim(\check{C}_{\underline{x},J} \otimes M/\mathfrak{a}^n M)) \\
&= H^i(\varprojlim(\check{C}_{\underline{x}} \otimes M/\mathfrak{a}^n M)) \\
&= \check{\mathfrak{F}}^i_{\mathfrak{a},I}(M).
\end{aligned}$$

□

Theorem 2.3. *Let I and J be ideals of R as before. Consider $\phi : R \to R'$ be a ring homomorphism and let M' be an R'-module. If $\phi(J) = JR'$, then there is a natural isomorphism $\check{\mathfrak{F}}^i_{\mathfrak{a},I,J}(M') \cong \check{\mathfrak{F}}^i_{\mathfrak{a}R',IR',JR'}(M')$.*

Proof. Let $I = (x_1, \ldots, x_s)R$ and $\phi(\underline{x}) = \phi(x_1), \ldots, \phi(x_s)$. Let $S_{x_i,J}$ a multiplicatively closed subset of R, described in the construction of Čech complex for an element. By hypothesis $\phi(S_{x_i,J}) = S_{\phi(x_i),JR'}$ for all i with $1 \leq i \leq s$. Thus we have $\check{C}_{\underline{x},J} \otimes M'/\mathfrak{a}^n M'$ homotopic to $\check{C}_{\phi(\underline{x}),JR'} \otimes M'/\mathfrak{a}R'^n M'$. Applying the inverse limit and cohomology we have the statement.

□

Is important to remember that the hypothesis $\phi(J) = JR'$ in the theorem above cannot be remove. For more details see [18, Remark 2.8].

Now, let the family of local cohomology modules $\{H^i_{I,J}(M/\mathfrak{a}^n M)\}_{n\in\mathbb{N}}$. For every n there is a natural homomorphism $H^i_{I,J}(M/\mathfrak{a}^{n+1}M) \to H^i_{I,J}(M/\mathfrak{a}^n M)$ such that the family forms a projective system. Their projective limit $\varprojlim H^i_{I,J}(M/\mathfrak{a}^n M)$ is called the i-th formal local cohomoloy of M with respect to a pair ideals I, J denoted by $\mathfrak{F}^i_{\mathfrak{a},I,J}(M)$.

The natural question is: When $\check{\mathfrak{F}}^i_{\mathfrak{a},I,J}(M)$ is isomorphic to $\mathfrak{F}^i_{\mathfrak{a},I,J}(M)$?. For try answer this question, following result is proved.

Proposition 2.4. *Using the notation preceding, there is the short exact sequence*

$$0 \to \varprojlim{}^1 H^{i-1}_{I,J}(M/\mathfrak{a}^n M) \to H^i(\varprojlim(\check{C}_{\underline{x},J} \otimes M/\mathfrak{a}^n M)) \to \varprojlim H^i_{I,J}(M/\mathfrak{a}^n M) \to 0$$

for all $i \in \mathbb{Z}$.

Proof. Let the natural epimorphism $M/\mathfrak{a}^{n+1}M \to M/\mathfrak{a}^n M$ and using the fact that Čech complex $\check{C}_{\underline{x},J}$ is a complex of flat R-modules, we have an R-morphism of R-complexes

$$\check{C}_{\underline{x},J} \otimes M/\mathfrak{a}^{n+1}M \to \check{C}_{\underline{x},J} \otimes M/\mathfrak{a}^n M$$

(which is degree-wise an epimorphism). By the definition of the projective limit, there is a short exact sequence of complexes

$$0 \to \varprojlim \big(\check{C}_{\underline{x},J} \otimes M/\mathfrak{a}^n M\big) \to \prod \big(\check{C}_{\underline{x},J} \otimes M/\mathfrak{a}^n M\big) \to \prod \big(\check{C}_{\underline{x},J} \otimes M/\mathfrak{a}^n M\big) \to 0.$$

So, there is the long exact cohomology sequence

$$\cdots \to H^i(\varprojlim(\check{C}_{\underline{x},J} \otimes M/\mathfrak{a}^n M)) \to H^i(K^\bullet) \to H^i(K^\bullet) \to \cdots$$

where $K^\bullet = \prod(\check{C}_{\underline{x},J} \otimes M/\mathfrak{a}^n M)$.

Applying [19, Theorem 3.5.8] in the complex $\mathbf{C}^\bullet : \check{C}_{\underline{x},J} \otimes M/\mathfrak{a}^n M$ follows

$$0 \to \varprojlim{}^1 H^{i-1}(\mathbf{C}^\bullet) \to H^i(\mathbf{C}^\bullet) \to \varprojlim H^i(\mathbf{C}^\bullet) \to 0.$$

Since $H^i_{I,J}(M/\mathfrak{a}^n M) \cong H^i(\check{C}_{\underline{x},J} \otimes M/\mathfrak{a}^n M)$(see [18, Theorem 2.4]), for all i integer, the statement is proved.

□

Proposition 2.5. *Let $(R, \mathfrak{m}, k)$ be a local Noetherian ring, $\underline{x} = x_1, \ldots, x_s$ elements of R, $I = (\underline{x})$, J ideals of R and M finitely generated R-module.*

(a) If M is J-torsion R-module then $H^i_{\mathfrak{m},J}(M)$ is an Artinian R-module .

(b) If M is J-torsion R-module and $\sqrt{I+J} = \mathfrak{m}$ then $H^i_{I,J}(M)$ is an Artinian R-module .

Proof. (a) Because $H^i_{\mathfrak{m},J}(M) \cong H^i_{\mathfrak{m}}(M)$ (see [18, Corollary 2.5])and the fact [3, Theorem 7.1.3], we have the statement.

(b) By Proposition 2.4. (vii) and (vi) in [13], $H^i_{I,J}(M) = H^i_{\sqrt{I+J},J}(M) = H^i_{\mathfrak{m},J}(M) = H^i_{\mathfrak{m}}(M)$ which is an Artinian R-module.

□

Corollary 2.6. *Let $(R, \mathfrak{m}, k)$ be a local Noetherian ring, $\underline{x} = x_1, \ldots, x_s$ elements of R, $I = (\underline{x})$, J ideals of R and M finitely generated R-module.*

(a) If M is a J- torsion R-module then, for all $i \in \mathbb{Z}$, $\check{\mathfrak{F}}^i_{\mathfrak{a},\mathfrak{m},J}(M) = \mathfrak{F}^i_{\mathfrak{a},\mathfrak{m},J}(M)$.

(b) If M is a J- torsion R-module and $\sqrt{I+J} = \mathfrak{m}$ then, for all $i \in \mathbb{Z}$, $\check{\mathfrak{F}}^i_{\mathfrak{a},I,J}(M) = \mathfrak{F}^i_{\mathfrak{a},I,J}(M)$.

(c) *If M is an Artinian R-module then $\check{\mathfrak{F}}^i_{\mathfrak{a},I,J}(M) = \mathfrak{F}^i_{\mathfrak{a},I,J}(M)$ for all $i \in \mathbb{Z}$.*

Proof. For proof of the statement (a) note that, because M is J-torsion R-module, $M/\mathfrak{a}^nM$ is too J-torsion R-module. Then, by proposition above, $H^i_{\mathfrak{m},J}(M/\mathfrak{a}^nM)$ is an Artinian R-module, for all $i \in \mathbb{Z}$. So the family $\{H^i_{\mathfrak{m},J}(M/\mathfrak{a}^nM)\}_{n\in\mathbb{N}}$, $i \in \mathbb{N}$ satisfies the Mittag-Leffler condition. By Proposition 2.4 and using the fact that $\varprojlim^1$ vanishes on the projective system of Artinian R-modules, which proves the statement. The proof of (b) is analogous.

For (c), use the remark above to proof that $H^i_{I,J}(M/\mathfrak{a}^nM)$ is an Artinian R-module, for all $i \in \mathbb{Z}$. Applying the previous idea finishes the proof. □

Corollary 2.7. *Let I and J be ideals of R as before. Consider $\phi : R \to R'$ be a ring homomorphism, M' be a finitely generated R'-module and $\phi(J) = JR'$.*

(a) *If M' is a J-torsion R'- module then, for all $i \in \mathbb{Z}$, $\mathfrak{F}^i_{\mathfrak{a},\mathfrak{m},J}(M') = \mathfrak{F}^i_{\mathfrak{a}R',\mathfrak{m}R',JR'}(M')$.*

(b) *If M' is a J-torsion R'-module and $\sqrt{I+J} = \mathfrak{m}$ then, for all $i \in \mathbb{Z}$, $\mathfrak{F}^i_{\mathfrak{a},I,J}(M') = \mathfrak{F}^i_{\mathfrak{a}R',IR',JR'}(M')$.*

(c) *If M' is an Artinian R'-module, then for all $i \in \mathbb{Z}$, $\mathfrak{F}^i_{\mathfrak{a},I,J}(M') = \mathfrak{F}^i_{\mathfrak{a}R',IR',JR'}(M')$.*

Proof. Since M' is too JR'-torsion R'-module, by previous corollary and Theorem 2.3 we have the proof of (a). The same idea can be used in assertions (b) and (c).

□

3 Cohomological dimension

Let a local Noetherian ring $(R, \mathfrak{m}, k)$, $\mathfrak{a}, I, J$ ideals of R and M be a R-module finitely generated. We now establish some preliminary results on cohomological dimension of R-module M with respect to a pair of ideals (I, J). First, is known that Divaani-Aazr, Naghipour and Tousi in [5] were the precursors on the term "cohomological dimension", defined by

$$\mathrm{cd}(\mathfrak{a}, M) = \sup\{i \in \mathbb{Z} : H^i_{\mathfrak{a}}(M) \neq 0\}.$$

In our context, Chu and Wang in [4] define the cohomological dimension of R-module M with respect to a pair of ideals (I, J), defined by

$$\operatorname{cd}(I, J, M) = \sup\{i \in \mathbb{Z} : H^r_{I,J}(M) \neq 0\}$$

and gives a characterization about this integer.

Chu and Wang in [4] too generalize the result of P. Schenzel [15, Lema 2.1], using this new concept. This result is gives below.

Proposition 3.1. *Let I be a proper of a commutative Noetherian ring R and M, N be a finitely generated R-modules such that $Supp_R N \subseteq Supp_R M$. Then $\operatorname{cd}(I, J, N) \leq \operatorname{cd}(I, J, M)$.*

Corollary 3.2. *Let M be a finitely generated R-module. Then*

$$\operatorname{cd}(I, J, M) = \operatorname{cd}(I, J, R/Ann_R M) = \max\{\operatorname{cd}(I, J, R/\mathfrak{p}) : \mathfrak{p} \in \operatorname{Min} M\}$$

Proof. The proof is similar to the [15, Corollary 2.2].

□

Lemma 3.3. *Let $(R, \mathfrak{m}, k)$ a local ring, $\underline{x} = x_1, \ldots, x_s$ be a system of elements of the ring R, $\mathfrak{a} = (\underline{x})$, J ideals of R and M be a finitely generated R-module. Then*

$$\operatorname{cd}((\mathfrak{a}, yR), J, M) \leq \operatorname{cd}(\mathfrak{a}, J, M) + 1$$

for any element $y \in \mathfrak{m}$.

Proof. By construction in [18], we can consider the $\check{C}$ech complex

$$\check{C}_{\underline{x},y,J} = (\bigotimes_{i=1}^{s} \check{C}_{x_i,J}) \bigotimes \check{C}_{y,J}.$$

Now, for the natural homomorphism $\check{C}_{\underline{x},J} \to \check{C}_{\underline{x},J} \otimes R_y$, let the complex $M(f) = \check{C}_{\underline{x},J} \oplus (\check{C}_{\underline{x},J} \otimes R_y[-1])$ namely mapping cone. Note that the mapping cone $M(f)$ is isomorphic to $\check{C}_{\underline{x},y,J}$, then we can consider the following exact sequence

$$0 \to \check{C}_{\underline{x},J} \otimes R_y[-1] \to \check{C}_{\underline{x},y,J} \to \check{C}_{\underline{x},J} \to 0.$$

By [16, Lemma 1.1] and using [18, Theorem 2.4], for all $n \in \mathbb{Z}$, there is a short exact sequence

$$0 \to H^1_{yR,J}(H^{n-1}_{\mathfrak{a},J}(M)) \to H^n_{(\mathfrak{a},yR),J}(M) \to H^0_{yR,J}(H^n_{\mathfrak{a},J}(M)) \to 0.$$

Let $j = \mathrm{cd}(\mathfrak{a}, J, M)$, then by the exact sequence previous and definition of cohomological dimension with respect to a pair of ideals, we have $H^{i+1}_{(\mathfrak{a},yR),J}(M) = 0$ for all $i > j$. Therefore $\mathrm{cd}((\mathfrak{a}, yR), J, M) \leq j + 1$, and the proof is completed. $\square$

Theorem 3.4. *Let $(R, \mathfrak{m}, k)$ a local ring, $\underline{x} = x_1, \ldots, x_s$ be a system of elements of the ring R and $\mathfrak{a}, I = (\underline{x})$ and J ideals of R. Let $0 \to A \to B \to C \to 0$ denote a short exact sequence of finitely generated R-modules. Then there is a long exact sequence*

$$\cdots \to \check{\mathfrak{F}}^i_{\mathfrak{a},I,J}(A) \to \check{\mathfrak{F}}^i_{\mathfrak{a},I,J}(B) \to \check{\mathfrak{F}}^i_{\mathfrak{a},I,J}(C) \to \check{\mathfrak{F}}^{i+1}_{\mathfrak{a},I,J}(A) \to \cdots$$

Proof. It is well known that the short exact sequence previous induces a projective system of short exact sequences

$$0 \to \check{C}_{\underline{x},J} \otimes A/B \cap \mathfrak{a}^n A \to \check{C}_{\underline{x},J} \otimes B/\mathfrak{a}^n B \to \check{C}_{\underline{x},J} \otimes C/\mathfrak{a}^n C \to 0$$

for all $n \in \mathbb{N}$. Because $\check{C}_{\underline{x},J}$ is a complex of flat R-modules and the maps

$$A/B \cap \mathfrak{a}^{n+1} A \to A/B \cap \mathfrak{a}^n A$$

are surjective, it follows that the projective system of R-complexes $\{\check{C}_{\underline{x},J} \otimes A/B \cap \mathfrak{a}^n A\}$ satisfies the Mittag-Leffler condition. Therefore, applying the inverse limit, we have the exact sequence of complexes

$$0 \to \varprojlim \check{C}_{\underline{x},J} \otimes A/B \cap \mathfrak{a}^n A \to \varprojlim \check{C}_{\underline{x},J} \otimes B/\mathfrak{a}^n B \to \varprojlim \check{C}_{\underline{x},J} \otimes C/\mathfrak{a}^n C \to 0$$

In the case $\{B \cap \mathfrak{a}^n A\}$ is equivalent to the $\mathfrak{a}$-adic topology on A and by Artin-Rees lemma [2, Ch. III,3, Cor. 1], we have

$$\cdots \to H^i(\varprojlim \check{C}_{\underline{x},J} \otimes A/\mathfrak{a}^n A) \to H^i(\varprojlim \check{C}_{\underline{x},J} \otimes B/\mathfrak{a}^n B) \to H^i(\varprojlim \check{C}_{\underline{x},J} \otimes C/\mathfrak{a}^n C) \to \cdots.$$

Using the definition of Formal local cohomology defined by a pair of ideals finishes the proof.

$\square$

Corollary 3.5. *Using the same hypothesis of theorem above, there is the long exact sequence:*

(a)

$$\cdots \to \mathfrak{F}^i_{\mathfrak{a},\mathfrak{m},J}(A) \to \mathfrak{F}^i_{\mathfrak{a},\mathfrak{m},J}(B) \to \mathfrak{F}^i_{\mathfrak{a},\mathfrak{m},J}(C) \to \mathfrak{F}^{i+1}_{\mathfrak{a},\mathfrak{m},J}(A) \to \cdots$$

if B is a J-torsion R-modules.

(b)

$$\cdots \to \mathfrak{F}^i_{\mathfrak{a},I,J}(A) \to \mathfrak{F}^i_{\mathfrak{a},I,J}(B) \to \mathfrak{F}^i_{\mathfrak{a},I,J}(C) \to \mathfrak{F}^{i+1}_{\mathfrak{a},I,J}(A) \to \cdots$$

if B is a J-torsion R-modules and $\sqrt{I+J} = \mathfrak{m}$.

(c)

$$\cdots \to \mathfrak{F}^i_{\mathfrak{a},I,J}(A) \to \mathfrak{F}^i_{\mathfrak{a},I,J}(B) \to \mathfrak{F}^i_{\mathfrak{a},I,J}(C) \to \mathfrak{F}^{i+1}_{\mathfrak{a},I,J}(A) \to \cdots$$

if B is an Artinian R-module.

Proof. For proof of all the cases, apply Corollary 2.6 and theorem previous. □

Proposition 3.6. *Let $(R, \mathfrak{m}, k)$ a local ring, $\underline{x} = x_1, \ldots, x_n$ be a system of elements of the ring R and $\mathfrak{a}, I = (\underline{x})$ and J ideals of R. Consider M a finitely generated R-module, $N \subseteq M$ be a R-module such that Supp $N \cap V(\mathfrak{a}) \subseteq V(\mathfrak{m})$ and $\overline{M} = M/N$. Then there is a short exact sequence*

$$0 \to N^{\mathfrak{a}} \to \mathfrak{F}^0_{\mathfrak{a},I,J}(M) \to \mathfrak{F}^0_{\mathfrak{a},I,J}(\overline{M}) \to 0$$

and isomorphisms $\mathfrak{F}^i_{\mathfrak{a},I,J}(M) \cong \mathfrak{F}^i_{\mathfrak{a},I,J}(\overline{M})$ for all $i \geq 1$.

Proof. Consider the short sequence exact $0 \to N \to M \to \overline{M} \to 0$. As well as in Theorem 3.4, there is the following long exact sequence

$$0 \to \check{C}_{\underline{x},J} \otimes N/\mathfrak{a}^n N \to \check{C}_{\underline{x},J} \otimes M/\mathfrak{a}^n M \to \check{C}_{\underline{x},J} \otimes \overline{M}/\mathfrak{a}^n \overline{M} \to 0$$

for all $n \in \mathbb{N}$. By view of the long exact cohomology sequence it follows that there is a long exact sequence

$$\cdots \to H^i(\check{C}_{\underline{x},J} \otimes N/\mathfrak{a}^n N) \to H^i(\check{C}_{\underline{x},J} \otimes M/\mathfrak{a}^n M) \to H^i(\check{C}_{\underline{x},J} \otimes \overline{M}/\mathfrak{a}^n \overline{M}) \to \cdots$$

By [18], $H^i(\check{C}_{\underline{x},J} \otimes X) \cong H^i_{I,J}(X)$ for all R-module X, so

$$\cdots \to H^i_{I,J}(N/\mathfrak{a}^n N) \to H^i_{I,J}(M/\mathfrak{a}^n M) \to H^i_{I,J}(\overline{M}/\mathfrak{a}^n \overline{M}) \to \cdots .$$

The assumption Supp $N \cap V(\mathfrak{a}) \subseteq V(\mathfrak{m})$ implies that $N/\mathfrak{a}^n N$ is an R-module of finite length, for all $n \in \mathbb{N}$. By Theorem 4.7 in [18], for all $i > 0$, $H^i_{I,J}(N/\mathfrak{a}^n N) = 0$. Therefore we have

$$0 \to H^0_{I,J}(N/\mathfrak{a}^n N) \to H^0_{I,J}(M/\mathfrak{a}^n M) \to H^0_{I,J}(\overline{M}/\mathfrak{a}^n \overline{M}) \to 0$$

and isomorphisms $H^i_{I,J}(M/\mathfrak{a}^n M) \cong H^i_{I,J}(\overline{M}/\mathfrak{a}^n \overline{M})$ for all $i > 0$. Note that the family $\{H^0_{I,J}(N/\mathfrak{a}^n N)\}_{n\in\mathbb{N}}$ of Artinian R-modules ([13], Theorem 3.7 show that $H^0_{I,J}(N/\mathfrak{a}^n N)$ is Artinian), satisfy the Mittag-Leffler condition. Passing to the projective limit in the exact sequence previous we have

$$0 \to \mathfrak{F}^0_{\mathfrak{a},I,J}(N) \to \mathfrak{F}^0_{\mathfrak{a},I,J}(M) \to \mathfrak{F}^0_{\mathfrak{a},I,J}(\overline{M}) \to 0$$

and isomorphisms $\mathfrak{F}^i_{\mathfrak{a},I,J}(M) \cong \mathfrak{F}^i_{\mathfrak{a},I,J}(\overline{M})$ for all $i \geq 1$. Now, as for all $i > 0$ $H^i_{I,J}(N/\mathfrak{a}^n N) = 0$, by Corollary 4.2 in [18], M is (I, J)-torsion R-module. Therefore $H^0_{I,J}(N/\mathfrak{a}^n N) = N/\mathfrak{a}^n N$ and $\mathfrak{F}^0_{\mathfrak{a},I,J}(N) = \varprojlim N/\mathfrak{a}^n N = N^{\mathfrak{a}}$.

□

Corollary 3.7. *Consider the same hypothesis of proposition previous.*

(a) If M is a J-torsion R-module there is a short exact sequence

$$0 \to \check{\mathfrak{F}}^0_{\mathfrak{a},\mathfrak{m},J}(N) \to \check{\mathfrak{F}}^0_{\mathfrak{a},\mathfrak{m},J}(M) \to \check{\mathfrak{F}}^0_{\mathfrak{a},\mathfrak{m},J}(\overline{M}) \to 0$$

and isomorphisms $\check{\mathfrak{F}}^i_{\mathfrak{a},\mathfrak{m},J}(M) \cong \check{\mathfrak{F}}^i_{\mathfrak{a},\mathfrak{m},J}(\overline{M}) = 0$ for all $i \geq 1$.

(b) If M is a J-torsion R-module and $\sqrt{I+J} = \mathfrak{m}$, there is a short exact sequence

$$0 \to \check{\mathfrak{F}}^0_{\mathfrak{a},I,J}(N) \to \check{\mathfrak{F}}^0_{\mathfrak{a},I,J}(M) \to \check{\mathfrak{F}}^0_{\mathfrak{a},I,J}(\overline{M}) \to 0$$

and isomorphisms $\check{\mathfrak{F}}^i_{\mathfrak{a},I,J}(M) \cong \check{\mathfrak{F}}^i_{\mathfrak{a},I,J}(\overline{M}) = 0$ for all $i \geq 1$.

Proof. Use the Corollary 2.6 and proposition above. □

Theorem 3.8. *Let M be a finitely generated R-module. Choose $x \in \mathfrak{m}$ an element such that $x \notin \mathfrak{p}$ for all $\mathfrak{p} \in Ass_R M \backslash \{\mathfrak{m}\}$, and let $M' = M/xM$.*

(a) If M is a J-torsion R-module there are short exact sequence

$$0 \to H_0(x; \varprojlim H^i_{\mathfrak{m},J} M/\mathfrak{a}^n M) \to \varprojlim H^i_{\mathfrak{m},J}(M'/\mathfrak{a}^n M') \to H_1(x; \varprojlim H^{i+1}_{\mathfrak{m},J} M/\mathfrak{a}^n M) \to 0$$

for all $i \in \mathbb{Z}$.

(b) If M is a J-torsion R-module and $\sqrt{I+J} = \mathfrak{m}$, there are short exact sequence

$$0 \to H_0(x; \varprojlim H^i_{I,J} M/\mathfrak{a}^n M) \to \varprojlim H^i_{I,J}(M'/\mathfrak{a}^n M') \to H_1(x; \varprojlim H^{i+1}_{I,J} M/\mathfrak{a}^n M) \to 0$$

for all $i \in \mathbb{Z}$.

Proof. We will proof of a), and b) is analogue. By the choice of x it follows that $0:_M x$ is an R-module of finite length. Moreover the multiplication by x induces an exact sequence

$$0 \to 0:_M x \to M \xrightarrow{x} M \to M' \to 0$$

breaks into two short exact sequences $0 \to N \to M \to \overline{M} \to 0$, where $N = 0:_M x$ and $\overline{M} = M/N$, and $0 \to \overline{M} \xrightarrow{x} M \to M' \to 0$.

The first of this sequences induces the isomorphisms $\varprojlim H^i_{\mathfrak{m},J}(M/\mathfrak{a}^n M) \cong \varprojlim H^i_{\mathfrak{m},J}(\overline{M}/\mathfrak{a}^n \overline{M})$ for all $i > 0$ and a short exact sequence

$$0 \to N^{\mathfrak{a}} \to \varprojlim H^0_{\mathfrak{m},J}(M/\mathfrak{a}^n M) \to \varprojlim H^0_{\mathfrak{m},J}(\overline{M}/\mathfrak{a}^n \overline{M}) \to 0$$

by Proposition 3.6. By Corollary 3.5, the second sequence induces a long exact sequence for the formal cohomology modules

$$\cdots \to \varprojlim H^i_{\mathfrak{m},J}(\overline{M}/\mathfrak{a}^n \overline{M}) \xrightarrow{x} \varprojlim H^i_{\mathfrak{m},J}(M/\mathfrak{a}^n M) \to \varprojlim H^i_{\mathfrak{m},J}(\overline{M'}/\mathfrak{a}^n \overline{M'}) \to \cdots .$$

With the isomorphisms above this proves the claim for $i > 0$. To this end one has to break up the long exact sequence into short exact sequences. For the proof in the case $i = 0$, the only remaining case, consider the composite of the above short exact sequence with the previous one for $i = 0$. Then this completes the proof for $i = 0$.

□

4 Non-vanishing

Let M be a finitely generated R-module. Let $\mathfrak{a}, I = (\underline{x}), J$ ideals in the local ring $(R, \mathfrak{m}, k)$. In this section, our purpose is to know the integers $\sup\{i \in \mathbb{Z} \mid \check{\mathfrak{F}}^i_{\mathfrak{a},I,J}(M) \neq 0\}$ and $\sup\{i \in \mathbb{Z} \mid \mathfrak{F}^i_{\mathfrak{a},I,J}(M) \neq 0\}$.

Proposition 4.1. *Consider an ideal* $\mathfrak{a}$ *such that* $\dim(M/\mathfrak{a}M) = 0$*. Then*

(a) $\mathfrak{F}^i_{\mathfrak{a},I,J}(M) = \begin{cases} 0 & \text{if } i \neq 0 \\ M^{\mathfrak{a}} & \text{if } i = 0, \end{cases}$

(b) $\mathfrak{F}^i_{\mathfrak{a},I,J}(M) = \check{\mathfrak{F}}^i_{\mathfrak{a},I,J}(M)$*, for all* $i \in \mathbb{Z}$*.*

Proof. For (a), by [18, Theorem 4.7], $H^i_{I,J}(M/\mathfrak{a}^nM) = 0$ for $i \neq 0$ and by [18, Corollary 4.2], $M/\mathfrak{a}^nM$ is (I,J)- torsion R-module. Then $H^0_{I,J}(M/\mathfrak{a}^nM) = \Gamma_{I,J}(M/\mathfrak{a}^nM) = M/\mathfrak{a}^nM$. Passing to the projective limit finishes the proof. For proof of (b), use Proposition 2.4.

□

Theorem 4.2. *Let M be a finitely generated module over a local ring $(R, \mathfrak{m}, k)$. Let $\mathfrak{a}, I, J$ ideals of R such that $J \neq R$ and $I+J$ is an $\mathfrak{m}$-primary ideal. Then,*

$$\dim_R M/(\mathfrak{a}+J)M = \sup\{i \in \mathbb{Z} \mid \mathfrak{F}^i_{\mathfrak{a},I,J}(M) \neq 0\}.$$

Proof. By [18, Theorem 4.3], $H^i_{I,J}(M/\mathfrak{a}^nM) = 0$ for any $i > \dim \frac{M/\mathfrak{a}^nM}{J(M/\mathfrak{a}^nM)}$. But, $\dim \frac{M/\mathfrak{a}^nM}{J(M/\mathfrak{a}^nM)} = \dim \frac{M}{(J+\mathfrak{a})M}$ for all $n \in \mathbb{N}$. Therefore

$$\dim_R M/(\mathfrak{a}+J)M \geq \sup\{i \in \mathbb{Z} \mid \mathfrak{F}^i_{\mathfrak{a},I,J}(M) \neq 0\}.$$

On the other hand, let $r = \dim(\frac{M/\mathfrak{a}^nM}{J(M/\mathfrak{a}^nM)}) = \dim(\frac{M}{(J+\mathfrak{a})M})$ for all $n \in \mathbb{N}$. Since $I+J$ is an $\mathfrak{m}$-primary ideal, we have $H^i_{I,J}(M) = H^i_{\mathfrak{m},J}(M)$ for any integer i. Thus we may assume $I = \mathfrak{m}$. Denote $\overline{M} = M/\mathfrak{a}^nM$, then the short exact sequence

$$0 \to J\overline{M} \to \overline{M} \to \overline{M}/J\overline{M} \to 0$$

induces an exact cohomology sequence

$$H^r_{\mathfrak{m},J}(\overline{M}) \to H^r_{\mathfrak{m},J}(\overline{M}/J\overline{M}) \to H^{r+1}_{\mathfrak{m},J}(J\overline{M}).$$

Since $\dim J\overline{M}/J^2\overline{M} \leq \dim \overline{M}/J^2\overline{M} = \dim \overline{M}/J\overline{M} = r$, by [18, Theoren 4.3], $H^{r+1}_{\mathfrak{m},J}(J\overline{M}) = 0$. Because $\overline{M}/J\overline{M}$ is a J-torsion R-module, by [18, Corollary 2.5] and Grothendieck's non-vanishing theorem

$$H^r_{\mathfrak{m},J}(\overline{M}/J\overline{M}) \cong H^r_{\mathfrak{m}}(\overline{M}/J\overline{M}) \neq 0.$$

Therefore $H^r_{\mathfrak{m},J}(\overline{M}) \neq 0$ and this implies that $\mathfrak{F}^r_{\mathfrak{a},I,J}(M) \neq 0$. This proof the statement.

□

Remark 4.3. If M be a finitely generated R-module then:

(a) $\mathfrak{F}^i_{\mathfrak{a},I,J}(M) = 0$ for any $i > \dim(M/\mathfrak{a}M)$. (see [18], Theorem 4.7)

(b) $\mathfrak{F}^i_{\mathfrak{a},I,J}(M) = 0$ for any $i > \dim(M/(\mathfrak{a}+J)M)$, if $J \neq R$. (see [18], Theorem 4.3)

(c) $\mathfrak{F}^i_{\mathfrak{a},I,J}(M) = 0$ for any $i > \dim R/J$. (see [18], Corollary 4.4)

(d) If M is (I,J)-torsion R, $\mathfrak{F}^i_{\mathfrak{a},I,J}(M) = 0$ for any i integer. (see [18], Corollary 1.13)

5 The Mayer-Vietoris sequence

Theorem 5.1. *Let $\mathfrak{a}, \mathfrak{b}, I, J$ ideals of a local ring $(R, \mathfrak{m}, k)$, $i \in \mathbb{Z}$ and M a finitely generated R-module. The there is the long exact sequence*

$$\cdots \to \check{\mathfrak{F}}^i_{\mathfrak{a}\cap\mathfrak{b},I,J}(M) \to \check{\mathfrak{F}}^i_{\mathfrak{a},I,J}(M) \oplus \check{\mathfrak{F}}^i_{\mathfrak{b},I,J}(M) \to \check{\mathfrak{F}}^i_{(\mathfrak{a},\mathfrak{b}),I,J}(M) \to \check{\mathfrak{F}}^{i+1}_{\mathfrak{a}\cap\mathfrak{b},I,J}(M) \to \cdots.$$

Proof. Let the following exact sequence

$$0 \to M/(\mathfrak{a}^n M \cap \mathfrak{b}^n M) \to M/\mathfrak{a}^n M \oplus M/\mathfrak{b}^n M \to M/(\mathfrak{a}^n, \mathfrak{b}^n)M \to 0.$$

Its induces a short exact sequence

$$0 \to \check{C}_{\underline{x},J} \otimes \frac{M}{(\mathfrak{a}^n M \cap \mathfrak{b}^n M)} \to (\check{C}_{\underline{x},J} \otimes \frac{M}{\mathfrak{a}^n M}) \oplus (\check{C}_{\underline{x},J} \otimes \frac{M}{\mathfrak{b}^n M}) \to \check{C}_{\underline{x},J} \otimes \frac{M}{(\mathfrak{a}^n, \mathfrak{b}^n)M} \to 0.$$

Because $\check{C}_{\underline{x},J}$ is a complex of flat R-modules and the maps

$$M/(\mathfrak{a}^{n+1} \cap \mathfrak{b}^{n+1})M \to M/(\mathfrak{a}^n \cap \mathfrak{b}^n)M$$

are surjective, it follows that the projective system of R-complexes $\{\check{C}_{\underline{x},J} \otimes M/\mathfrak{a}^n M \cap \mathfrak{b}^n M\}$ satisfies the Mittag-Leffler condition. Therefore, applying the inverse limit, we have the exact sequence of complexes

$$0 \to \varprojlim \check{C}_{\underline{x},J} \otimes \frac{M}{(\mathfrak{a}^n M \cap \mathfrak{b}^n M)} \to \varprojlim(\check{C}_{\underline{x},J} \otimes \frac{M}{\mathfrak{a}^n M}) \oplus \varprojlim(\check{C}_{\underline{x},J} \otimes \frac{M}{\mathfrak{b}^n M}) \to$$

$$\to \varprojlim \check{C}_{\underline{x},J} \otimes \frac{M}{(\mathfrak{a}^n, \mathfrak{b}^n)M} \to 0.$$

We can observe that the $(\mathfrak{a}^n, \mathfrak{b}^n)$-adic filtration is equivalent to the filtration $\{(\mathfrak{a}^n, \mathfrak{b}^n)M\}_{n\in\mathbb{N}}$. Then to finish the proof we have to show the $(\mathfrak{a}\cap\mathfrak{b})$-adic filtration on M is equivalent to the filtration $\{(\mathfrak{a}^n \cap \mathfrak{b}^n)M\}_{n\in\mathbb{N}}$. Note that $(\mathfrak{ab})^n M \subseteq (\mathfrak{a}^n \cap \mathfrak{b}^n)M \subseteq \mathfrak{a}^n M \cap \mathfrak{b}^n M$ for all $n \in \mathbb{N}$. Let $m \in \mathbb{N}$ denote a given integer. By Artin-Rees Lemma [2, Ch. III,3, Cor. 1], there exists an $k \in \mathbb{N}$ such that $\mathfrak{a}^n N \cap \mathfrak{b}^m N \subseteq \mathfrak{a}^{n-k}\mathfrak{b}^m N$ for all $n \geq k$. Note too that the $\mathfrak{ab}$-adic and the $\mathfrak{a}\cap\mathfrak{b}$-adic topology on M are equivalent. If consider the long exact cohomology sequence and the definition of Formal local cohomology defined by a pair of ideals finishes the proof.

□

Corollary 5.2. *Let $\mathfrak{a}, \mathfrak{b}, I, J$ ideals of a local ring $(R, \mathfrak{m}, k)$, $i \in \mathbb{Z}$ and M be a finitely generated R-module.*

(a) If M is J-torsion R-module, there is a long exact sequence

$$\cdots \to \mathfrak{F}^i_{\mathfrak{a}\cap\mathfrak{b},\mathfrak{m},J}(M) \to \mathfrak{F}^i_{\mathfrak{a},\mathfrak{m},J}(M) \oplus \mathfrak{F}^i_{\mathfrak{b},\mathfrak{m},J}(M) \to \mathfrak{F}^i_{(\mathfrak{a},\mathfrak{b}),\mathfrak{m},J}(M) \to \mathfrak{F}^{i+1}_{\mathfrak{a}\cap\mathfrak{b},\mathfrak{m},J}(M) \to \cdots$$

(b) If M is J-torsion R-module and $\sqrt{I+J} = \mathfrak{m}$, there is a long exact sequence

$$\cdots \to \mathfrak{F}^i_{\mathfrak{a}\cap\mathfrak{b},I,J}(M) \to \mathfrak{F}^i_{\mathfrak{a},I,J}(M) \oplus \mathfrak{F}^i_{\mathfrak{b},I,J}(M) \to \mathfrak{F}^i_{(\mathfrak{a},\mathfrak{b}),I,J}(M) \to \mathfrak{F}^{i+1}_{\mathfrak{a}\cap\mathfrak{b},I,J}(M) \to \cdots$$

(c) If M is Artinian R-module, there is a long exact sequence

$$\cdots \to \mathfrak{F}^i_{\mathfrak{a}\cap\mathfrak{b},I,J}(M) \to \mathfrak{F}^i_{\mathfrak{a},I,J}(M) \oplus \mathfrak{F}^i_{\mathfrak{b},I,J}(M) \to \mathfrak{F}^i_{(\mathfrak{a},\mathfrak{b}),I,J}(M) \to \mathfrak{F}^{i+1}_{\mathfrak{a}\cap\mathfrak{b},I,J}(M) \to \cdots.$$

Proof. We will go show the proof of a) and the other cases are analogous. Because M is J-torsion, any quotient of M is too J-torsion. Then, by Corollary 2.6 and theorem previous we have the statement.

□

6 Local duality for an pair of ideals

Let $(R, \mathfrak{m}, \mathbb{K})$ be a d-dimensional Cohen-Macaulay local ring with a canonical module ω. Then, for $0 \leq i \leq d$, it is well known the existence of isomorphisms

$$H^i_{\mathfrak{m}}(M) = \operatorname{Ext}^{d-i}_R(M, \omega)^\vee$$

where $(-)^\vee = \operatorname{Hom}_R(-, E_R(\mathbb{K}))$ and $H^d_{\mathfrak{m}}(R) \cong \omega^\vee$. This result is called local duality Theorem. There is a generalization of this result in [18, Theorem 5.1].

The purpose of this section is give a another proof of Local Duality Theorem for a pair of ideals and, in our context, obtain any results about formal local cohomology defined by a pair of ideals.

Lemma 6.1. *Let $(R, \mathfrak{m})$ denote a local ring, $\underline{x} = x_1, \cdots, x_n$ be a system of elements of R such that $\mathfrak{m} = (\underline{x})$ and J ideal of R. If M a finitely generated R-module then, for all $i \in \mathbb{Z}$, there are the isomorphisms*

$$H^i_{\mathfrak{m},J}(M) \cong \operatorname{Hom}_R(H^{-i}(\operatorname{Hom}_R(M, D_{\underline{x},J})), E)$$

where E denotes the injective hull of $R/\mathfrak{m}$ and $D_{\underline{x},J} = \operatorname{Hom}_R(\check{C}_{\underline{x},J}, E)$.

Proof. Proceeding analogously the construction made in [16, Theorem 1.7], change $D_{\underline{x}}$ by $D_{\underline{x},J}$ we obtain the result. □

Lemma 6.2. *Let $(R, \mathfrak{m}, \mathbb{K})$ be a local ring of dimension d, J be a perfect ideal of R of grade t, i.e, $\operatorname{pd}_R R/J = \operatorname{grade}(J, R) = t$. If R is Gorenstein then*

$$H^i_{\mathfrak{m},J}(R) = \begin{cases} 0 & \text{if } \; i \neq d-t \\ \bigoplus\limits_{\substack{ht\mathfrak{p}=d-t \\ \mathfrak{p}\in W(\mathfrak{m},J)}} E_R(R/\mathfrak{p}) & \text{if } \; i = d-t \end{cases}.$$

Proof. Let $I^\bullet$ be a minimal injective resolution of R. Since R is Gorenstein, for each i one has an isomorphism

$$I^i = \bigoplus_{ht\mathfrak{p}=i} E_R(R/\mathfrak{p}).$$

Applying the functor $\Gamma_{\mathfrak{m},J}(-)$ and using the Proposition 1.11 in [18] follows the complex

$$0 \to \bigoplus_{\substack{ht\mathfrak{p}=0 \\ \mathfrak{p}\in W(\mathfrak{m},J)}} E_R(R/\mathfrak{p}) \to \bigoplus_{\substack{ht\mathfrak{p}=1 \\ \mathfrak{p}\in W(\mathfrak{m},J)}} E_R(R/\mathfrak{p}) \to \bigoplus_{\substack{ht\mathfrak{p}=2 \\ \mathfrak{p}\in W(\mathfrak{m},J)}} E_R(R/\mathfrak{p}) \to \cdots$$

Now, by Corollary 4.4 and Lemma 5.2 in [18], $H^i_{\mathfrak{m},J}(R) = 0$ for $i \neq d-t$ and $H^{d-t}_{\mathfrak{m},J}(R) = \bigoplus\limits_{\substack{ht\mathfrak{p}=d-t \\ \mathfrak{p}\in W(\mathfrak{m},J)}} E_R(R/\mathfrak{p})$. This finishes the proof.

□

Theorem 6.3. *Let $(R, \mathfrak{m}, \mathbb{K})$ be a Gorenstein local ring of dimension d, J be a perfect ideal of R of grade t, i.e, $\operatorname{pd}_R R/J = \operatorname{grade}(J, R) = t$. If M is a finitely generated R-module, there are isomorphisms*

$$H^i_{\mathfrak{m},J}(M) \cong \operatorname{Ext}_R^{d-t-i}(M, S)^\vee$$

for all $0 \leq i \leq d-t$, where $(-)^\vee = \operatorname{Hom}_R(-, E_R(\mathbb{K}))$ and $S = H^{d-t}_{\mathfrak{m},J}(R)^\vee$.

Proof. Let $\underline{x} = x_1, \cdots, x_n$ elements of R such that $\mathfrak{m} = (\underline{x})$. Since $H^i(\check{C}_{\underline{x},J}) \cong H^i_{\mathfrak{m},J}(R)$, by Lemma 6.2 follows

$$H^i(\check{C}_{\underline{x},J}) = \begin{cases} 0 & \text{if } \; i \neq d-t \\ \displaystyle\bigoplus_{\substack{ht\mathfrak{p}=d-t \\ \mathfrak{p}\in W(\mathfrak{m},J)}} E_R(R/\mathfrak{p}) & \text{if } \; i = d-t \end{cases}.$$

Denote $\overline{E} = \displaystyle\bigoplus_{\substack{ht\mathfrak{p}=d-t \\ \mathfrak{p}\in W(\mathfrak{m},J)}} E_R(R/\mathfrak{p})$, follows that $\check{C}_{\underline{x},J}$ is a flat resolution of $\overline{E}$ shifted $d-t$ places to the right. Therefore $D_{\underline{x},J} = \operatorname{Hom}_R(\check{C}_{\underline{x},J}, E_R(\mathbb{K}))$ is an injective resolution of $\operatorname{Hom}_R(\overline{E}, E)$ shifted $d-t$ places to the right. Since

$$H^{-i}(\operatorname{Hom}_R(M, D_{\underline{x},J})) \cong \operatorname{Ext}_R^{d-t-i}(M, \operatorname{Hom}_R(\overline{E}, E))$$

and $\operatorname{Hom}_R(\overline{E}, E) = H^i_{\mathfrak{m},J}(R)^\vee$ by Lemma 6.2, applying Lemma 6.1 we have the statement. □

The natural question is : The same theorem is true when R is Cohen Macaulay?.
For answer this we need a preliminary observations. Let R be a commutative noetherian ring, I, J two ideals of R and M be a R-module. Let

$$\operatorname{depth}(I, J, M) = \inf\{i \in \mathbb{N}_0 \; ; \; H^i_{I,J}(M) \neq 0\}.$$

If we consider M is a finitely generated module over a local ring $(R, \mathfrak{m})$ and $J \neq R$, by Theorem 4.5 in [18] and definition above, we have $H^i_{\mathfrak{m},J}(M) \neq 0$ for all

$$\operatorname{depth}(\mathfrak{m}, J, M) \leq i \leq \dim M/JM.$$

When $\operatorname{depth}(\mathfrak{m}, J, M) = \dim M/JM$, the R-module $M \neq 0$ is called $(\mathfrak{m}, J)$-Cohen Macaulay (or if $M = 0$). If R itself is an $(\mathfrak{m}, J)$-Cohen-Macaulay R-module we say that R is an $(\mathfrak{m}, J)$-Cohen Macaulay ring. In this definition its obvious that $J \neq 0$. Note too that if $J = 0$ this natural definition of $(\mathfrak{m}, J)$-Cohen-Macaulay coincides with definition of Cohen-Macaulay R-modules. The same definition can be made for any I, J two ideals of R and for this, for more details we recommend see [1]. Under this comments, we will go answer the question previous.

Theorem 6.4. *Let $M \neq 0$ be a finitely generated module over a local ring $(R, \mathfrak{m}, \mathbb{K})$. Suppose that R is (I, J)-Cohen-Macaulay where $I + J$ is an $\mathfrak{m}$-primary ideal. Then, there are isomorphisms*

$$H^i_{I,J}(M)^\vee \cong \operatorname{Ext}^{\widehat{d}-i}_R(M, S)$$

for all $0 \le i \le \widehat{d}$, where $(-)^\vee = \operatorname{Hom}_R(-, E_R(\mathbb{K}))$, $S = H^{\widehat{d}}_{I,J}(R)^\vee$ and $\widehat{d} := \dim(M/JM)$.

Proof. First note that since $I + J$ is is an $\mathfrak{m}$-primary ideal, by Proposition 1.4 (6),(7) in [18] we have $H^i_{I,J}(R) = H^i_{\mathfrak{m},J}(R)$ for any i integer, i.e, in this case R is (I, J)-Cohen Macaulay if and only if M is $(\mathfrak{m}, J)$-Cohen-Macaulay. Thus we may assume that $I = \mathfrak{m}$.

Let $\underline{x} = x_1, \cdots, x_n$ elements of R such that $\mathfrak{m} = (\underline{x})$. Since $H^i(\check{C}_{\underline{x},J}) \cong H^i_{\mathfrak{m},J}(R)$ and R is $(\mathfrak{m}, J)$-Cohen-Macaulay, $\check{C}_{\underline{x},J}$ is a flat resolution of $H^{\widehat{d}}_{\mathfrak{m},J}(R)$ shifted $\widehat{d}$ places to the right because $H^i_{\mathfrak{m},J}(R) = 0$ for all $i \neq \widehat{d}$ (see [18, Theorem 4.5] or [1, Corollary 4.13]). Now,

$$H^i_{\mathfrak{m},J}(M) \cong H^i(\check{C}_{\underline{x},J}[-\widehat{d}] \otimes_R M) \cong H_{\widehat{d}-i}(\check{C}_{\underline{x},J} \otimes_R M) \cong \operatorname{Tor}^R_{\widehat{d}-i}(H^{\widehat{d}}_{\mathfrak{m},J}(R), M).$$

Let $K^\bullet$ be a free resolution of M. Thus, as $H_{\widehat{d}-i}(K^\bullet \otimes_R H^{\widehat{d}}_{\mathfrak{m},J}(R)) \cong \operatorname{Tor}^R_{\widehat{d}-i}(M, H^{\widehat{d}}_{\mathfrak{m},J}(R))$, follows $H^i_{\mathfrak{m},J}(M) \cong H_{\widehat{d}-i}(K^\bullet \otimes_R H^{\widehat{d}}_{\mathfrak{m},J}(R))$. Therefore, for all i, we have

$$\begin{aligned} H^i_{\mathfrak{m},J}(M)^\vee &\cong H_{\widehat{d}-i}(K^\bullet \otimes_R H^{\widehat{d}}_{\mathfrak{m},J}(R))^\vee \\ &\cong H^{\widehat{d}-i}((K^\bullet \otimes_R H^{\widehat{d}}_{\mathfrak{m},J}(R))^\vee) \\ &\cong H^{\widehat{d}-i}(\operatorname{Hom}_R(K^\bullet \otimes_R H^{\widehat{d}}_{\mathfrak{m},J}(R), E_R(\mathbb{K}))) \\ &\cong H^{\widehat{d}-i}(\operatorname{Hom}_R(K^\bullet, H^{\widehat{d}}_{\mathfrak{m},J}(R)^\vee)) \\ &\cong \operatorname{Ext}^{\widehat{d}-i}_R(M, H^{\widehat{d}}_{\mathfrak{m},J}(R)^\vee). \end{aligned}$$

□

Remark 6.5. Note that this theorem is a generalization of Theorem 5.1 in [18] because, if $(R, \mathfrak{m})$ is a Cohen-Macaulay complete local ring of dimension d and J be a perfect ideal of R such that $\operatorname{grade}(J, R) = t$, then $\dim R/J = d-t$. Therefore

$$H^i_{\mathfrak{m},J}(M)^\vee \cong \operatorname{Ext}^{d-t-i}_R(M, H^{\widehat{d}}_{\mathfrak{m},J}(R)^\vee)$$

for all integer i by theorem above.

Remark 6.6. Which the same hypothesis of theorem above and suppose that R is (I,J)-torsion R-module we obtain, by Corollary 1.13 in [18], that R/J is an Artinian R-module. Therefore $\Gamma_{I,J}(R) \cong \Gamma_{I,J}(R)^{\vee}$.

We are interested here now, using this previous results, is an characterization of formal local cohomology defined by a pair of ideals using local cohomology and Matlis duality functor. The next result show this relation.

Theorem 6.7. *Let $(R,\mathfrak{m})$ denote a local ring, $\underline{x} = x_1, \cdots, x_n$ be a system of elements of R such that $\mathfrak{m} = (\underline{x})$ and J ideal of R. If M is a finitely generated R-module then, for all $i \in \mathbb{Z}$, there are the isomorphisms*

$$\mathfrak{F}^i_{\mathfrak{a},\mathfrak{m},J}(M) \cong \operatorname{Hom}_R(H^{-i}_{\mathfrak{a}}(\operatorname{Hom}_R(M, D_{\underline{x},J})), E_R(\mathbb{K})).$$

Proof. By Lemma 6.1, for $n \in \mathbb{N}$, there are the isomorphisms

$$H^i_{\mathfrak{m},J}(M/\mathfrak{a}^n M) \cong \operatorname{Hom}_R(H^{-i}(\operatorname{Hom}_R(M, D_{\underline{x},J})), E_R(\mathbb{K}))$$

for all $i \in \mathbb{Z}$. By passing the projective limit and using the fact that $\varinjlim \operatorname{Hom}_R(M/\mathfrak{a}^n M, D_{\underline{x},J}) \cong \Gamma_{\mathfrak{a}}(\operatorname{Hom}_R(M/\mathfrak{a}^n M, D_{\underline{x},J}))$ we obtain the statement. □

Remark 6.8. In the other hand, using the same hypothesis in Theorem 6.4 we obtain

$$\mathfrak{F}^i_{\mathfrak{a},I,J}(M) \cong \operatorname{Hom}_R(\varinjlim \operatorname{Ext}^{\widehat{d}-i}_R(M/\mathfrak{a}^n M, S), E_R(\mathbb{K})).$$

Note that, for all $i \in \mathbb{Z}$, $\varinjlim \operatorname{Ext}^{\widehat{d}-i}_R(M/\mathfrak{a}^n M, S)$ is exactly the generalized local cohomology with respect to $\mathfrak{a}$ (denoted by $H^{\widehat{d}-i}_{\mathfrak{a}}(M,S)$), introduced by Herzog in [9]. Therefore

$$\mathfrak{F}^i_{\mathfrak{a},I,J}(M) \cong H^{\widehat{d}-i}_{\mathfrak{a}}(M,S)^{\vee}$$

where $(-)^{\vee} = \operatorname{Hom}_R(-, E_R(\mathbb{K}))$, $i \in \mathbb{Z}$. This show the relation between the formal local cohomology defined by a pair of ideals and the Matlis' dual of certain generalized local cohomology with respect to $\mathfrak{a}$.

For the next result we first need any considerations. Using the natural homomorphism $R \to \widehat{R}$, where $(\widehat{R}, \widehat{\mathfrak{m}})$ denote the $\mathfrak{m}$-adic completion of $(R, \mathfrak{m}, \mathbb{K})$, by Theorem 2.3 we may assume the existence of the complex

$D_{\underline{x},J} = \mathrm{Hom}_R(\check{C}_{\underline{x},J}, E_R(\mathbb{K}))$. Now if $x \in \mathfrak{m}$, we are interested to relate how the $\mathfrak{a}$-formal local cohomology and $(\mathfrak{a}, x)$-formal local cohomology, both defined by a pair of ideals, are connected. The long exact sequence below show this relation.

Theorem 6.9. *Let $(R, \mathfrak{m})$ denote a local ring, $\underline{x} = x_1, \cdots, x_n$ and $\underline{y} = y_1, \cdots, y_n$ system of elements of R such that $\mathfrak{m} = (\underline{x})$, $\mathfrak{a} = (\underline{y})$ and J ideal of R. If M a finitely generated R-module and $x \in \mathfrak{m}$ element of R, there is the long exact sequence*

$$\cdots \to \mathrm{Hom}_R(R_{x,J}, \mathfrak{F}^i_{\mathfrak{a},\mathfrak{m},J}(M)) \to \mathfrak{F}^i_{\mathfrak{a},\mathfrak{m},J}(M) \to \mathfrak{F}^i_{(\mathfrak{a},x),\mathfrak{m},J}(M) \to \cdots$$

for all $i \in \mathbb{Z}$.

Proof. By comment above, let the complex $D_{\underline{x},J}$ and $\check{C}_{x,J}$ the $\check{C}$ech complex for an element $x \in \mathfrak{m}$. So there is the short exact sequence of flat R-modules.

$$0 \to R_{x,J}[-1] \to \check{C}_{x,J} \to R \to 0.$$

Let $\widetilde{\mathrm{H}} = \mathrm{Hom}_R(M, D_{\underline{x},J})$. Tensoring the exact sequence above with $\check{C}_{\underline{y},J} \otimes \widetilde{\mathrm{H}}$, it induces the following exact sequence of R-modules

$$0 \to \check{C}_{\underline{y},J} \otimes \widetilde{\mathrm{H}} \otimes R_{x,J}[-1] \to \check{C}_{\underline{y},x,J} \otimes \widetilde{\mathrm{H}} \to \check{C}_{\underline{y},J} \otimes \widetilde{\mathrm{H}} \to 0.$$

Now, seeing the long exact cohomology sequence together with Theorem 2.4 in [18] we obtain, for all $i \in \mathbb{Z}$,

$$\cdots \to H^i_{(\mathfrak{a},xR),J}(\widetilde{\mathrm{H}}) \to H^i_{\mathfrak{a},J}(\widetilde{\mathrm{H}}) \to H^i_{\mathfrak{a},J}(\widetilde{\mathrm{H}}) \otimes R_{x,J} \to \cdots .$$

By applying the functor $\mathrm{Hom}_R(-, E_R(\mathbb{K}))$ and the Theorem 6.7 we obtain the result. $\square$

The natural consequence and application of this Theorem follow taking $\mathfrak{a} = 0$. This result relates the formal local cohomology with respect to an ideal generated by a single element and local cohomology , both defined by a pair of ideals.

Corollary 6.10. *With the same hypothesis of Theorem above, there is a short exact sequence*

$$\cdots \to \mathrm{Hom}_R(R_{x,J}, H^i_{\mathfrak{m},J}(M)) \to H^i_{\mathfrak{m},J}(M) \to \mathfrak{F}^i_{xR,\mathfrak{m},J}(M) \to \cdots$$

for all $i \in \mathbb{Z}$.

7 Formal grade with respect to a pair of ideals

Let $(R, \mathfrak{m})$ is a local ring, $I, J, \mathfrak{a}$ ideals as above and M denote a finitely generated R-module. The concept of formal grade was introduced by Peskine and Szpiro in [14] and not so much is known about this tool. I our approach, since some any cases $\mathfrak{F}^i_{\mathfrak{a},I,J}(M) \cong \check{\mathfrak{F}}^i_{\mathfrak{a},I,J}(M)$, we need to give two definitions for formal grade, different for the approach given by Schenzel in [15].

Definition 7.1. *For an ideal $\mathfrak{a}$ of R define by*

$$\operatorname{fgrade}(\mathfrak{a}, I, J, M) = \inf\{i \in \mathbb{Z} : \mathfrak{F}^i_{\mathfrak{a},I,J}(M) \neq 0\}$$

and

$$\check{f}\mathrm{grade}(\mathfrak{a}, I, J, M) = \inf\{i \in \mathbb{Z} : \check{\mathfrak{F}}^i_{\mathfrak{a},I,J}(M) \neq 0\}.$$

Theorem 7.2. *Let $(R, \mathfrak{m}, \mathbb{K})$ be a Cohen-Macaulay complete local ring of dimension d and let $J \neq 0$ be a perfect ideal of R of grade t, i.e,* $\operatorname{pd}_R(R/J) = \operatorname{grade}(J, R) = t$. *Then, for M be a finitely generated R-module,*

$$\operatorname{fgrade}(\mathfrak{a}, \mathfrak{m}, J, M) + \operatorname{cd}_{\mathfrak{a}}(M, S) + \operatorname{grade}(J, R) = \dim R,$$

where $S = H^{d-t}_{\mathfrak{m},J}(R)^\vee$.

Proof. By Theorem 6.4 $H^i_{\mathfrak{m},J}(M) \cong \operatorname{Hom}_R(\operatorname{Ext}^{d-t-i}_R(M, S), E_R)$. Thus

$$\begin{aligned}
\mathfrak{F}^i_{\mathfrak{a},\mathfrak{m},J}(M) &= \varprojlim H^i_{\mathfrak{m},J}(M/\mathfrak{a}^n M) \\
&\cong \varprojlim \operatorname{Hom}_R(\operatorname{Ext}^{d-t-i}_R(M/\mathfrak{a}^n M, S), E_R(\mathbb{K})) \\
&= \operatorname{Hom}_R(\varinjlim \operatorname{Ext}^{d-t-i}_R(M/\mathfrak{a}^n M, S), E_R(\mathbb{K}))
\end{aligned}$$

and since $H^i_{\mathfrak{a}}(M, S) = \varinjlim \operatorname{Ext}^{d-t-i}_R(M/\mathfrak{a}^n M, S)$ (see [9]), for all $i \in \mathbb{Z}$, there are isomorphisms

$$\mathfrak{F}^i_{\mathfrak{a},\mathfrak{m},J}(M) \cong \operatorname{Hom}_R(H^{d-t-i}_{\mathfrak{a}}(M, S), E_R(\mathbb{K})).$$

Therefore

$$\begin{aligned}
\inf\{i \in \mathbb{Z} : \mathfrak{F}^i_{\mathfrak{a},\mathfrak{m},J}(M) \neq 0\} &= \inf\{i \in \mathbb{Z} : H^{d-t-i}_{\mathfrak{a}}(M, S) \neq 0\} \\
&= \inf\{d - t - j : H^j_{\mathfrak{a}}(M, S) \neq 0\} \\
&= d - t - \sup\{j : H^j_{\mathfrak{a}}(M, S) \neq 0\} \\
&= \dim R - \operatorname{grade}(J, R) - \operatorname{cd}_{\mathfrak{a}}(M, S).
\end{aligned}$$

□

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Victor Hugo Jorge Pérez
Universidade de São Paulo - ICMC
email: vhjperez@icmc.usp.br

Thiago Henrique de Freitas
Universidade de São Paulo - ICMC
email: tfreitas@icmc.usp.br